The Origin of the Divergences of Quantum Field Theory

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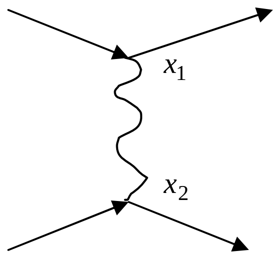
1. The Usual Story

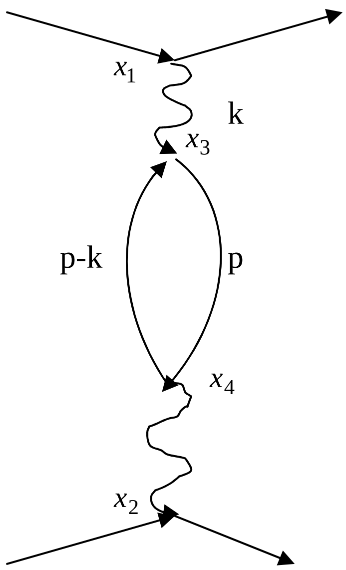
Why is quantum field theory (QFT) divergent? The neophyte, having mastered the arcane formalism of QFT, comes across the divergences for the first time when (s)he tries to calculate the second order term in a scattering amplitude. The infinite result presents itself in the form of a momentum space integral which is divergent. For example, in quantum electrodynamics (QED), the lowest order term in (say) electron-electron scattering is given by the Feynman diagram,

Figure 1



which is perfectly finite (and produces a good enough result to be proud of in most circumstances, perhaps accurate to ~1% or so). However, one of the Feynman graphs contributing to the second order correction is,

Figure 2



The virtual electron-positron loop which has entered the virtual photon line contributes to the scattering amplitude due to this graph a factor of,

$$\alpha\int d^4 p \cdot D(p)D(p-k) = -\alpha\int \frac{d^4 p}{(\not{p}-m)(\not{p}-\not{k}-m)} \qquad (1)$$

where $D(p)=\frac{i}{\not p - m}$ is the (spinor valued) Feynman propagator for the electron or positron ($\not p \equiv \gamma^\mu p_\mu$) and α is the fine structure constant (~1/137). But (1) is, of course, divergent since its large-p behaviour is $\sim \int \frac{d^4 p}{p^2} \sim \int pdp$, since $d^4 p \sim p^3 dp$.

Why does this happen? Where did we go wrong?

The purpose of this brief essay is to trace the source of this problem. We shall argue that this debacle is inherent in the most fundamental aspects of the formulation of quantum mechanics. The problem originates from the canonical commutation relations - that conjugate variables behave thus: $[p, x] = i\hbar$. These are the seeds of our destruction. Let us start with that.

2. Lifting the Carpet

Consider any pair of dynamically conjugate observables, represented by Hermetian operators $\hat{P}$ and $\hat{Q}$ on some Hilbert space. QM tells us they will obey the canonical commutation relation, $[\hat{P}, \hat{Q}] = i\hbar$. Initially let us assume that both observables take values from a discrete set, $\{p_i\}$ and $\{q_i\}$, i.e., they have discrete eigenvalues. A contradiction immediately results from the usual formulation, as follows.

Putting $\hat{Q}|q_j\rangle = q_j|q_j\rangle$, the Hermetian nature of $\hat{Q}$ means that the eigenvalues are real and $\langle q_i|\hat{Q} = \langle q_i|q_i$. It also means that the eigenvalues are orthogonal (and can be assumed normalised), so that $\langle q_i|q_j\rangle = \delta_{ij}$. Now consider the matrix element of the commutator between two q-states,

$$\langle q_i|(\hat{P}\hat{Q} - \hat{Q}\hat{P})|q_j\rangle = \langle q_i|(\hat{P}q_j - q_i\hat{P})|q_j\rangle = (q_j - q_i)\langle q_i|\hat{P}|q_j\rangle = i\hbar\delta_{ij} \qquad (2)$$

For i = j this results in the contradiction that the RHS is non-zero but the LHS is zero.

Have we really been so foolish as to erect our supposedly most fundamental theory of physics upon foundations so obviously and seriously flawed? It may well be that the answer is “yes”. But elementary QM must implicitly include some means of avoiding this disaster.

It does. But it is not spelled out in texts with which I am familiar. QM generally avoids the problem by not considering conjugate pairs of observables which both have discrete sets of eigenvalues. To see how this is used to circumvent the difficulty consider a free particle in infinite Euclidean space. Equ.(2) becomes,

$$(\bar{r} - \bar{r}')\langle\bar{r}'|\hat{\bar{P}}|\bar{r}\rangle = i\hbar\delta^3(\bar{r}' - \bar{r}) \qquad (3)$$

It is not obvious that this has helped, until it is remembered that the matrix elements of $\hat{\bar{P}} = i\hbar\bar{\nabla}$ with respect to the free states $\exp(i\bar{p}\cdot\bar{r})$ are,

$$\langle\bar{r}'|\hat{\bar{P}}|\bar{r}\rangle = i\hbar\bar{\nabla}\delta^3(\bar{r}' - \bar{r}) \qquad (4)$$

and also that,

$$\bar{\nabla}\delta^3(\bar{r}' - \bar{r}) \equiv -\frac{\delta^3(\bar{r}' - \bar{r})}{\bar{r}' - \bar{r}} \qquad (5)$$

So (3) is correct. Both sides are zero when $\bar{r}' \neq \bar{r}$ and when $\bar{r}' = \bar{r}$, $\langle \bar{r}' | \hat{\bar{P}} | \bar{r} \rangle$ is so strongly divergent that it overcomes the zero in $(\bar{r} - \bar{r}')$ and the product is still divergent!

Interpreting P and Q the other way around requires,

$$(\bar{p} - \bar{p}')\langle \bar{p}' | \hat{\bar{r}} | \bar{p} \rangle = i\hbar\delta^3(\bar{p}' - \bar{p}) \tag{6}$$

In infinite space this is consistent for the same reason, i.e., we have,

$$\langle \bar{p}' | \hat{\bar{r}} | \bar{p} \rangle = \delta'(p'_x - p_x)\delta'(p'_y - p_y)\delta'(p'_z - p_z) \equiv -\frac{\delta^3(\bar{p}' - \bar{p})}{\bar{p}' - \bar{p}} \tag{7}$$

were δ' denotes the derivative of the Dirac delta function. So far, so good. But things start to go wrong even when we consider something as innocent as imposing periodic boundary conditions, so that the wavefunctions are required to be periodic with a period of *L*, say. Confining attention to the 1D case, the periodic wavefunctions are $\exp(ip_n x)$ with $p_n = 2\pi n / L$ (note that we have put $\hbar = 1$). This is consistent with orthonormality and completeness relations over the region $x \in [0, L]$,

$$\langle x' | x \rangle = \delta(x' - x) \qquad \text{and} \qquad \langle p_n | p_m \rangle = \delta_{nm} \tag{8}$$

We can also show that (4) remains true, so that (2) is again consistent due to the severe δ' divergence in $\langle \bar{r}' | \hat{\bar{P}} | \bar{r} \rangle$. However, when we consider (2) in the form,

$$(p_n - p_m)\langle p_m | \hat{x} | p_n \rangle = i\hbar\delta_{nm} \tag{9}$$

we run into trouble again. For $n \neq m$, both sides of (9) are zero – fine. But now we find that for $n = m$, $\langle p_n | \hat{x} | p_n \rangle$ is finite. This is inevitable since it is evaluated as the integral of finite quantities over a finite domain. Specifically, (possibly ignoring some constant factors),

$$\langle p_m | \hat{x} | p_n \rangle = \int_0^L x \cdot \exp\left\{ i \frac{2\pi(n-m)}{L} x \right\} \cdot \frac{dx}{L} \tag{10}$$

which is zero for $n = m$, as required, but,

$$\langle p_n | \hat{x} | p_n \rangle = \int_0^L x \cdot \frac{dx}{L} \propto L \tag{11}$$

Consequently for $n = m$ the LHS of (9) is $0 \times L$ and the RHS is $\sim \hbar$. A sort of sense can be restored to this as long as the periodicity, *L*, is large. The accuracy with which $(p_n - p_m)$ can be measured to be zero is limited, in a box of side *L*, to be within $\hbar / L$. So, in a crude heuristic sense, the LHS of (9) is then $\sim \frac{\hbar}{L} \times L \approx \hbar$, and hence compatible with the RHS.

However, this explanation relies upon introducing a rather *ad hoc* physical rationalisation into what should be a clean mathematical position. Am I alone in finding this grossly unsatisfactory? And why is this problem not exposed in the standard texts?

A further example is angular momentum. Surely this provides an instance of clean, finite good sense? Certainly the derivation of the eigen structure of $\left[L_j, L_k\right] = i\hbar \in_{jkn} L_n$ is faultless and finite. But derivation of the eigen structure does not require the variable conjugate to $\overline{L}$. If one brings the angular position variable into play, the same problem occurs as exposed in (2) and (9). The usual 'explanation' of this is that the periodic nature of the angular coordinate means that either it is multiple valued, or, if confined by *fiat* to the region $[0,2\pi]$, then it is discontinuous. But how impressed should we be by such an excuse?

Quantum mechanics is all about discreteness. That is its essence. In quantum mechanics, systems are ***fully*** described by a finite set of integers. For free particles, discrete states result from topological compactness. The periodicity of rotations results in the discrete angular momentum states, and imposing translational periodicity results in discrete linear momentum states. But these are the very cases where (2) becomes inconsistent.

So is there something rotten in the state of quantum mechanics? It seems that to avoid the problem posed by (2) we need to adopt the continuum. This is contrary to the finite spirit of quantum systems, and, ultimately, cannot be physical if systems truly are fully specified by finite sets of integers. The continuum exists only in mathematics not in the physical world. But nevertheless the continuum has been embedded at the heart of quantum mechanics since Dirac introduced his delta function. This was our compact with the Devil and our downfall.

3. Field Theory: Commutators and Propagators

Field theory, of course, re-emphasises the point. A field is, by definition, defined over the spacetime continuum. In view of the previous discussion, adoption of a field description may seem perfectly sensible. After all, the only case considered above which was free of the contradiction posed by (2) was the infinite continuum. Accordingly, then, field theory is based upon the assumption that a field and its conjugate field obey the canonical (equal time) commutation relations, thus,

$$[\phi_r(\overline{r},t), \pi_s(\overline{r}',t)] = i\hbar\delta_{rs}\delta^3(\overline{r}-\overline{r}') \tag{12}$$

where,
$$\pi_s \equiv \frac{\partial L}{\partial \phi_{s,0}} \tag{13}$$

The subscripts on ϕ_r, π_s may denote different scalar fields or the different components of a field with spin, it does not matter which. The canonical commutation relations of the fields are equivalent to those of Fock space creation and annihilation operators,

$$\left[a_r(\overline{k}), a_s^+(\overline{k}')\right] = V\delta^3(\overline{k}-\overline{k}')\delta_{rs} \qquad \left[b_r(\overline{k}), b_s^+(\overline{k}')\right] = V\delta^3(\overline{k}-\overline{k}')\delta_{rs} \tag{14}$$

other commutators being zero, and where the fields are expanded in terms of plane waves (Fourier terms) as,

$$\phi_r(x) = \int \sqrt{\frac{V\hbar c^2}{2\omega_{\overline{k}}}} \cdot \left\{a_r(\overline{k})e^{-ikx} + b_r^+(\overline{k})e^{ikx}\right\} d^3k \tag{15}$$

In (14) and (15), V is some suitably large normalisation volume which will factor out of any observable quantities. *I may have dropped some factors of* $(2\pi)^3$ *from either*

(14) or (15)?? In the exponents, $kx \equiv k_\mu x^\mu = (Et - \bar{p}\cdot\bar{r})/\hbar$, and the integral is carried out on the understanding that it is "on mass shell", $k_0 = E/\hbar = \omega_{\bar{k}} = c\sqrt{|\bar{k}|^2 + (mc/\hbar)^2}$.

From the effect of the Fock operators on the vacuum, specifically that $a(\bar{k})|0\rangle = 0$, and from the expansion (15), we readily derive the vacuum expectation values of products of field operators. For example,

$$\langle 0|\phi_r(x)\phi_s(x')|0\rangle = i\hbar c\delta_{rs}\Delta^+(x-x') \tag{16}$$

where,

$$\Delta^\pm(x-x') \equiv \pm\frac{\hbar c^2}{2(2\pi)^3}\int\frac{d^3k}{\omega_{\bar{k}}}e^{\mp ik(x-y)} \tag{17}$$

In (16, 17) we have now assumed that the fields are scalars for simplicity of exposition. Otherwise the $\Delta^\pm$ functions would have spin labels, *rs*, e.g., they would be spinor valued for fermions. This is without loss of generality as regards the burden of our argument. We will see that the QFT divergences arise due to the spacetime structure of the fields. Consequently any spin/vector/tensor labels on these quantities would just go along for a free ride and make no substantive difference to the observations below.

Note, however, that the functions (17) will differ for different scalar fields due to the mass dependence implicit in the on-shell $\omega_{\bar{k}}$.

In practice it is the vacuum expectation values of the time-ordered products of fields which are required to derive physical quantities. These give rise to the Feynman propagator,

$$\langle 0|T\{\phi_r(x)\phi_s(x')\}|0\rangle = i\hbar c\delta_{rs}\Delta_F(x-x') \tag{18}$$

Where,

$$\Delta_F(x) \equiv \theta(t)\Delta^+(x) - \theta(-t)\Delta^-(x) \tag{19}$$

and $\theta(t)$ is the unit step function. Note that $\Delta_F(-x) = \Delta_F(x)$ and that when $t=0$, $\Delta^\pm(x)$ are real, equal and opposite and non-zero. Consequently Δ_F is continuous over $t=0$ and non-zero.

The key observation is that the functions $\Delta^\pm(x), \Delta_F(x)$ are divergent at $x=0$. This follows immediately from (17) since $\int\frac{d^3k}{\omega_{\bar{k}}} \to \int kdk \to \infty$. This will shortly be seen to be the source of all the divergences in QFT. It is appropriate, therefore, to pause to consider just why these propagators are divergent. There are two perspectives on this.

The first is to note that the fields must obey the Euler-Lagrange equations of motion, and, for the usual Lagrangians, this implies that the free fields have the form of plane waves, as in (15). Provided that we also assume that the creation/annihilation operators have the commutation relations (14) – and this would appear to be necessary in order to produce the 'right' commutation relations for the fields, (12) – then (16-19) follow as a consequence.

There is, however, a more basic motivation for this conclusion, which does not depend upon the fields having a creation/annihilation structure. This starts by noting that the canonical commutation relations, (12), are not explicitly Lorentz covariant.

This is because they have been written at equal times, which is a frame specific concept. How can they be made covariant? Start by noting that,

$$\delta^3(\bar{r}) = \frac{1}{(2\pi)^3}\int e^{-i\bar{k}\cdot\bar{r}} \cdot d^3k \tag{20}$$

The obvious nearest equivalent of the RHS of (20) which comprises Lorentz invariant parts is $\int e^{-ikx} \cdot d^4k$. There are two problems with this. Firstly it produces a factor of $\delta(t)$, which is not compatible with the RHS of (12). Secondly, it is an off-mass-shell integral, effectively integrating over all particle masses as well as their momenta. The latter problem can be rectified by inserting $\delta(k^2 - m^2)$ into the integrand. This makes the integral purely on mass shell, *m*. Moreover, because $k^2 - m^2$ is a Lorentz scalar it preserves the invariance of the integral. Finally, note that the Dirac delta function has the property,

$$\int g(x)\delta(f(x))dx = \sum \left.\frac{g}{f'}\right|_{f=0} \tag{21}$$

The RHS sums over all the zeros of the function $f(x)$. Thus we have,

$$\delta(k^2 - m^2) \equiv \frac{c}{2\omega_{\bar{k}}}\{\delta(k_0 - \omega_{\bar{k}}/c) + \delta(k_0 + \omega_{\bar{k}}/c)\} \tag{22}$$

Consequently the desire to re-write the equal time commutation relations in a covariant form results in,

$$\begin{aligned}[\phi_r(x), \pi_s(x')] &\sim \delta_{rs}\int e^{-ik(x-x')} \cdot \frac{d^4k}{2\omega_{\bar{k}}}\{\delta(k_0 - \omega_{\bar{k}}/c) + \delta(k_0 + \omega_{\bar{k}}/c)\} \\ &\sim \int \frac{\mathrm{d}^3k}{2\omega_{\bar{k}}} e^{-ik(x-x')} + \int \frac{\mathrm{d}^3k}{2\omega_{\bar{k}}} e^{+ik(x-x')} \\ &\sim \Delta^+(x-x') + \Delta^-(x-x')\end{aligned} \tag{23}$$

[*In (23) the ~ just means I've been lazy about bothering to track the right constant factors*]. So we see that it is really the Lorentz covariant extension of the canonical, equal time, commutation relations that appears to lead inexorably to these singular $\Delta^{\pm}$ functions.

As an aside, note that (22) displays the true origin of the denominator of $\omega_{\bar{k}}$ which appears in the definition of the fields, (15). Some authors are guilty of stating that this is merely, "a normalisation factor introduced for later convenience". This is appallingly misleading. Actually the denominator of $\omega_{\bar{k}}$ is crucial to the Lorentz transformation properties of the fields, as the above derivation shows. But also, its presence is crucial in the definition of the propagators, (17). Without the denominator of $\omega_{\bar{k}}$ the propagators would be different and the predictions of field theory for scattering amplitudes, decay times, etc., would be different – and plain wrong!

4. Transition Amplitudes

The bulk of our argument is already done. It remains only to show how terms of second order and above in α generally involve products of Feynman propagators at *the same spacetime point*. This is the fatal feature. Single factors of functions like

$\delta(x-x')$ or $\Delta^{\pm}(x-x')$ are tolerable in integrands since the integrals will generally be finite despite the singularity in the integrands. But something like $\int (\delta(x-x'))^2 dx$ is catastrophic since it produces (heuristically speaking) "$\delta(0)$", i.e., it is divergent. But scattering amplitudes in field theory of second order and above generally include factors which are quadratic, or of higher order, in Δ_F evaluated at the same spacetime point. Note that it is the evaluation at the same spacetime point (or interval) which is the problem (or, as we see later, a sequence around any closed loop). An expression like $\int \delta(x-x_1)\delta(x-x_2)dx = \delta(x_1-x_2)$ is sensible when $x_1 \neq x_2$.

What remains is just standard text book stuff, deriving the Feynman diagrams / integrals from the interaction Hamiltonian. However, here we shall work in spacetime not momentum space. Consider an interaction Hamiltonian between two scalar fields of the form,

$$H_I = e\psi^2\phi \tag{24}$$

Hence ψ is a scalar surrogate for the electron, and ϕ is a scalar surrogate for the photon. In the interaction picture, the time development of a state $|t\rangle$ is given by,

$$i\hbar\frac{d}{dt}|t\rangle = H_I|t\rangle \tag{25}$$

Hence, $i\hbar|t_2\rangle = i\hbar|t_1\rangle + \int_{t_1}^{t_2} H_I(t')|t'\rangle dt'$ and by repeated substitution into the integrand we get,

$$\begin{aligned} |out\rangle &= \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{\infty} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \ldots\ldots \int_{t_0}^{t_{n-1}} dt_n \cdot H_I(t_1)H_I(t_2)H_I(t_3)\ldots.H_I(t_n)|in\rangle \\ &= \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n \cdot \frac{1}{n!}\int_{t_0}^{\infty} dt_1 \int_{t_0}^{\infty} dt_2 \int_{t_0}^{\infty} dt_3 \ldots\ldots \int_{t_0}^{\infty} dt_n \cdot T\{H_I(t_1)H_I(t_2)H_I(t_3)\ldots.H_I(t_n)\}|in\rangle \end{aligned} \tag{26}$$

In (26), t_0 is some time before which the interaction was not active. We note in passing that it is causality which requires the first form of (26), and hence requires the time-ordered product of the second – and hence leads to the Feynman propagator being the relevant function in QFT.

The terms of order n in (26) correspond to contributions to the scattering amplitude of order $e^n = \alpha^{n/2}$, with n vertices in the corresponding Feynman diagram. Consider electron-electron scattering in lowest order, as given by the Feynman diagram of Figure 1. We want to evaluate,

$$\langle in|out\rangle\big|_{1st\_order} = \langle \bar{p}_1', \bar{p}_2'|\left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{\infty} dt_1 \int_{t_0}^{\infty} dt_2 T\{H_I(t_1)H_I(t_2)\}|\bar{p}_1, \bar{p}_2\rangle \tag{27}$$

Now the integrand is quadratic in H_I and hence of order 6 in the fields as a consequence of (24). The product of the four ψ fields is required to first annihilate the incoming electrons, and then to re-create a pair of outgoing electrons. This will give a non-zero value for (27) so long as the vacuum expectation value of the product of two ϕ fields is non-zero. This vacuum expectation value of the product of the two ϕ

fields is just the Feynman propagator, (18). Consequently we get, in a spacetime formulation, (28)

$$\langle in|out\rangle\big|_{1st\_order} = \frac{1}{2}\left(\frac{-ie}{\hbar}\right)^2 \int d^4x_1 d^4x_2 N\left\{\overline{\psi}^-(x_1)\psi^+(x_1)\cdot\overline{\psi}^-(x_2)\psi^+(x_2)\right\}\cdot\Delta_F(x_1-x_2)$$

In (28) the superscripts +/- denote the creation & annihilation parts of the 'electron' field respectively, and the Feynman propagator refers to the 'photon'. Of course, (28) is not terribly enlightening as it stands. Each of the fields is of the form

$\psi^+(x) = \sqrt{\frac{V\hbar c^2}{2\omega_{\overline{p}}}}\cdot b_r^+(\overline{p})e^{ipx}$, where p is one of the relevant electron 4-momenta.

Substitution of these into (28) and explicitly carrying out the x-integrals produces delta functions in k-space for every vertex of the Feynman diagram representing the conservation of energy and momentum at each vertex. The Feynman propagator can be written as its k-space integral, (17,19), but the 4-momentum over which this integral is carried out also appears in the delta functions expressing energy-momentum conservation. Hence, the integral is trivial and the momentum-space expression for the scattering amplitude becomes essentially,

$$\langle in|out\rangle\big|_{1st\_order} \sim \frac{e^2}{(p_1+p_2)^2 - m^2} \tag{29}$$

Modulo a delta function representing overall energy-momentum conservation. Had we carried this through properly for spin ½ electrons and the massless vector photon the result would have been,

$$\langle in|out\rangle\big|_{1st\_order} \sim e^2\,\frac{\overline{u}(\overline{p}_1')\gamma^\alpha u(\overline{p}_1)g_{\alpha\beta}\overline{u}(\overline{p}_2')\gamma^\beta u(\overline{p}_2)}{(p_1+p_2)^2} \tag{30}$$

What concerns us here, however, is the scattering amplitude for the second order diagram of Figure 2. Contrary to the normal practice, our message is that it is only necessary to consider the spacetime expression for this amplitude to see that it is divergent. This diagram corresponds to the $n = 4$ term in (26). This involves a factor of 8 'electron' fields and 4 'photon' fields. Four of the electron fields are used up as before in annihilating then re-creating the incoming electron pair. This leaves 4 electron fields and 4 photon fields which contribute to the scattering amplitude via their vacuum expectation values. Since pairs of vacuum expectation values result in a Feynman propagator, we get two electron propagators and two photon propagators. This is correct, as can be seen from Figure 2 (the internal lines being the propagators). The spacetime expression for the fourth order contribution to scalar 'electron-electron' scattering is thus, (31)

$$\langle in|out\rangle\big|_{2nd\_order} = \frac{1}{4!}\left(\frac{-ie}{\hbar}\right)^4 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 N\left\{\overline{\psi}^-(x_1)\psi^+(x_1)\cdot\overline{\psi}^-(x_2)\psi^+(x_2)\right\}\times$$
$$\Delta_F^\phi(x_1-x_3)\Delta_F^\psi(x_3-x_4)\Delta_F^\psi(x_4-x_3)\Delta_F^\phi(x_4-x_2)$$

In (31) we have distinguished between the Feynman propagators for the 'electron' and the 'photon' with suitable superscripts. Note that the spacetime positions over which the integrals are carried out are shown on Figure 2. The crucial feature of (31) is that, as a consequence of the two 'electron' propagators in Figure 2 connecting the same pair of spacetime points, and thus forming a closed loop, the above scattering amplitude contains the factor,

$$\Delta_F^{\psi}(x_3 - x_4)\Delta_F^{\psi}(x_4 - x_3) = \left(\Delta_F^{\psi}(x_3 - x_4)\right)^2 \qquad (32)$$

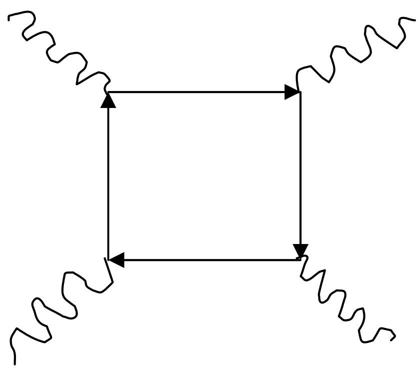
Consequently it is clear that the second order contribution to the scattering process is divergent. Of course, we can express (31) in momentum space in the usual way and come to the same conclusion because,

$$\int \frac{d^4 p}{\left(p^2 - m^2\right)\left((p-k)^2 - m^2\right)} \to \sim \int \frac{p^3 dp}{p^4} \to \infty \qquad (33)$$

which is the scalar equivalent of the more correct Equ.(1).

Before closing note that above it was stated that terms of second order and above *generally* involve products of Feynman propagators at the same spacetime point. Terms of this sort do occur in almost all higher order scattering amplitudes. This is because any internal or external electron line in an nth order expression can be replaced by a line including an added virtual photon (electron self-energy) to generate an (n+1)th order term. Similarly, any photon line can have an electron-positron virtual pair loop added (vacuum polarisation). However, I can think of one process whose lowest order diagram has 4 vertices, namely photon-photon scattering. (This is a phenomenon which is impossible classically since the linearity of the classical Maxwell equations mean that two e/m fields cannot interact). The diagram is,

Figure 3



This diagram is divergent and requires renormalisation to provide a finite prediction. In momentum space this is because,

$$\int \frac{d^4 p}{(\not p - m)(\not p - \not k_1 - m)(\not p - \not k_1' - m)(\not p - \not k_1 - \not k_2 - m)} \sim \int \frac{p^3 dp}{p^4} \to \infty \qquad (34)$$

In the spacetime formulation we have a term in,

$$\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Delta_F(x_1 - x_2)\Delta_F(x_2 - x_3)\Delta_F(x_3 - x_4)\Delta_F(x_4 - x_1) \times \text{fields} \qquad (35)$$

This does not actually involve a product of propagators at the same spacetime interval. However, the singularity of the propagators at 0 means that there is a non-zero contribution to such integrals from this point. In other words the propagators behave in a similar manner to delta functions. This means that the integral over, say, x_4 will include a part which is roughly,

$$\int d^4x_1 d^4x_2 d^4x_3 \Delta_F(x_1 - x_2)\Delta_F(x_2 - x_3)\Delta_F(x_3 - x_1) \times \text{fields} \tag{36}$$

Similarly, carrying out the x_3 integral will include a part which is roughly,

$$\int d^4x_1 d^4x_2 d^4x_3 \Delta_F(x_1 - x_2)\Delta_F(x_2 - x_1) \times \text{fields} \tag{37}$$

This now contains a product of propagators at the same spacetime interval, and is hence divergent. Consequently "exceptional" graphs like Figure 3 are not really terribly different.

Summary

The source of the divergences in QFT can be traced to the product of propagators evaluated at the same spacetime point. Since the propagators are distributions, akin to the Dirac delta function rather than finite functions, such products are singular even when integrated. The requirement to adopt such singular propagators can be traced to the combination of the canonical commutation relations for the quantum fields and the requirement for Lorentz covariance. The creation of a finite QFT, with no need of renormalisation, would therefore require abandonment or modification of either the canonical commutation relations or Lorentz covariance.