Derivation of the LEFM Crack Tip Stress Fields

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There are various ways of deriving the LEFM fields, including proving their uniqueness. This is well covered in the standard texts. Consequently all that is presented here is an outline. We start with the 2D problem formulated in terms of an Airy function, φ. This obeys the biharmonic equation,

$$\nabla^4 \varphi = 0 \tag{1}$$

(see other Notes on this site). In 2D polars, the stress components are given in terms of the Airy function by,

$$\sigma_\theta = \varphi_{,rr} \qquad \sigma_r = \frac{1}{r}\varphi_{,r} + \frac{1}{r^2}\varphi_{,\theta\theta} \qquad \sigma_{r\theta} = -\partial_r\left(\frac{1}{r}\varphi_{,\theta}\right) \tag{2}$$

The general solution of the harmonic equation (i.e. Laplace's equation) is any analytic function of $z = x + iy$. It is a reasonable bet that the general solution of the biharmonic equation involves two independent analytic functions. This turns out to be true. The general solution to Equ.(1) can thus be written,

$$\varphi = \Re\left(Z_1(z) + \xi Z_2(z)\right) \tag{3}$$

where ξ is something which is not analytic in z, and which therefore prevents the two analytic functions Z_1 and Z_2 merely being combined into a new analytic function. The simplest choices for ξ are x or y or z*. We choose $\xi = y$. The analytic functions which reproduce the correct boundary conditions turn out to be,

$$Z_1 = z^{3/2} \qquad \text{and} \qquad Z_2 = z^{1/2} \tag{4}$$

These are analytic except on a branch cut, which is chosen to lie along the negative x-axis. This branch cut is the crack. The existence of the branch cut is essential in permitting the solution to be discontinuous over the crack. This is crucial since otherwise the two crack faces could not deflect in opposite directions under load (which is pretty much the essence of being a crack). Introducing a constant proportional to load, the Mode I and Mode II solutions are,

Mode I:
$$\varphi = k_1 \Re\left(z^{3/2} - \frac{3i}{2} y z^{1/2}\right) \tag{5}$$

Mode II:
$$\varphi = k_2 \Re\left(y z^{1/2}\right) \tag{6}$$

Substituting (5) into the first of Equs.(2) gives, after using the appropriate trig identity,

$$\sigma_\theta = \frac{3k_1}{4\sqrt{r}}\left[\frac{1}{4}\cos\frac{3\theta}{2} + \frac{3}{4}\cos\frac{\theta}{2}\right] \qquad (7)$$

There would be nothing to stop us defining the constant k_1 as the stress intensity factor – except convention. By convention, the stress intensity factor is normalised so that the relation between K_I and stress ahead of the crack, on $\theta = 0$, is $\sigma_\theta = \frac{K_I}{\sqrt{2\pi r}}$. Hence we replace k_1 with,

$$k_1 = \frac{4}{3\sqrt{2\pi}} K_I \qquad (8)$$

Substituting (5) into the other Equs.(2) thus yields the LEFM fields in their conventional form.,

$$\sigma_\theta = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\frac{3\theta}{2} + \frac{3}{4}\cos\frac{\theta}{2}\right] \qquad (9a)$$

$$\sigma_r = \frac{K_I}{\sqrt{2\pi r}}\left[-\frac{1}{4}\cos\frac{3\theta}{2} + \frac{5}{4}\cos\frac{\theta}{2}\right] \qquad (9b)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\frac{3\theta}{2} + \frac{1}{4}\sin\frac{\theta}{2}\right] \qquad (9c)$$

Now substituting (6) into Equs.(2), and adopting the convention that which defines the Mode II SIF via $\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}$ on $\theta = 0$, we find that we need,

$$k_2 = -\frac{2}{\sqrt{2\pi}} K_{II} \qquad (10)$$

and the Mode II LEFM crack tip stresses are,

$$\sigma_\theta = -\frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{3}{4}\sin\frac{3\theta}{2} + \frac{3}{4}\sin\frac{\theta}{2}\right] \qquad (11a)$$

$$\sigma_r = \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{3}{4}\sin\frac{3\theta}{2} - \frac{5}{4}\sin\frac{\theta}{2}\right] \qquad (11b)$$

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{3}{4}\cos\frac{3\theta}{2} + \frac{1}{4}\cos\frac{\theta}{2}\right] \qquad (11c)$$

The Cartesian components of stress can be obtained simply via the usual coordinate transformation.

Mode III

In pure Mode III the solution is much simpler because the only non-zero displacement near the crack tip is the out-of-plane displacement, u_z. This displacement varies in the (x,y) plane, but is constant in the z-direction. A Note on this web site derives the equation obeyed by the displacements in the general 3D case. The z-displacement obeys,

$$\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)u_{z,zz}+\nu u_{y,yz}+\nu u_{x,xz}\right]+\frac{E}{2(1+\nu)}\left[u_{z,yy}+u_{y,zy}+u_{z,xx}+u_{x,zx}\right]=b_z \quad (12)$$

In the case of Mode III near the crack tip this becomes simply,

$$u_{z,xx}+u_{z,yy}=\nabla^2 u_z=0 \quad (13)$$

where ∇^2 is here the 2D Laplacian in the (x,y) plane. Hence, the z-displacement is an analytic function of x+iy. A solution with an appropriate branch cut is $u_z \propto \mathrm{Im}(\sqrt{x+iy})$, i.e. $u_z \propto \sqrt{r}\sin(\theta/2)$. This permits the z-displacement to be discontinuous over the crack, which lies at $\theta=\pi$. The only non-zero stresses are the xz and yz shears, which are given by,

$$\sigma_{xz}=Gu_{z,x} \quad \text{and} \quad \sigma_{yz}=Gu_{z,y} \quad (14)$$

which give,

$$\sigma_{xz}=\frac{K_{III}}{\sqrt{2\pi r}}\sin\left(\frac{\theta}{2}\right) \quad \text{and} \quad \sigma_{yz}=\frac{K_{III}}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right) \quad (15)$$

where the definition of the Mode II SIF, K_{III}, again follows the convention that the dominant stress component (in this case σ_{yz}) equals $\frac{K_{III}}{\sqrt{2\pi r}}$ directly ahead of the crack tip. The yz shear stress is, of course, zero on the crack faces, whilst the xz shear is non-zero but equal and opposite on the two faces. This discontinuity in stress across $\theta=\pi$ can arise only because we have considered a solution with a branch cut.

For completeness the following pages give the LEFM stresses and displacements in both polar and Cartesian coordinates…

Polar Coordinates:-

Mode I

Eqs.(16)

$$\sigma_r = \frac{K_I}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[-\cos\frac{3\theta}{2} + 5\cos\frac{\theta}{2}\right] \qquad u_r = \frac{K_I(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\left[\left(\frac{5}{2} - 4\bar{\nu}\right)\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3\theta}{2}\right]$$

$$\sigma_\theta = \frac{K_I}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[\cos\frac{3\theta}{2} + 3\cos\frac{\theta}{2}\right] \qquad u_\theta = \frac{K_I(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\left[\left(-\frac{7}{2} + 4\bar{\nu}\right)\sin\frac{\theta}{2} + \frac{1}{2}\sin\frac{3\theta}{2}\right]$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}\right]$$

Mode II

Equs.(17)

$$\sigma_r = \frac{K_{II}}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[3\sin\frac{3\theta}{2} - 5\sin\frac{\theta}{2}\right] \qquad u_r = \frac{K_{II}(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\left[\left(-\frac{5}{2} + 4\bar{\nu}\right)\sin\frac{\theta}{2} + \frac{3}{2}\sin\frac{3\theta}{2}\right]$$

$$\sigma_\theta = -\frac{K_{II}}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[3\sin\frac{3\theta}{2} + 3\sin\frac{\theta}{2}\right] \qquad u_\theta = \frac{K_{II}(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\left[\left(-\frac{7}{2} + 4\bar{\nu}\right)\cos\frac{\theta}{2} + \frac{3}{2}\cos\frac{3\theta}{2}\right]$$

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[3\cos\frac{3\theta}{2} + \cos\frac{\theta}{2}\right]$$

In Equs.(16, 17), $\bar{\nu} = \frac{\nu}{1+\nu}$ in plane stress, but $\bar{\nu} = \nu$ in plane strain.

Mode III

Equs.(18)

$$\sigma_{rz} = \frac{K_{III}}{\sqrt{2\pi r}}\sin\left(\frac{\theta}{2}\right) \qquad \sigma_{\theta z} = \frac{K_{III}}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right) \qquad u_z = \frac{4K_{III}(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\sin(\theta/2)$$

Cartesian Coordinates :-

Mode I Equs.(19)

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \qquad u_x = \frac{2(1+\nu)K_I}{E}\sqrt{\frac{r}{2\pi}}\left[1-2\bar{\nu}+\sin^2\frac{\theta}{2}\right]\cos\frac{\theta}{2}$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \qquad u_y = \frac{2(1+\nu)K_I}{E}\sqrt{\frac{r}{2\pi}}\left[2-2\bar{\nu}-\cos^2\frac{\theta}{2}\right]\sin\frac{\theta}{2}$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

Mode II Equs.(20)

$$\sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right) \qquad u_x = \frac{2(1+\nu)K_{II}}{E}\sqrt{\frac{r}{2\pi}}\left[2-2\bar{\nu}+\cos^2\frac{\theta}{2}\right]\sin\frac{\theta}{2}$$

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} \qquad u_y = \frac{2(1+\nu)K_{II}}{E}\sqrt{\frac{r}{2\pi}}\left[-1+2\bar{\nu}+\sin^2\frac{\theta}{2}\right]\cos\frac{\theta}{2}$$

$$\sigma_{xy} = \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

In Equs.(19, 20), $\bar{\nu} = \frac{\nu}{1+\nu}$ in plane stress, but $\bar{\nu} = \nu$ in plane strain.

Mode III Equs.(21)

$$\sigma_{xz} = -\frac{K_{III}}{\sqrt{2\pi r}}\sin\left(\frac{\theta}{2}\right) \qquad \sigma_{yz} = \frac{K_{III}}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right) \qquad u_z = \frac{4K_{III}(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\sin(\theta/2)$$