The Proper Derivation of J
(which is a vector, by the way)

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What ***is*** J? Frequently it is presented as some arcane integral without any clue whence it originated. I blame Rice. His name is wrongly attached to J. Precedent goes to John Eshelby (an English physicist). Note only did Eshelby derive J first, but he did so in a manner which illuminates its physical meaning. It does not spring from nowhere, but out of one of the most profound theorems in mathematical physics, namely Noether's Theorem.

This theorem states that, for a very broad class of physical theories, every symmetry leads to a conservation law. Thus, the homogeneity of space (symmetry under translation) leads directly to the conservation of momentum. The conservation of energy is seen to be a consequence of the fact that the laws of physics are the same today as yesterday (symmetry under translation in time). And the isotropy of space (symmetry under rotation) leads to the conservation of angular momentum. A humble fracture parameter may seem somewhat prosaic compared to these profound insights. However, the path independence of J derives from precisely the same mathematics. The path independence of J will be seen to be a result of the homogeneity of the material (translational symmetry).

The path independence of J is exact for non-linear elasticity. The stress-strain relation can be as non-linear as you wish, providing that the behaviour is reversible. In other words, providing that energy is not dissipated. The identification of the path independence of J with the homogeneity of the material provides physical insight into why path independence no longer holds for irreversible plasticity. The yielded zone no longer behaves like unyielded material. So the material is no longer homogeneous. Path independence holds in plastic materials only so long as the loads are monotonically increasing[1] (and hence the situation is equivalent to that for a non-linear elastic material with the same monotonic stress-strain relation).

Finally, before we get started on the derivation, note that the familiar J is really just one component of a vector quantity. I do not mean merely that there are three path independent contour integrals, J_1, J_2 and J_3, although this is true. I mean that they transform under a rotation of the coordinate system in the way that any good vector should. Why are there three Js? What do they mean?

In the general case, the demonstration of Noether's Theorem would require the theory to be formulated in terms of Lagrangian dynamics. However, it is sufficient for our purposes to consider static non-linear elasticity, formulated as a problem in minimising the total potential energy. Because non-linear elasticity is conservative (i.e. non-dissipative) there exists a strain energy density, W, which can be written as a function of the strains. The work done in increasing the strain by $\delta\varepsilon_{ij}$ over a region of volume δV where the stress is σ_{ij} is $\sigma_{ij}\delta\varepsilon_{ij}\delta V$ (where repeated indices are summed unless otherwise indicated). This

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[1] and, I should strictly add, provided that proportional loading applies.

must equate to the increase in the energy density times the volume, $\delta W \delta V$, for a conservative system. Hence,

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \tag{1}$$

Since the strains are given in terms of the displacement gradients by,

$$\varepsilon_{ij} = \frac{1}{2} u_{(i,j)} \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2}$$

we can equally regarded W as a function of the $u_{i,j}$. The total strain energy is,

$$U = \int W dV \tag{3}$$

For a given set of applied loads, displacements and restraints, the strains arrange themselves so as to minimise U subject to these constraints. If the body is in some force field (e.g. gravity) then its potential energy will depend upon its position. More generally, the change in the potential energy of each element of volume with respect to the undeformed state will depend upon its displacement. Hence, W depends upon both the displacements and the displacement gradients in general. The displacements and their gradients are required to minimise the total strain-plus-other-potential energy, i.e.,

$$\delta \int W\left(u_i, u_{i,j}\right) dV = 0 \tag{4}$$

The condition that the integral be stationary with respect to variations in the displacement field is easily shown to result in the Euler-Lagrange equations,

$$\partial_j \left(\frac{\partial W}{\partial u_{i,j}} \right) = \frac{\partial W}{\partial u_i} \tag{5}$$

Equ.(5) represents three equations, for the three possible values for i, noting that the j is summed. But thanks to (1) and (2) the LHS of (5) is simply $\sigma_{ij,j}$, whereas the gradient of the energy density on the RHS is just (minus) the force per unit volume, b_i, being applied by the external force field. Hence, (5) are just the familiar equations of equilibrium,

$$\sigma_{ij,j} = -b_i \tag{6}$$

Having shown that the formalism makes sense, we can now go on to investigate the implications of the material being homogeneous. For a body under load, the energy density, W, will generally vary from point to point. In what sense, then, is the material homogeneous?

What is meant by homogeneity is that the underlying *theory* is not biased to one point compared with another. Any particular *solution* to the equations, i.e. for a given set of boundary conditions, will generally be inhomogeneous in the sense that the stresses and strains will vary from place to place. The distinction between the homogeneity of the theory, as contrasted with the spatially varying specific solution, is manifest in the distinction between *partial* derivatives and *total* derivatives. The homogeneity of the material requires that the energy density, W, has no *explicit* dependence upon position. Hence its partial derivatives will vanish, i.e.,

$$\partial_i W \equiv \frac{\partial W}{\partial x_i} \equiv 0 \qquad (7)$$

On the other hand, for a body under some arbitrary loading, the magnitude of the energy density will obviously vary from place to place in general. The reason is that W depends upon the strains, and the strains will vary from place to place. This is written in terms of the total derivative as,

$$\frac{dW}{dx_i} \neq 0 \text{ in general} \qquad (8)$$

and the total derivative can be written,

$$\frac{dW}{dx_i} = \frac{\partial W}{\partial x_i} + \frac{\partial W}{\partial u_j} \cdot \frac{\partial u_j}{\partial x_i} + \frac{\partial W}{\partial u_{j,k}} \cdot \frac{\partial u_{j,k}}{\partial x_i} \qquad (9)$$

But using (5) to write $\frac{\partial W}{\partial u_j} = \partial_k \left(\frac{\partial W}{\partial u_{j,k}} \right)$ this becomes,

$$\frac{dW}{dx_i} = \frac{\partial W}{\partial x_i} + \partial_k \left(\frac{\partial W}{\partial u_{j,k}} \right) \cdot \frac{\partial u_j}{\partial x_i} + \frac{\partial W}{\partial u_{j,k}} \cdot \frac{\partial u_{j,k}}{\partial x_i} = \frac{\partial W}{\partial x_i} + \partial_k \left[\frac{\partial W}{\partial u_{j,k}} \cdot \frac{\partial u_j}{\partial x_i} \right] \qquad (10)$$

Re-arranging and introducing the Kronecker delta, δ_{ik}, we have,

$$\frac{d}{dx_k} \left[W\delta_{ik} - \frac{\partial W}{\partial u_{j,k}} \cdot u_{j,i} \right] = \frac{\partial W}{\partial x_i} \qquad (11)$$

But homogeneity means that this is zero, by Equ.(7). Defining the tensor T_{ik} by,

$$T_{ik} = W\delta_{ik} - \frac{\partial W}{\partial u_{j,k}} \cdot u_{j,i} \qquad (12)$$

the homogeneity of the material is thus expressed by the fact that this tensor has zero divergence, i.e.,

$$T_{ik,k} = 0 \tag{13}$$

The alert reader will have spotted in Equ.(12) the origin of the arcane-seeming integrand which defines J. The tensor T clearly has units of energy density. So, we first define a vector quantity by integrating the divergence of T over some arbitrary volume, R, within the body,

$$\Im_i = \int_R T_{ij,j} dV = 0 \tag{14}$$

So $\Im_i$ has units of energy/length. But the volume integral of a divergence is just the surface integral over the boundary of the region, δR. Hence we can write,

$$\Im_i = \oiint_{\delta R} T_{ij} dS_j = 0 \tag{15}$$

where the notation emphasises that this is an integral over (any) closed surface lying completely within the material.

Well, this is all very merry, but isn't a quantity which is identically zero rather useless? And in any case, shouldn't we have a crack somewhere? We shall now introduce a crack, and also restrict attention to 2-dimensional cases to begin with. The vanishing of $\Im$ will be seen to be the origin of the path independence of J.

As is conventional, place the crack along the negative x-axis, with the tip at the origin, so that the y-stress is the Mode I opening stress. Because the problem is now 2D, the closed surface, δR, is prismatic. It consists of some closed boundary Γ in the x,y plane (actually a prismatic surface), together with the two "ends caps" at some pair of constant values of z.

Consider i = 1 in (12). Substituting in (15) gives,

$$\Im_1 = \oiint_{\delta R} \left[W\delta_{1k} - \frac{\partial W}{\partial u_{j,k}} \cdot \frac{\partial u_j}{\partial x} \right] dS_k = \oiint_{\delta R} \left[W\delta_{1k} - \sigma_{jk} \cdot \frac{\partial u_j}{\partial x} \right] dS_k = 0 \tag{16}$$

Consider firstly the end caps (at some z = constant). The element of surface area dS on the ends caps is parallel to the z-axis, and hence has no x-component. Hence the first term in the integrand of (16) is zero. The second term involves the traction $\sigma_{jk} dS_k$ acting over the element of area dS. But for a 2D plane stress problem the traction on the ends caps is zero. Alternatively, for a 2D plane strain, or engineering plane strain, problem, the z-displacement is uniform, so that $\frac{\partial u_z}{\partial x} = 0$. The third possibility is anti-plane strain. In this case the tractions $\sigma_{xz} dS_z$ and $\sigma_{yz} dS_z$ will be non-zero, but will be equal and opposite on the two end caps, giving zero net contribution. In any 2D case, therefore, the end caps contribute nothing to $\Im_1$.

Consider an element (dx,dy) of the boundary Γ in the x,y plane. The corresponding element of surface area is dS = t(dy,-dx), where t is the thickness of the body in the z-direction. Hence, the first term in the integrand of (16) involves $dS_1 = tdy$. The second term can be written in terms of the unit vector normal to the surface, $\hat{n}$, where $d\overline{S} = \hat{n}tds$. Hence we get,

$$\Im_1 = \oiint_\Gamma \left[tWdy - \sigma_{jk}\hat{n}_k \cdot \frac{\partial u_j}{\partial x} tds \right] = 0 \qquad (17)$$

where ds is an element of length along the boundary Γ, and dy is its y-component. The thickness, t, factors out and we finally define the conventional J integral by,

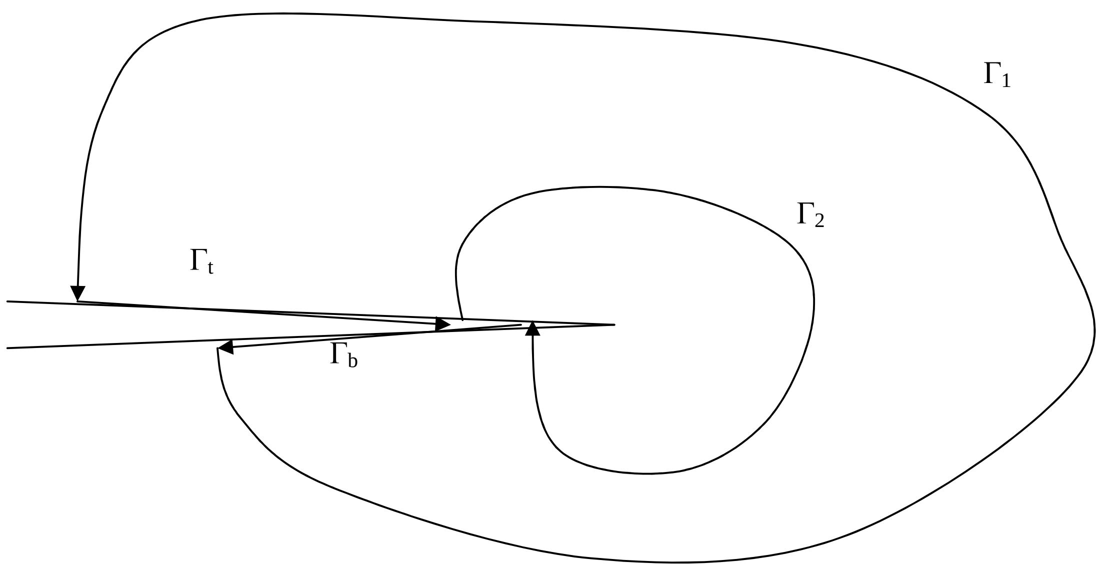
$$J \equiv J_1 = \frac{\Im_1}{t} \qquad (18)$$

So J has units of energy/area, as required, and is given by the usual expression,

$$J_1 = \oiint_\Gamma \left[Wdy - \sigma_{jk}\hat{n}_k \cdot \frac{\partial u_j}{\partial x} ds \right] = 0 \qquad (19)$$

But, this is still identically zero I hear you cry! A mere detail. This is due to the choice of boundary, Γ, namely a closed boundary. Consider now a closed boundary constructed in the following special manner:-

- Γ_1 is a boundary which begins on the lower negative x-axis, and ends on the upper negative x-axis;
- Γ_2 is a boundary which begins on the upper negative x-axis, and ends on the lower negative x-axis, and lies entirely within Γ_1;
- Γ_t runs along the upper negative x-axis and joins Γ_1 to Γ_2;
- Γ_b runs along the lower negative x-axis and joins Γ_2 to Γ_1.



The final step consists of noting that dy is zero along Γ_t and Γ_b, and that the traction $\sigma_{jk}\hat{n}_k$ is also zero on these free surfaces of the crack. Hence Γ_t and Γ_b contribute nothing to the integral. The fact that the closed boundary integral is zero therefore means that the integrals along Γ_1 and Γ_2 are equal and opposite. But note that the contour Γ_2 is defined as clockwise, whereas Γ_1 is anti-clockwise. If we redefined Γ_2 to run anti-clockwise as well, then the integrals over both would be the same. Hence, we finally have the true, contour independent definition of the J integral,

$$J_1 = \int_{\Gamma_1}\left[W dy - \sigma_{jk}\hat{n}_k \cdot \frac{\partial u_j}{\partial x} ds\right] = \int_{-\Gamma_2}\left[W dy - \sigma_{jk}\hat{n}_k \cdot \frac{\partial u_j}{\partial x} ds\right] \qquad (20)$$

The traction vector is $T_j = \sigma_{jk}\hat{n}_k$ so an alternative notation is,

$$J_1 = \int_{\Gamma_1}\left[W dy - \overline{T} \cdot \frac{\partial \overline{u}}{\partial x} ds\right] \qquad (21)$$

Note that the reason why the integral is not zero is because it is not really a closed contour. The presence of the crack prevents the contour being closed, since one end of the contour lies on the top crack face and the other end lies on the bottom crack face. If we began and ended the contour on the same crack face, the integral would be zero.

Note that the path independence results from the fact that the integral over a truly closed boundary, formed by combining the contour Γ_1 with other contours as in the diagram, is zero. And the integral over a truly closed boundary is zero because of the homogeneity of the material.

The J_2 and J_3 Integrals

Consider i = 2 in (12). Substituting in (15) gives,

$$\Im_2 = \oiint_{\delta R}\left[W\delta_{2k} - \frac{\partial W}{\partial u_{j,k}} \cdot \frac{\partial u_j}{\partial y}\right] dS_k = \oiint_{\delta R}\left[W\delta_{2k} - \sigma_{jk} \cdot \frac{\partial u_j}{\partial y}\right] dS_k = 0 \qquad (22)$$

For a 2D problem and a prismatic surface, the ends caps again give no contribution to the integral for the same reasons as before. Since dS = t(dy,-dx) the first term in the integrand now involves –tdx. The second term involves the traction over the surface element, as before. It is clear that, in analogy to Equ.(21), we end up with,

$$J_2 = -\int_{\Gamma_1+\Gamma_t+\Gamma_b}\left[W dx + \overline{T} \cdot \frac{\partial \overline{u}}{\partial y} ds\right] \qquad (23a)$$

Note, however, that (23) has retained the contribution to the integral along the top and bottom crack faces. The second term in the integrand is zero along these paths, because the crack faces are assumed traction-free. But, unlike J_1, it is not clear that the first term is necessarily zero on Γ_t and Γ_b. It would appear that in order to produce a contour integral characteristic of the crack tip fields we need to allow the inner contour, Γ_2, to shrink to zero. But actually things are not so bad. Consider the explicit form of the LEFM fields for arbitrary 2D loading, i.e. an arbitrary combination of Mode I and Mode II. It turns out that W is the same at corresponding points on the top and bottom surface of the crack. So, at least near the crack tip, and for the linear elastic case, the contributions from Γ_t and Γ_b cancel. (NB: this is because the two paths run in opposite directions). In fact, the same is found to be true for the elastic-plastic crack tip fields, the HRR fields. Is this true?

However, we still cannot be sure that this cancellation will apply at greater distances, where the crack tip fields may not be indicative. And we may wish to start and finish the contour Γ_1 at different positions along the crack front, in which case the Γ_t and Γ_b integrals will not cancel. So, we still write in general,

$$J_2 = -\int_{\Gamma_1}\left[Wdx + \overline{T}\cdot\frac{\partial \underline{u}}{\partial y}ds\right] - \int_{\Gamma_t+\Gamma_b} Wdx \qquad (23b)$$

but noting that the second term will have a vanishing contribution from near the crack tip due to cancellation. If we restrict Γ_1 to start and end at the same distance from the crack tip, and if we are willing to believe that the crack tip fields are indicative of the cancellation of the second term even at large distances, then we can make the approximation,

$$J_2 \approx -\int_{\Gamma_1}\left[Wdx + \overline{T}\cdot\frac{\partial \underline{u}}{\partial y}ds\right] \qquad (23c)$$

Note that, as for J_1, we have defined: $J_2 = \frac{\Im_2}{t}$ (24)

Finally, consider i = 3 in (12). Substituting in (15) gives,

$$\Im_3 = \oiint_{\delta R}\left[W\delta_{3k} - \frac{\partial W}{\partial u_{j,k}}\cdot\frac{\partial u_j}{\partial z}\right]dS_k = \oiint_{\delta R}\left[W\delta_{3k} - \sigma_{jk}\cdot\frac{\partial u_j}{\partial z}\right]dS_k = 0 \qquad (25)$$

Consider firstly the case of a 2D problem, such as plane stress, plane strain or engineering plane strain. The tractions on the end caps are either zero or in the z-direction. In the former case the second term in the integrand is zero on the ends caps. In the latter case the second term involves $\frac{\partial u_z}{\partial z} = \varepsilon_{zz}$, which is the uniform out-of-plane strain. It follows that, whilst the contribution of one end cap may be non-zero, it is equal and opposite to the contribution of the other end cap (since the direction of the surface element is reversed, i.e. $dS_z = dxdy$ on one end cap but $dS_z = -dxdy$ on the other). The same is true for the W term. Hence, overall the end caps make no contribution to the integral. Considering now the integrals over the in-plane boundary Γ, the first term is clearly zero since $dS_z = 0$ on this boundary. But the second term is also zero because the in-plane displacements, u_x and u_y, do not vary in the out-of-plane direction, z, i.e. $\frac{\partial \bar{u}}{\partial z} = 0$. Hence we conclude that J_3 is identically zero for 2D problems of plane stress, plane strain or engineering plane strain. Surmising that J_3 is related to the Mode III stress intensity factor (to be shown later), this is hardly surprising.

It remains, therefore, to consider J_3 in the case $u_x = u_y = 0$ and with u_z dependent upon (x,y) but constant in the z-direction, i.e. anti-plane strain. The only non-zero strains, and hence stresses, are the xz and yz shear components. It follows that the in-plane boundary Γ gives no contribution to the integral. Note that the second term is zero because $\frac{\partial u_z}{\partial z} = 0$
The first term in the integrand cancels out between the two ends caps, for the same reason as before. The second term involves dS_z and hence either σ_{zx} or σ_{zy}, and hence either $\frac{\partial u_x}{\partial z}$ or $\frac{\partial u_y}{\partial z}$, both of which are zero. Hence, J_3 is also identically zero for the case of anti-plane strain loading.

So, is J_3 always zero? Or does it require some in-plane and some out-of-plane loading in order to be non-zero? It turns out that the latter is the case. This will be shown next.

Evaluation of J_1, J_2, J_3 for the LEFM Fields
We can shed some light on the meaning of this vectorial J by evaluating each component in the LEFM case. In principle this is simply accomplished by substituting the explicit LEFM expressions for the three stress intensity factors, K_I, K_{II} and K_{III}, and carrying out the integrals. The algebra is nasty, though. In 30 years I have never previously bothered to carry out this calculation, and I soon regretted not waiting another 30 years. I have now carried out this explicit integration, with the help of MAPLE. The LEFM stresses and displacements which enter the calculation are given in polar coordinates below. The resulting expressions for $\bar{J}$ follow. There are some notes in an Appendix at the end for the assistance anyone wanting to carry out the same exercise for themselves.

Mode I

Eqs.(26)

$$\sigma_r = \frac{K_I}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[-\cos\frac{3\theta}{2} + 5\cos\frac{\theta}{2}\right] \qquad u_r = \frac{K_I(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\left[\left(\frac{5}{2} - 4\bar{\nu}\right)\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3\theta}{2}\right]$$

$$\sigma_\theta = \frac{K_I}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[\cos\frac{3\theta}{2} + 3\cos\frac{\theta}{2}\right] \qquad u_\theta = \frac{K_I(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\left[\left(-\frac{7}{2} + 4\bar{\nu}\right)\sin\frac{\theta}{2} + \frac{1}{2}\sin\frac{3\theta}{2}\right]$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}\right]$$

Mode II

Equs.(27)

$$\sigma_r = \frac{K_{II}}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[3\sin\frac{3\theta}{2} - 5\sin\frac{\theta}{2}\right] \qquad u_r = \frac{K_{II}(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\left[\left(-\frac{5}{2}+4\bar{\nu}\right)\sin\frac{\theta}{2} + \frac{3}{2}\sin\frac{3\theta}{2}\right]$$

$$\sigma_\theta = -\frac{K_{II}}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[3\sin\frac{3\theta}{2} + 3\sin\frac{\theta}{2}\right] \qquad u_\theta = \frac{K_{II}(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\left[\left(-\frac{7}{2}+4\bar{\nu}\right)\cos\frac{\theta}{2} + \frac{3}{2}\cos\frac{3\theta}{2}\right]$$

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[3\cos\frac{3\theta}{2} + \cos\frac{\theta}{2}\right]$$

In Equs.(26,27), $\bar{\nu} = \frac{\nu}{1+\nu}$ in plane stress, but $\bar{\nu} = \nu$ in plane strain. (Yes, this is the opposite way around to what you might have expected). Note that some sources give an incorrect expression for u_r in Mode II (which caused me a great deal of grief in attempting to derive Equ.(29), below).

Mode III Equs.(28)

$$\sigma_{xz} = -\frac{K_{III}}{\sqrt{2\pi r}}\sin\left(\frac{\theta}{2}\right) \qquad \sigma_{yz} = \frac{K_{III}}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right) \qquad u_z = \frac{4K_{III}(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\sin(\theta/2)$$

The first two components of the vector J evaluate to:-

$$EJ_1 = \lambda\left(K_I^2 + K_{II}^2\right) + (1+\nu)K_{III}^2 \qquad (29)$$

$$EJ_2 = -2\lambda K_I K_{II} \qquad (30)$$

where $\lambda = 1$ in plane stress and $\lambda = 1 - \nu^2$ in plane strain. In the case of 2D problems (i.e. when $K_{III} = 0$), knowledge of J_1 and J_2 suffices to deduce K_I and K_{II}, apart from their absolute sign (which is generally obvious).

It turns out, of course, that J_1 equals the energy release rate for self-similar crack extension. This forms an algebraically simpler, and physically more enlightening, way to derive Equ.(29). It is relatively easy to show that the RHS of Equ.(29) is the energy release rate in the LEFM case. And it can be shown that, as a consequence of its definition, J_1 is the self-similar energy release rate in the general case, i.e. for non-linear material behaviour. Hence, Equ.(29) follows without the pain of explicit integration. Energy release rate will be the subject of a separate Note.

There remains the J_3 integral. As we observed above, this can only be non-zero if there are both in-plane and out-of-plane loadings. Suppose the loading is a combination of a 2D loading (plane stress or strain) together with anti-plane shears. J_3 reduces to,

$$J_3 = -\int_{\Gamma_1} \overline{T} \cdot \frac{\partial \overline{u}}{\partial z} ds = -\int_{\Gamma_1} T_z \cdot \frac{\partial u_z}{\partial z} ds = -\int_{\Gamma_1} T_z \cdot \varepsilon_z ds \qquad (31)$$

noting that the end caps again give zero net contribution. Thus if the nature of the loading ensures that there is no out-of-plane strain, then J_3 is zero. However, if the 2D part of the loading approximates to plane stress, then the z-strain is just $-\nu(\sigma_x + \sigma_y)$, and the integral evaluates in the LEFM case to,

(Plane stress): $$EJ_3 = -\nu K_{II} K_{III} \qquad (32a)$$

(Plane strain): $$EJ_3 = 0 \qquad (32b)$$

In the general 3D case things become more complicated. The closed surface can be defined arbitrarily. Even when it approximates to a prismatic surface, it is no longer true in general that the ends caps will give zero contribution. Moreover, if the surface extends over a large length of the crack front, the resulting J integrals must represent some sort of average fracture parameter along this portion of the crack front. It is therefore more useful to employ surfaces which are thin prismatic slices perpendicular to the crack front. We can then hope that the ends caps again give zero net contribution, due to cancellation, and that the J integrals can be interpreted as indicated above for the 2D cases. However, it seems to me that the interpretation of J_3 will always be rather ambiguous for an arbitrary state of constraint.

I have failed to find a discussion of J_2 and J_3 in the standard texts. Consequently be cautious about the accuracy of the above remarks.

Appendix: Notes on the Integration to find J_1

If you work in polar coordinates do not fall into the trap of thinking that the second term in the integrand can be written,

$$\int \overline{T} \cdot \frac{\partial \overline{u}}{\partial x} ds = \int \left(T_r \frac{\partial u_r}{\partial x} + T_\theta \frac{\partial u_\theta}{\partial x} \right) ds \qquad (\textit{wrong!})$$

This is wrong because the polar unit vectors are not constant in the x-direction, and so we have,

$$\partial_x \overline{u} = \partial_x \left(\hat{r} u_r + \hat{\theta} u_\theta \right) = \hat{r} \partial_x u_r + \hat{\theta} \partial_x u_\theta + u_r \partial_x \hat{r} + u_\theta \partial_x \hat{\theta} \qquad (33)$$

and it is simple to show that,

$$\partial_x \hat{r} = -\frac{\sin\theta}{r} \hat{\theta} \qquad \text{and} \qquad \partial_x \hat{\theta} = \frac{\sin\theta}{r} \hat{r} \qquad (34)$$

You will also need: $$\partial_x = \cos\theta \cdot \partial_r - \frac{\sin\theta}{r} \partial_\theta \qquad (35)$$

With these expressions, the corrected expansion of the T-term is,

$$\int \overline{T} \cdot \frac{\partial \overline{u}}{\partial x} ds = \int \left(T_r \left[u_{r,r} \cos\theta - \left(u_{r,\theta} - u_\theta \right) \frac{\sin\theta}{r} \right] + T_\theta \left[u_{\theta,r} \cos\theta - \left(u_{\theta,\theta} + u_r \right) \frac{\sin\theta}{r} \right] \right) ds \qquad (36)$$

On reflection it is probably easier to work in Cartesian coordinates.

As a guide, the three terms contributing to J_1, the W-term, the T_r term and the T_θ term individually evaluated to (in plane stress):-

$$[W] = \frac{1}{4} - \frac{1}{4}\nu \qquad [T_r] = -\frac{11}{16} - \frac{7}{16}\nu \qquad [T_r] = -\frac{1}{16} + \frac{3}{16}\nu \qquad (37)$$

times K_I^2 / E in each case. J_1 is then $[W] - [T_r] - [T_\theta]$, which agrees with Equ.(29).