Stress Intensity Factors for Very Deep Cracks: (2)Axial Cracks in Cylinders

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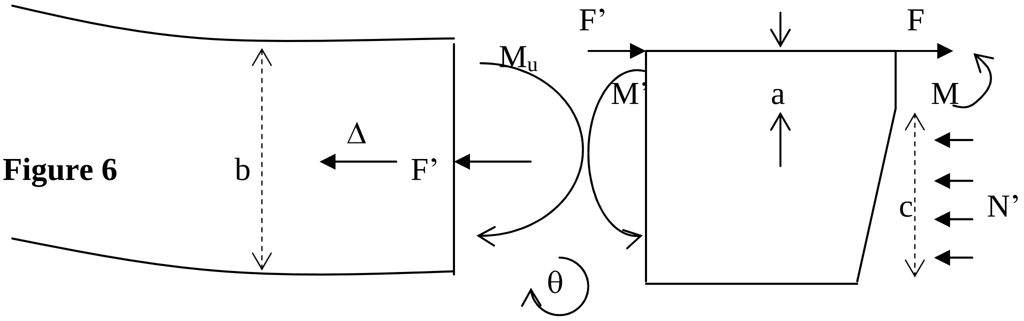
4.1 Membrane Stress

Figure 6



In the above Figure we have used the Bueckner equivalent loading. Thus, the uncracked stress N/b has been applied to the crack face. Hence, the load on the crack face is N’ = (c/b)N. This is equilibrated by the sum of the reaction loads on the ligament, F, and at the junction with the bulk of the cylinder, F’. Hence,

$$F + F' = N' = \frac{c}{b}N \tag{4.1.1}$$

The directions of all the forces and moments in Figure 6 are in the expected sense, i.e. we expect all the quantities to be positive. In particular, the expected direction of rotation, θ, is in the same sense as the moment M_u acting on the bulk of the cylinder, but is in the opposite sense to the moment M’ acting on the end of the small piece. The requirement for zero net moment on th small piece gives,

$$M + M' = \frac{a+b}{2}N' = \frac{(a+b)c}{2b}N \tag{4.1.2}$$

Equal and opposite moments on either side of the junction gives,

$$M_u = M' - F'\frac{b}{2} \tag{4.1.3}$$

The rotation of the small piece is given by Bentham and Koiter’s result (i.e. Equ.2) in terms of the load resultants acting on the ligament, thus,

$$\theta = 7.904\frac{M}{Ea^2} - 5.817\frac{F}{Ea} \tag{4.1.4}$$

For the rotation and displacement of the rest of the cylinder under the action of the moment M_u and load F’ we use standard formulae (e.g. from Roark – but see my separate derivation). Thus, the rotation is,

$$\theta = \frac{12\pi r M_u}{Eb^3} + s\frac{12\pi r^2 F'}{Eb^3} \tag{4.1.5}$$

where s = 1 for external cracks and s = -1 for internal cracks. Figure 6 has been drawn for an external crack. In this case a positive F' would cause a positive rotation. To envisage an internal crack, consider the cylinder curving the other way. It is clear that in this case a positive F' will cause a negative rotation, hence s = -1 is applicable. This subtlety was ignored by Heaton in the original derivation.

Similarly, the circumferential displacement of the bulk of the cylinder (in the direction shown in Figure 6) is,

$$\Delta = s\frac{12\pi r^2 M_u}{Eb^3} + \frac{18\pi r^3 F'}{Eb^3} \qquad (4.1.6)$$

In this case it is the sign of the bending moment's contribution which varies according to whether the crack is internal or external.

To summarise, we have seven unknowns: θ, Δ, F, F', M, M' and M_u. So far we have six equations relating them to the known N (or N'), namely equations (4.1.1-6). Now because the small piece of the cylinder including the crack is very short, and hence very stiff, we can assume that Δ is zero. This provides the seventh required equation to allow us to solve for all the unknowns. Firstly, (4.1.6) with $\Delta = 0$ gives,

$$F' = -s\frac{2}{3r}M_u \qquad (4.1.7)$$

Next we substitute (4.1.7) into (4.1.5), which gives (using $s^2 = 1$),

$$\theta = \frac{4\pi r M_u}{Eb^3} \qquad (4.1.8)$$

Next substitution of (4.1.7) into (4.1.3) gives,

$$M' = \left(1 - s\frac{b}{3r}\right)M_u \qquad (4.1.9)$$

Next substitution of (4.1.7) into (4.1.1) gives,

$$F = N' + s\frac{2}{3r}M_u \qquad (4.1.10)$$

Next substitution of (4.1.9) into (4.1.2) gives,

$$M = \frac{a+b}{2}N' - \left(1 - s\frac{b}{3r}\right)M_u \qquad (4.1.11)$$

Hence (4.1.7-11) express all the remaining non-zero variables in terms of just M_u (and the known quantities N or N'). Finally we equate the rotation in (4.1.8) to (4.1.4). and substitute for M and F in the resulting expression from (4.1.10) and (4.1.11). On re-arranging we get,

$$\frac{M_u}{N} = \frac{c}{b} \cdot \frac{7.904\left(\frac{a+b}{2a^2}\right) - \frac{5.817}{a}}{\frac{4\pi r}{b^3} + 3.878\frac{s}{ar} + 7.904\left(\frac{1 - s\frac{b}{3r}}{a^2}\right)} \qquad (4.1.12)$$

To find the normalised SIFs we again split the loading into a 'bending restrained' part, plus the remainder of the moment. The singular stress for the bending restrained part (Equ.3) determines the corresponding normalised SIF, k_{BR}, i.e.,

$$\sigma(r) = k_{BR}\frac{N}{b} \cdot \frac{\sqrt{\pi c}}{\sqrt{2\pi r}} = 0.826\frac{F}{\sqrt{2ar}} \Rightarrow k_{BR} = 0.826\frac{b}{\sqrt{ac}} \cdot \frac{F}{N} \qquad (4.1.13)$$

Substituting for F from (4.1.10) gives,

$$k_{BR} = 0.826\frac{b}{\sqrt{ac}} \cdot \left[\frac{c}{b} + s\frac{2}{3} \cdot \frac{M_u}{rN}\right] \qquad (4.1.14)$$

The net moment still to be accounted for is,

$$M_N = M - 0.736Fa \qquad (4.1.15)$$

(since a moment of 0.736Fa is implicit within the 'bending restrained' part, see Section 2). The normalised bending SIF, k_B, is found from the singular stress (Equ.8) as follows,

$$\sigma(r) = k_B\frac{N}{b} \cdot \frac{\sqrt{\pi c}}{\sqrt{2\pi r}} = \frac{2.244}{a^2}\sqrt{\frac{a}{2r}} \cdot M_N \Rightarrow k_B = 2.244\frac{b}{a\sqrt{ac}} \cdot \frac{M_N}{N} \qquad (4.1.16)$$

Adding (4.1.14) and (4.1.16) gives the total (normalised) SIF to be,

$$k_m = k_{BR} + k_B = 0.826\sqrt{\frac{c}{a}} + Z \qquad (4.1.17)$$

where, $$Z = 0.826\frac{b}{\sqrt{ac}} \cdot \frac{2s}{3r} \cdot \frac{M_u}{N} + 2.244\frac{b}{a\sqrt{ac}} \cdot \frac{M_N}{N} \qquad (4.1.18)$$

Next we use (4.1.10) and (4.1.11) in (4.1.15) to find M_N,

$$M_N = \left(\frac{c}{2} + 0.264a\right)N' - \left(1 - \frac{s}{3r}[c - 0.472a]\right)M_u \qquad (4.1.19)$$

By substituting (4.1.19) into (4.1.18) we find,

$$Z = 2.244\sqrt{\frac{c}{a}}\left(\frac{c}{2a} + 0.264\right) + \frac{b}{\sqrt{ac}} \cdot \frac{M_u}{N}\left(0.198\frac{s}{r} + 0.748\frac{sc}{ar} - \frac{2.244}{a}\right) \qquad (4.1.20)$$

It now remains only to substitute (4.1.12) for M_u/N into (4.1.20) to find the final expression for Z. This involves some nasty algebra, but eventually simplifies by various cancellations to leave the following expression for the normalised SIF,

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$$k_m = \left\{0.826 + \frac{\left(2.244 + 0.346s\frac{b^3}{ar^2}\right)\left(\frac{b}{2a} - 0.236\right)}{1 + 0.6292\left(\frac{b^3}{a^2 r}\right)\left[1 + 0.49s\frac{a}{r} - s\frac{b}{3r}\right]}\right\}\sqrt{\frac{c}{a}} \qquad (4.1.21)$$

This is identical to Heaton's expression if we assume s = -1, i.e. for an *internal* crack. For an external crack, which Heaton appears not to have considered, three of the terms in the above expression reverse sign. However, in practice, this makes negligible difference to the results.

Figure 6 compares (4.1.21) with FEA results for r/b = 35.7 and c/b in the range 0.6 to 0.9. The agreement is excellent.

Figure 7 compares (4.1.21) with FEA results for r/b = 69.1 and c/b in the range 0.52 to 0.92. The agreement is again excellent.

However, we note that for c/b > 0.94 (4.1.21) behaves in an unexpected manner, reaching a local maximum around 0.94 – 0.96. It then has a local minimum before turning up again to diverge as expected.

Looking now at the limiting behaviour of (4.1.21) we note first of all that the term $0.346s\frac{b^3}{ar^2}$, which might have been expected to be part of the dominant term as $a \to 0$ is actually negligible for reasonable values of a. For, say, r/b = 35.7, this term only becomes comparable with the constant term, 2.244, for a < 0.00012. Similarly, for r/b = 69.1, this term only becomes comparable with 2.244 for a/b < 0.00003. Finally, for r/b = 5 it becomes comparable only for a/b < 0.006. So this term can be ignored for all reason values of a and for reasonable thin shells. In fact, a reasonable approximation to (4.1.21) is provided by the much simpler expression,

$$k_m \approx \left\{0.826 + \frac{1.122\frac{b}{a}}{1 + 0.6292\left(\frac{b^3}{a^2 r}\right)}\right\}\sqrt{\frac{c}{a}} \qquad (4.1.22)$$

This approximation is sufficient to reproduce the turning point behaviour noted above. For extremely deep cracks (c/b > 0.999) the SIF diverges as per SENT with bending restrained, i.e. we have $k_m \to 0.826\sqrt{c/a}$. However, it would be misleading to use this to extrapolate FE solutions for other geometries, say for which the greatest crack depth analysed was 0.8. This is because both (4.1.21) and (4.1.22) show that the SIF increase *faster* than $\propto a^{-1/2}$ in the range, say, c/b = 0.8 to 0.9. In fact the SIF increases by a factor of nearly 2 over this range for r/b = 69.1, whereas the $\propto a^{-1/2}$ prescription would give a factor of only $\sqrt{2}$ = 1.414.

Hence, for axial cracks, the best advice is to scale assuming behaviour proportional to that of (4.1.21) or (4.1.22). The only simpler and safe rule of thumb is to assume the SIF is proportional to $\propto a^{-3/2}$ for c/b > 0.7 or so. This is demonstrated for r/b = 105 in Figure 8. This compares an extrapolation beyond c/b = 0.7 based on $\propto a^{-3/2}$ with (4.1.21). The approximation is reasonable for moderate extrapolations (say up to c/b = 0.85) but is then very conservative. On the other hand, using a less steep divergence (say $\propto$1/a) is not conservative and hence cannot be recommended. This is shown clearly in Figure 8. Note that the use of the theoretical (strict) limiting variation, namely $\propto a^{-1/2}$, would be grossly non-conservative for axial cracks.

This contrasts with the case for circumferential where where Figure 1 shows that extrapolating on the basis of $\propto a^{-1/2}$ is reasonable/conservative. This, however, is expected from the exact solution (Equ.????).