Stress Intensity Factors for Very Deep Cracks:
(1)Edge-Cracked Plates and Circumferentially Cracked Cylinders

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1. Introduction

It is common when reporting critical crack depths for results to be capped at, say, 80% of the section thickness. The explanation given is that this is the validity limit for the stress intensity factor (SIF) solution employed. This is ironic in view of the fact that SIF solutions are mathematically more tractable in the limit of very deep cracks (>80%) than for intermediate crack depths (say, 20% to 80%). The reason is the existence of asymptotic analytic solutions which are applicable when the remaining ligament is sufficiently small compared with the section thickness (Bentham & Koiter, Ref.1). In 1981, Heaton used the results of Ref.1 to derive SIFs for deep extended cracks in cylinders (Refs.2, 3, 4). Using such solutions in the deep crack regime, together with standard solutions for less deep cracks, enables critical crack sizes to be calculated for any fraction of the section thickness. (NB: Limit load solutions are generally applicable to very deep cracks as well as less deep, but possibly subject to changes in the degree of constraint).

Unfortunately only the results were reported in Refs.2, 3, 4, not their derivation nor any validation. Moreover, the results were reported only in the form of internal letters within the CEGB. This Note re-derives the simple closed-form expressions for deep, fully circumferential and extended axial cracks under membrane or wall-bending stressing. The expressions are expected to be accurate for cracks ~80% of the wall or deeper. Note that there is no distinction between the *normalised* SIFs for internal and external cracks in this limit (though the analyst may need to include crack-face pressure loading for internal cracks, which will mean the *absolute* SIFs are different).

Use of Roark formulae, based on shell theory, will limit accuracy to cylinders which are not too thick. Some limited validation against FE results is included. This suggests the results are accurate for radius:thickness ratios ≥ 5.

For other geometries the same general method will potentially be applicable. To apply it, enough must be known about the stiffness characteristics of the structure to permit the ratio of the load resultants on the uncracked ligament, i.e. M : N, to be found in terms of the structural dimensions.

2. The Basic Asymptotic Solutions – Edge Cracked Plates

Ref.1 is not easy to digest. This Section presents a simplified account of the relevant matters. All results apply for sufficiently deep cracks only (perhaps c/b ≥ 0.8).

A number of cases were considered by Bentham & Koiter, but we only need the solutions for an "edge dam between two quarter-planes" from Ref.1. These concern an edge crack of depth c on a section of thickness b, leaving a uncracked ligament of a = b – c. The essential result of Ref.1 is that, for sufficiently small a/b, the form of the singular Mode I stress near the crack tip depends only upon the two load resultants N and M acting on the uncracked ligament, where,

$$N = \int_0^a \sigma_I(x)dx \qquad M = \int_0^a x\sigma_I(x)dx \qquad \text{(Equ.1)}$$

where x is the distance across the section from the uncracked face. Note the unconventional definition of M with respect to the outer edge of the section rather than the centre of the ligament.

A bending moment, M, may occur on the ligament even when a membrane stress only is applied to the uncracked structure. This is due to the offset between the applied force and the centre of the ligament. Consider a edge-cracked plate…

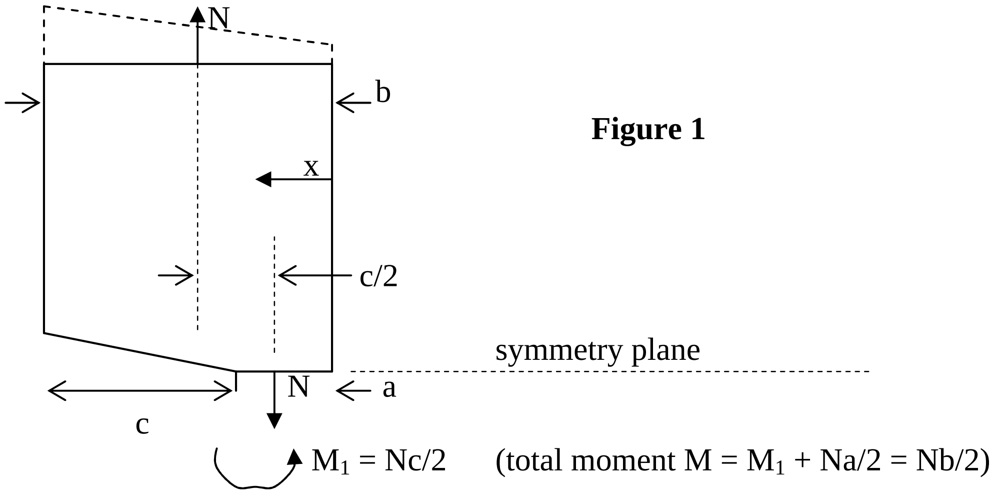


Figure 1

The moment $M_1 = Nc/2$ is required for equilibrium. Its sense would cause the ligament to rotate anti-clockwise with respect to the end of the plate. Since the ligament is on the symmetry plane this means the end of the plate rotates clockwise, as shown by the dashed line.

Alternatively, we may position the applied force to cause bending in the opposite sense…

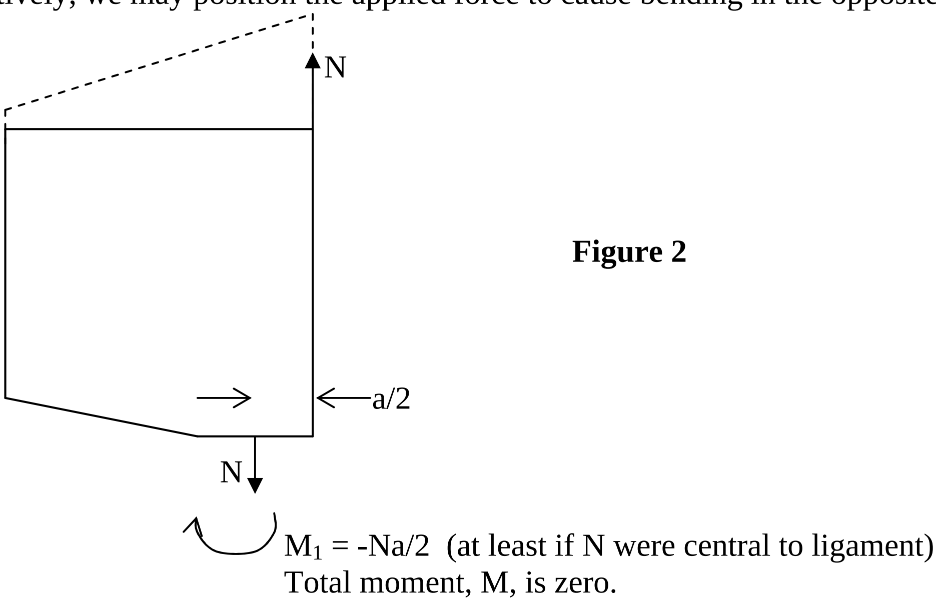


Figure 2

Hence, with carefully chosen positioning of the applied force, the rotation of the end of the plate can be made to be zero. This is the familiar "edge cracked plate, bending restrained"…

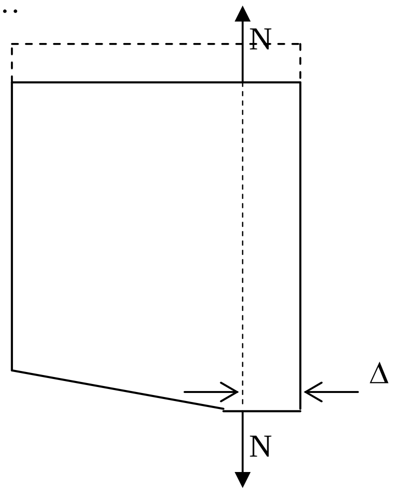


Figure 3

Usually the zero rotation of the end of the plate is simply stated as a boundary condition, i.e. a uniform displacement is applied. However, considering this condition to come about from an applied point force raises an interesting question: "at what position must N be applied so that the rotation is zero?". The answer is not at the centre of the ligament (x = a/2). Actually it is at $x = \Delta = 0.736a$.

This is not obvious but derives from the asymptotic solution. Ref.1, Equ.3.46, gives an expression for the rotation of the end of the plate for arbitrary ligament load resultants N and M. When numerical values are substituted for the various terms it becomes,

$$\text{Rotation} = -5.817\frac{N}{Ea} + 7.904\frac{M}{Ea^2} \qquad \text{(Equ.2)}$$

Hence, the rotation is zero for $M / N = 0.736a$. Assuming this M:N ratio, i.e. the bending restrained case, Ref.1, Equ.3.50, gives the solution for the singular stress on the ligament and at a distance r from the crack tip to be,

Bending restrained: $$\sigma_I = 0.826\frac{N}{\sqrt{2ar}} \qquad \text{(Equ.3)}$$

The conventional definitions of the SIF (K) and the normalised SIF (or 'compliance factor', k) are,

$$k = K / K_0, \quad K_0 = \frac{N}{b}\sqrt{\pi c}, \quad \sigma_I = \frac{K}{\sqrt{2\pi r}} \qquad \text{(Equ.4)}$$

Equs.3 and 4 therefore give the 'compliance factor' solution for deep cracks to be,

Bending restrained: $$k = 0.826\frac{b}{\sqrt{ac}} \qquad \text{(Equ.5)}$$

Hence the bending restrained SIF has a singularity of order $(a/b)^{-1/2}$ as $a \to 0$.

Now consider the case of pure bending, i.e. N = 0. In this case Ref.1, Equ.(3.52), gives the singular stress on the ligament and at a distance r from the crack tip to be,

N = 0: $$\sigma_I = \frac{1.122}{3} \cdot \frac{6M}{a^2} \cdot \sqrt{\frac{a}{2r}} \qquad \text{(Equ.8)}$$

If this comes about due to an applied bending moment, then the conventional definitions of the bending SIF and its 'compliance factor' are,

$$k = K/K_0, \quad K_0 = \sigma_b \sqrt{\pi c}, \quad \sigma_b = \frac{6M}{b^2}, \quad \sigma_I = \frac{K}{\sqrt{2\pi r}} \qquad \text{(Equ.9)}$$

and hence Equs.8 and 9 give,

Pure bending: $$k = \frac{1.122}{3}\left(\frac{b}{a}\right)^2 \sqrt{\frac{a}{c}} \qquad \text{(Equ.10)}$$

The pure bending SIF has a singularity of order $(a/b)^{-3/2}$ as $a \to 0$, i.e. a stronger order of singularity than the bending restrained SIF.

Finally, there is the case of an applied force, N, only – but with bending <u>un</u>restrained. In this case there is a total moment Nb/2 acting on the ligament (see Figure 1). However, recalling that, in the case of restrained bending, the centre of force is at x = 0.736a, which corresponds to a moment of 0.736Na, it is convenient to regard unrestrained bending to be the addition of a moment of N(b/2 – 0.736a) to the case of restrained bending. For this moment part, the singular stress is again given by Equ.8 but the compliance factor is defined by Equ.4, giving,

$$\frac{1.122}{3} \cdot \frac{6}{a^2} \cdot \sqrt{\frac{a}{2r}} \left(\frac{b}{2} - 0.736a\right) N = k_B \frac{N}{b} \sqrt{\pi c}$$

i.e. $$k_B = 1.122 \frac{b}{\sqrt{ac}} \left(\frac{b}{a} - 1.472\right)$$

To this must be added the bending restrained 'compliance', which is just Equ.5, giving,

Bending unrestrained: $$\begin{aligned} k &= 1.122 \frac{b}{\sqrt{ac}} \left(\frac{b}{a} - 0.736\right) \\ &= 1.122 \left(\frac{b}{a} + 0.264 \frac{b}{c}\right) \sqrt{\frac{c}{a}} \end{aligned} \qquad \text{(Equ.11)}$$

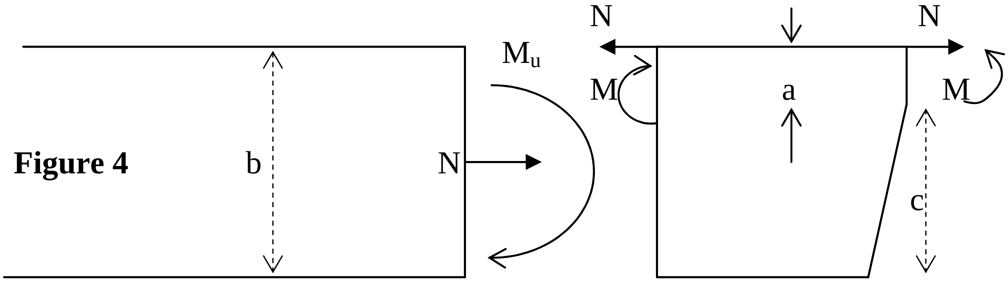
This agrees with the result given by Bentham & Koiter, Equ.(3.121). Note the factor of b/c which does not occur in the expression given by Heaton.

3. The Asymptotic Method Applied to Circumferential Cracks in Cylinders

3.1 Membrane Stressing

The half-cylinder on one side of the crack (symmetry plane) is imagined to consist of two parts: a short region near the crack plus the remaining semi-infinite part. The two are joined by the requirements of equilibrium and compatibility:-

Figure 4



Strictly, it should not be assumed that the moment (per unit circumference), M, is the same on the two ends of the short cylinder containing the crack. This is because an axisymmetric moment is self-equilibrating and as a result M decays exponentially with axial distance. However, for sufficiently short lengths, L, of the cylinder, it can be shown that M is proportional to L^4 / r^2t^2, where t is the effective thickness (and hence lies between a and b in our case) and r is the mean radius. Consequently, over the very short length of cylinder envisaged we can assume M does not decay much.

The other thing to observe from Figure 4 is that the bending moment M_u acting on the semi-infinite part is opposite in sign to that acting on the ligament. This is essential for the rotations of the mating parts to be compatible. It comes about due to the offset bending caused by N about the ligament. We have,

$$M_u = \frac{b}{2}N - M \qquad \text{(Equ.12)}$$

We now impose compatibility of rotations between the two parts. For the short cylinder we assume that the Bentham and Koiter analysis is valid and that the rotation is therefore given in terms of N and M by Equ.2. For the long cylinder, assumed sufficiently thin, the rotation is given by Roark Table 29 case 10 as,

$$\text{Rotation} = \frac{M_u}{D\lambda}, \qquad \text{where} \quad D = \frac{Eb^3}{12(1-\nu^2)}, \ \lambda = \frac{[3(1-\nu^2)]^{1/4}}{\sqrt{rb}} \qquad \text{(Equ.13)}$$

Hence, substituting Equ.12 into Equ.13 and equating to Equ.2 gives,

$$M\left[1 + 7.904\frac{[3(1-\nu^2)]^{1/4}}{12(1-\nu^2)} \cdot \frac{b^{5/2}}{r^{1/2}a^2}\right] = N\left[\frac{b}{2} + 5.817\frac{[3(1-\nu^2)]^{1/4}}{12(1-\nu^2)} \cdot \frac{b^{5/2}}{r^{1/2}a}\right] \qquad \text{(Equ.14)}$$

Using $\nu = 0.3$, as we shall do throughout from now on, gives,

$$M = \beta N \qquad \text{where, } \beta = \frac{\dfrac{b}{2} + 0.6847\dfrac{b^{5/2}}{r^{1/2}a}}{1 + 0.93\dfrac{b^{5/2}}{r^{1/2}a^2}} \qquad \text{(Equ.15)}$$

It is convenient, following the method used for SENT with unrestrained bending, to consider the loads acting on the ligament to be the sum of a restrained bending part plus an additional moment to account for bending being unrestrained. Since restrained bending implicitly include the effects of a moment of 0.736Na, the additional moment required is M – 0.736Na, where M is as given by Equ.15. For this moment part, the singular stress is again given by Equ.8 but the compliance factor is defined using the membrane stress, i.e. as per Equ.5, giving,

$$\frac{1.122}{3} \cdot \frac{6}{a^2} \cdot \sqrt{\frac{a}{2r}} \cdot (\beta - 0.736a)N = k_B \frac{N}{b}\sqrt{\frac{\pi c}{2\pi r}}$$

which, on substitution of Equ.15 for β gives,

$$k_B = \frac{1.122\left(\dfrac{b}{a} - 1.472\right)}{1 + 0.93\dfrac{b^{5/2}}{r^{1/2}a^2}} \cdot \frac{b}{\sqrt{ac}} \qquad \text{(Equ.16)}$$

To this must be added the bending restrained part as given by Equ.5. Using the identities,

$$\frac{b}{\sqrt{ac}} = \frac{b}{c}\sqrt{\frac{c}{a}} = \left(1 + \frac{a}{c}\right)\sqrt{\frac{c}{a}}$$

the total normalised SIF (‘compliance factor’) becomes,

Circumferential Crack, Membrane Stress

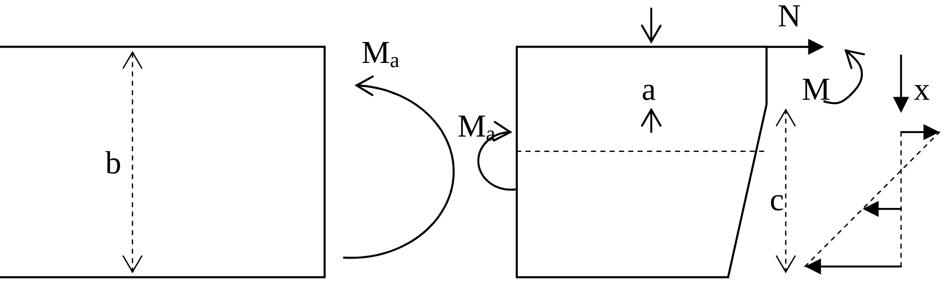
$$k_m = \left[0.826\frac{b}{c} + \frac{1.122\dfrac{b}{a} - 0.53\dfrac{b}{c}}{1 + 0.93\dfrac{b^{5/2}}{r^{1/2}a^2}}\right]\sqrt{\frac{c}{a}} \qquad \text{(Equ.17)}$$

This differs from Heaton’s solution, which had unity in the place of the two factors of b/c. This difference may be traced to Heaton’s use of the equivalent Bueckner crack face loading technique. Whilst the Bueckner method is perfectly valid, the invalid step (I believe) is to replace the Bueckner loads on the crack face with ligament loads with which the Bueckner loads are in equilibrium and thence to apply the results of Bentham and Koiter. By doing this a error of a factor of b/c occurs in the membrane load. Thus, a remotely applied force N produces a stress of N/b. If this stress is considered as acting on the crack face, as per Bueckner, the load required on the ligament to equilibrate it is Nc/b. This is at odds with the true load on the ligament, which is clearly just N. Hence, the 'N' results obtained from a Bueckner approach will be too small by this spurious factor of c/b, i.e. the parts of Heaton's results derived from the 'N' term should be multiplied by b/c to correct them, as seen in Equ.17.

The results from both Equ.17 and Heaton's origin version are compared with finite element evaluations of the membrane SIFs in Figures 1 to 4. These cover a range of cylinder radii from r/b = 4.5 to r/b = 105. Equ.17 always produces a larger SIF than Heaton, and the finite element results always lie between the two solutions. Purely on the grounds of conservatism this motivates the use of Equ.17 rather than Heaton's original (in the case of doubt as to which is correct). However, we note that the FE results tend to favour Equ.17 at moderate crack size (say around b/c ~ 0.5), but are in better agreement with Heaton for deeper cracks (say b/c ≥ 0.8). In the Author's view this suggests a degree of numerical inaccuracy in the FE results – which would tend to increase for deeper cracks since such cases are computationally more challenging. The finite element will, in general, tend to underestimate SIFs slightly – since, at heart, all FE methods of SIF evaluation depend upon how well a singularity is reproduced by an essentially finite model. The correct solution is therefore expected to be slightly higher than the FE results, especially for deeper cracks. This observation also favours Equ.17.

3.2 Through-Wall Bending

Figure 4



The moment per unit circumference acting on the uncracked section is M_a. We consider the Bueckner equivalent loading of the crack face, as shown above. This stress is $\sigma(x) = \sigma_b(2x/b - 1)$, and the crack face lies at x = [a, b]. Integrating the net load on the crack face thus gives,

$$N = \int_a^b \sigma_b \left(\frac{2x}{b} - 1 \right) dx = \frac{ac}{b} \sigma_b = \frac{6ac}{b^3} M_a \qquad (18)$$

Similarly, integrating the moment caused by the Bueckner crack face loads *about the uncracked face (x = 0)* gives,

$$M' = \int_a^b \sigma_b \left(\frac{2x}{b} - 1 \right) x dx = \frac{c}{6b} \sigma_b \left(b^2 + ab + 4a^2 \right) = \frac{c}{b} \left(1 + \frac{a}{b} + 4\frac{a^2}{b^2} \right) M_a \qquad (19)$$

The moment acting on the ligament is required to equilibrate this M' together with the applied moment M_a. Taking account of the signs this gives $M = M' - M_a$. The rotation of the end of the long cylinder, using Roark, is thus,

$$\text{rotation of long cylinder} = \frac{M_a}{D\lambda} = \frac{M' - M}{D\lambda} \tag{20}$$

The rotation of the short cylinder is given by Bentham and Koiter (using Equ.2) to be,

$$\text{rotation of short cylinder} = -5.817\frac{N}{Ea} + 7.904\frac{M}{Ea^2} \tag{21}$$

Equating (20) and (21) and using (18) and (19) for N and M' results in,

$$\frac{M}{M_a} = \frac{\frac{c}{b}\left(1 + \frac{a}{b} + 4\frac{a^2}{b^2}\right) + 5.817\left(\frac{D\lambda}{Ea}\right)\frac{6ac}{b^3}}{1 + 7.904\left(\frac{D\lambda}{Ea^2}\right)} \tag{22}$$

To calculate the normalised SIF we regard the ligament loading to be the sum of a restrained bending part, plus the rest of the moment. The restrained bending part accounts for N as well as 0.736Na of the moment. The singular stress due to the restrained bending part is given by Equ.(3), whereas the SIF is normalised by the bending stress. Using Equ.(18) this leads to,

$$k\sigma_b\sqrt{\pi c} = 0.826\frac{N}{\sqrt{2ar}} \cdot \sqrt{2\pi r} \Rightarrow k\frac{6M_a}{b^2}\sqrt{c} = \frac{0.826}{\sqrt{a}} \cdot \frac{6ac}{b^3} M_a \tag{23}$$

Hence,

$$k = 0.826\frac{\sqrt{ac}}{b} \tag{24}$$

Note that this k reduces to zero as the ligament vanishes ($a \rightarrow 0$), as a consequence of the fact that N becomes zero.

The net moment which is left to account for is $M_N = M - 0.736Na$. Using (22) this is given by,

$$\frac{M_N}{M_a} = \frac{\frac{c}{b}\left(1 + \frac{a}{b} + 4\frac{a^2}{b^2}\right) + 5.817\left(\frac{D\lambda}{Ea}\right)\frac{6ac}{b^3}}{1 + 7.904\left(\frac{D\lambda}{Ea^2}\right)} - 0.736 \times \frac{6a^2c}{b^3} \tag{25}$$

simplification of which results in,

$$\frac{M_N}{M_a} = \frac{\frac{c}{b}\left(1 + \frac{a}{b} - 0.416\frac{a^2}{b^2}\right)}{1 + 0.93\frac{b^{5/2}}{r^{1/2}a^2}} \tag{25}$$

The bending ‘compliance’ is found from the singular stress of Equ.(8), giving,

$$k_B \frac{6M_a}{b^2}\sqrt{\pi c} = \frac{1.122}{3} \cdot \frac{6M_N}{a^2}\sqrt{\frac{a}{2r}} \cdot \sqrt{2\pi r} \tag{26}$$

i.e.

$$k_B = 0.374\frac{b^2}{a\sqrt{ac}} \cdot \frac{M_N}{M_a} \tag{27}$$

Hence, using (25) gives,

$$k_B = \frac{0.374\frac{c}{b}\sqrt{\frac{a}{c}}\left(\frac{b^2}{a^2} + \frac{b}{a} - 0.416\right)}{1 + 0.93\frac{b^{5/2}}{r^{1/2}a^2}} \tag{28}$$

Thus, adding the bending restrained part from Equ.(24) gives finally,

$$k_B = \left\{0.826 + \frac{0.374\left(\frac{b^2}{a^2} + \frac{b}{a} - 0.416\right)}{1 + 0.93\frac{b^{5/2}}{r^{1/2}a^2}}\right\}\frac{\sqrt{ac}}{b} \tag{29}$$

Heaton presented his result for a linear stress distribution (zero stress at x = 0). This may be converted to a conventional bending stress distribution using $2k_{linear} = k_m + k_b$. When this is done, Heaton’s solution is seen to be identical to Equ.(29).