Why is the Average Day Longer Than 12 Hours?

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Sunrise and sunset times can be found on the internet. The difference is taken to define the length of the day. At a latitude of 51.45^o (Bristol) I find the average day is 12 hours 16 minutes, to the nearest minute.

First Order Theory of the Length of the Day

A little geometry establishes the length of the longest and shortest days to be,

$$L = 24\left(\frac{\pi \pm 2\phi}{2\pi}\right) \text{ (hours),} \qquad \text{where, } \sin\phi = \tan\theta_i \tan\theta_L \qquad (1)$$

where θ_i is the inclination of the earth's axis to the ecliptic normal (23.44^o) and θ_L is the latitude of the point of observation (taken as 51.45^o). To the nearest minute, the above formula gives, in comparison with internet data,

	Equation (1)	**Actual**
Shortest Day	7:36	7:45
Longest Day	16:24	16:43

We see that both the shortest and the longest days are actually longer than Equ.(1) predicts, by 9 and 19 minutes respectively.

Corrections to the First Order Theory

I have considered four effects which might contribute to the average day exceeding 12 hours:-

1) The Sidereal Day: This is irrelevant. Daylight is controlled by the solar day, which is 24 hours.

2) The Finite Diameter of the Sun: Equation (1) assumes that exactly one hemisphere of the earth is illuminated at any instant. Because of the non-zero diameter of the sun, and its finite distance, actually more than a hemisphere of the earth is illuminated. The angular extent of the illuminated region extends beyond the hemisphere by α *on both sides* (morning and evening), where α is the apparent angular diameter of the sun, namely 0.533^o. It is not right to estimate the effect of this on the length of the day simply as 2 x 0.533/360 x 24 hours ~ 4 minutes. The reason is that the earth's tilt further amplifies its effect. Modifying the trigonometry that led to Equation (1) gives the corrected version to be,

$$L = 24\left(\frac{\pi \pm 2\tilde{\phi}}{2\pi}\right) \text{ (hours)} \qquad (2a)$$

$$\sin\phi = \tan\theta_i \tan\theta_L \pm \frac{\sin\alpha}{\cos\theta_i \cos\theta_L} \qquad (2b)$$

Both upper signs apply for the longest day and both the lower signs for the shortest day. Equation (2) produces corrected predictions as follows,

	Equation (1)	Equation (2)	Actual
Shortest Day	7:36	7:45	7:45
Longest Day	16:24	16:33	16:43

This is a substantial improvement, by ~9 minutes, and hence suggests an average day of duration 12 hours and about 9 minutes. But this is not yet sufficient to explain the whole discrepancy.

3) Varying Orbital Speed (Elliptical Orbit): The northern hemisphere summer solstice occurs at aphelion, i.e. the greatest distance from the sun (152.10 Mkm). The northern hemisphere winter solstice occurs at perihelion, i.e. the smallest distance from the sun (147.09 Mkm)[1]. The orbital speeds are respectively 29.29 km/s and 30.29 km/s, i.e. the earth is moving more slowly during northern hemisphere summers. This will bias the average length of the day in the northern hemisphere to the longer summer days. The speed varies by ±1.678% about its mean.

Avoiding messy elliptical geometry, it is a fair bet that, to first order in small quantities, the orbital speed will vary sinusoidally with position in the orbit. Hence, the orbital speed will be a factor

$$w(d) = 1 - 0.01678\cos(2\pi d / 365) \qquad (3)$$

times its mean value, where d is the day (0 to 365). The slower the speed, the larger is the number of days spent in that region of the orbit. Hence, to correctly bias the average, the length of the day must be divided by this factor. Using the first order approximation, the length of the dth day (generalising Equ.1) is,

$$L(d) = 24\left(\frac{\pi + 2\phi\cos(2\pi d/365)}{2\pi}\right) \text{ (hours), where, } \sin\phi = \tan\theta_i \tan\theta_L \qquad (4)$$

[A further refinement could employ the corrected version, based on Equations (2), but this would be a third order correction]. Hence, the average day length is found as,

$$\langle L \rangle = \frac{1}{366}\sum_{d=0}^{365}\frac{L(d)}{w(d)} \qquad (5)$$

I have assumed a leap year because my internet data was for 2008. Numerical evaluation shows that Equation (5) exceeds 12 hours by ~3 minutes. (This is in addition to the corrections due to the sun's diameter).

Hence, so far, corrections (2) and (3) together imply an average day of 12 hours and about 12 minutes. There remains a discrepancy of about 4 minutes still to be explained. This brings us to the final correction…

4) Optical Effects in the Atmosphere (Refraction): A light ray will be diffracted by the atmosphere. The outer layers of the atmosphere are so tenuous that they have little effect. At lower altitudes the density of the air is greater and hence so is the refractive index. The result is a trajectory which curves downward (convex from above). This is akin to the mirage effect, but upside-down. Calculation of the overall angular deflection of the light ray is straightforward.

[1] I presume this is why southern hemisphere summers are even hotter than northern hemisphere summers.

Consider two adjacent layers of atmosphere which differ slightly in refractive index. Snell's law in differential form gives,

$$\frac{n+dn}{n} = \frac{\sin\theta}{\sin(\theta - d\theta)} \tag{6}$$

The angle of refraction is reduced with respect to the angle of incidence by the deflection $d\theta$. This simplifies to,

$$\frac{dn}{n} = \tan\theta \cdot d\theta \tag{7}$$

which integrates to,

$$\cos\theta_b = \frac{n_a}{n_b}\cos\theta_a \tag{8}$$

Equ.(8) gives the final angle of refraction, θ_b, as the light emerges from the last layer of refractive index n_b, in terms of the initial angle of incidence, θ_a, at the top of the atmosphere where the refractive index is n_a. If the light ray has, overall, deflected through an angle α, then $\theta_b = \theta_a - \alpha$ and we can write an implicit equation for this deflection as,

$$\cos(\theta_a - \alpha) = \frac{n_a}{n_b}\cos\theta_a \tag{9}$$

It is interesting that the total deflection only depends upon the top and bottom layers' refractive index, and not on its variation in between. This fact makes the present calculation rather more robust than it might have been.

At the top of the atmosphere we have $n_a = 1$, and we re-write n_b simply as n.

Assuming that the deflection, α, is small, Equ.(9) reduces to,

$$\cos\alpha \approx \frac{1}{n} \tag{10}$$

Now consider an incoming ray from the sun in a reference 'horizontal' direction (see diagram on next page). Avoiding messy exact geometry, the actual curved trajectory of the diffracted ray is replaced by a straight line from the top of the atmosphere, but with a slope equal to the average slope of the actual curve, taken as $\alpha/2$. Sunrise or sunset occurs where this diffracted ray is tangential to the surface of the earth. Hence, the 'light front' has been moved by an angle of $\alpha/2$ with respect to its uncorrected position (see diagram). Since this occurs at both sunrise and sunset the total correction to the length of the day is,

$$2 \times \frac{\alpha/2}{360^o} \times 24\text{hours} \tag{11}$$

The refractive index of air at STP is 1.00028. Contamination by water vapour will make little difference since the refractive index of water vapour corrected to STP is not much less (~1.00024). Hence, α is ~1.36^o and (11) evaluates to ~5.4 minutes. This nicely accounts for the final ~4 minutes. It is not terribly surprising that it is not exact. Assuming the average slope of the diffracted ray to be $\alpha/2$ is rather crude. This would be exact if the refractive index varied linearly with height. If it actually varies parabolically then the average slope would be $\alpha/3$, and the correction would become 3.6 minutes.

Conclusion

The combined effects of the non-zero diameter of the sun, the variation of the speed of the earth around its elliptical orbit, and the refractive effects of the atmosphere can successfully account for the average day exceeding 12 hours by ~16 minutes (at a latitude of ~51 degrees north).