Proof of Convexity & Normality in Plasticity

The cunning thing is to recognise that the agency causing the yielding may not be responsible for the whole state of stress.

Suppose a body is initially fully elastic but has various loads, residual stresses, etc acting, so that the state of stress at a certain point is σ_{ij}^{0}, within the yield surface. These stresses are in equilibrium with the applied loads, so that $\sigma_{ij,j}^{0} + b_i^0 = 0$, where $\bar{b}^{0}$ is the force per unit volume applied by the initial loads.

Now an external agency, which we call the "yielding agency", causes additional stressing, $\Delta\sigma_{ij}$, and takes the stress to $\sigma_{ij} = \sigma_{ij}^{0} + \Delta\sigma_{ij}$ which lies on the yield surface. The external agency now causes some plastic straining, $\delta\varepsilon_{ij}^{p}$. Because the yielding agency has applied only those forces which give rise to $\Delta\sigma_{ij}$, the work done by the yielding agency (per unit volume) is $\Delta\sigma_{ij}\delta\varepsilon_{ij}^{p} = \left(\sigma_{ij} - \sigma_{ij}^{0}\right)\delta\varepsilon_{ij}^{p}$.

The irreversibility of plasticity means that it is only possible for the yielding agency to do work on the material. It is not possible for any element of the material to do work on the external agency whilst undergoing plastic straining, since this would be synonymous with recovery. Consequently $\Delta\sigma_{ij}\delta\varepsilon_{ij}^{p} = \left(\sigma_{ij} - \sigma_{ij}^{0}\right)\delta\varepsilon_{ij}^{p}$ is everywhere positive indefinite, and this holds for an arbitrary choice of σ_{ij}^{0} within the yield surface and σ_{ij} on the yield surface.

A little thought shows that this gives us both normality and convexity. The "vector" $\Delta\sigma_{ij}$ can be drawn from any point within the yield surface to end at any point on the yield surface, but it must always lie "behind" the strain increment "vector", $\delta\varepsilon_{ij}^{p}$, where "behind" means making an angle >90^{o}. This is only possible if both normality and convexity apply.