The Strain Compatibility Equations in Polar Coordinates

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In 2D there is just one compatibility equation. In 2D polars it is,

$$r\gamma_{r\theta,r\theta} + \gamma_{r\theta,\theta} = \varepsilon_{rr,\theta\theta} + r^2\varepsilon_{\theta\theta,rr} - r\varepsilon_{rr,r} + 2r\varepsilon_{\theta\theta,r} \qquad \text{(Equ.1)}$$

where γ denotes the engineering shear (twice the tensorial shear), and indices after a comma denote partial differentiation with respect to the corresponding coordinate. This equation is simple enough to derive using trial and error starting from the explicit expressions in polars for the strains in terms of the polar displacements.

The same cannot be said for 3D (spherical) polars. However, they can be derived from the Riemann tensor. Curiously, this method does not require the explicit expressions for the strains in terms of the polar displacements to be known. In 3D there are six independent compatibility equations. It turns out that one of them is identical to Equ.1 (where θ is now the polar angle to the z-axis, and the azimuthal angle is φ). As an example, one of the other equations is,

(Equ.2)

$$\begin{aligned}\sin\theta\left(r\gamma_{r\phi,r\phi} + \gamma_{r\phi,\phi}\right) = {} & \varepsilon_{rr,\phi\phi} + r^2\sin^2\theta\cdot\varepsilon_{\phi\phi,rr} + 2r\sin^2\theta\cdot\varepsilon_{\phi\phi,r} - r\sin^2\theta\cdot\varepsilon_{rr,r} - r\sin\theta\cos\theta\cdot\gamma_{r\theta,r} \\ & + \sin\theta\cos\theta\left(\varepsilon_{rr,\theta} - \gamma_{r\theta}\right)\end{aligned}$$

Nasty, isn't it! The other four equations may be derived, if desired, using the same method (details below). However, it is a tedious exercise and personally I cannot be bothered. On the other hand, the manner in which the problem is properly formulated does have intrinsic interest.

Derivation of the Compatibility Equations in Full Generality

This is a purely geometrical problem. An arbitrary coordinate system is imagined as embedded within a medium. Upon deforming the medium, the coordinate system also deforms to what may be considered as a new coordinate system. Specifying a displacement field clearly defines the new coordinate system and the associated strain field. On the other hand, a strain field cannot be specified arbitrarily and still be compatible with a displacement field. Mathematically, if we work in terms of strains, we must ensure that the strains satisfy certain integrability conditions which guarantee the existence of a compatible displacement field. The problem may also be posed in terms of the intrinsic curvature of the deformed coordinate system. Due to the physical manner in which we have defined our deformed coordinate system, it is clear that there exists a continuous coordinate transformation to a Cartesian system. The intrinsic curvature is zero because the intrinsic curvature of the underlying space is zero[1]. Conversely, a strain field which was not compatible with displacements from an initial Euclidean space would possess intrinsic curvature. But intrinsic curvature is measured by the Riemann tensor. Hence, the compatibility equations are simply the requirement that the Riemann tensor vanish. In terms of the metric tensor of the deformed coordinate system ($g_{\alpha\beta}$), the Riemann tensor can be written,

$$R^{\alpha}{}_{\mu\beta\nu} = \Gamma^{\alpha}{}_{\mu\nu,\beta} - \Gamma^{\alpha}{}_{\beta\mu,\nu} - \Gamma^{\alpha}{}_{\tau\beta}\Gamma^{\tau}{}_{\nu\mu} + \Gamma^{\alpha}{}_{\tau\nu}\Gamma^{\tau}{}_{\beta\mu} \qquad (3)$$

where,

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\tau}\left(g_{\beta\gamma,\tau} - g_{\beta\tau,\gamma} - g_{\gamma\tau,\beta}\right) \qquad (4)$$

and where the repeated index τ implies summation (over τ = 1, 2, 3). The compatibility equations are then simply,

$$R^{\alpha}{}_{\mu\beta\nu} = 0 \qquad \text{for all } \alpha, \mu, \beta, \nu \,(= 1, 2, 3) \qquad (5)$$

Equ.(5) is the most general expression of the compatibility equations, applicable for an arbitrary coordinate system and for arbitrarily large strains.

Before turning to particular special cases, firstly consider how many independent equations Equ.5 represents. Since each of the indices takes 3 values, it might appear that Equ.5 is actually 3 x 3 x 3 x 3 = 81 equations! Of course this is not so (it would be massively over-constrained). The expressions (3, 4) imply certain symmetries amongst the 81 components of the Riemann tensor. These are most simply expressed in terms of the fully covariant tensor ($R_{\alpha\mu\beta\nu} = g_{\alpha\delta} R^{\delta}{}_{\mu\beta\nu}$), as follows,

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[1] Assuming that we are not dealing with a region of intense gravitational field, otherwise space(time) really would be curved. Incidentally, this would provide an approach to the design of the hull of a spacecraft intended for close approach to an event horizon. If the usual compatibility equations are written X = 0, then we would replace this with X = Y, where Y represents the gravitationally induced curvature.

$$R_{\alpha\mu\beta\nu} = -R_{\mu\alpha\beta\nu}; \qquad R_{\alpha\mu\beta\nu} = -R_{\alpha\mu\nu\beta}; \qquad R_{\alpha\mu\beta\nu} = R_{\beta\nu\alpha\mu} \tag{6}$$

Thus, the Riemann tensor is antisymmetric in both the first and second pair of indices, and also symmetric with respect to exchange of the two pairs. In 3D space this means that the Riemann tensor has only 6 independent components[2]. This, of course, is the correct number of compatibility equations in 3D. In 2D, there is only one independent component of the Riemann tensor, again the correct number.

Hold Up – Where Are The Strains?

Subtract the unit matrix from the metric tensor and there you have the strain tensor – in its full, large strain, glory. However, the general formulation of Riemannian geometry permits any markers to be used as coordinates. The consequence of this is that the components of the strain tensor which result from this prescription will not necessarily conform to the engineering definition. This is because differentiating with respect to an angle (say θ) is not the same as differentiating with respect to a coordinate with the dimensions of length in the direction of θ. But the engineering definition requires that strains be dimensionless, with displacements properly differentiated with respect to quantities with the dimensions of length.

There is a further complication. Riemannian geometry contains two versions of the metric tensor: the covariant and the contravariant forms. Which are we to use to define the strains? The prescription used here is firstly to define covariant tensorial strains $\lambda_{\alpha\beta}$ as follows,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \lambda_{\alpha\beta} \tag{7}$$

We now restrict attention to small strains. In this case the contravariant metric can be written, to first order in strain,

$$g^{\alpha\beta} = \eta^{\alpha\beta} - \lambda^{\alpha\beta} + O(\varepsilon^2) \tag{8}$$

Note that the undeformed metric, $(\eta_{\alpha\beta})$, is not necessarily the unit matrix, but depends upon the coordinate system chosen. Multiplication of (7) by (8) shows that the deformed contravariant metric is the inverse of the deformed covariant metric, to first order in the strain, as required. We now restrict attention to spherical polar coordinates, in terms of which the undeformed metric tensor can be written,

$$(\eta_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \quad \text{and} \quad (\eta^{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & (r\sin\theta)^{-2} \end{pmatrix} \tag{9}$$

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[2] Relativists might wonder why I have not included the familiar cyclic symmetry condition: $R_{\alpha\eta\beta\gamma} + R_{\beta\eta\gamma\alpha} + R_{\gamma\eta\alpha\beta} = 0$. This is because in 3D this equation may be derived from Equs.(6). Only in four or more dimensions does this become an independent symmetry. In 4D spacetime, this, together with Equs.(6), imply that the Riemann tensor has 20 independent components.

The dimensionless (engineering) strains in spherical polars are,

$$\varepsilon_{rr} = \lambda_{rr} = \lambda^{rr}; \quad \varepsilon_{\theta\theta} = \frac{\lambda_{\theta\theta}}{r^2} = r^2\lambda^{\theta\theta}; \quad \varepsilon_{\phi\phi} = \frac{\lambda_{\phi\phi}}{r^2 \sin^2\theta} = r^2 \sin^2\theta \cdot \lambda^{\phi\phi} \qquad (10)$$

$$\varepsilon_{r\theta} = \frac{\lambda_{r\theta}}{r} = r\lambda^{r\theta}; \quad \varepsilon_{r\phi} = \frac{\lambda_{r\phi}}{r\sin\theta} = r\sin\theta\cdot\lambda^{r\phi} \quad \varepsilon_{\theta\phi} = \frac{\lambda_{\theta\phi}}{r^2\sin\theta} = r^2\sin\theta\cdot\lambda^{\theta\phi}$$

Compatibility Equations in Cartesian Coordinates

Do Equs. (5) reduce to the familiar Cartesian equations? Yes, they do. For example, we find,

$$R_{1212} = g_{22,11} + g_{11,22} - 2g_{21,12}$$

And,

$$R_{1223} = g_{23,12} + g_{21,23} - g_{31,22} - g_{22,13}$$

These, and their symmetrical equivalents, reduce to the usual Cartesian compatibility equations when the Riemann tensor is equated to zero. Note that, in Cartesian coordinates and small strain theory, the terms quadratic in Γ in Equ.(3) can be ignored.

Compatibility Equations in 2D (Cylindrical) Polars

Do Equs. (5) reduce to Equ.(1)? Yes, they do. But since it turns out that the 3D (spherical) polar expression for $R^r{}_{\theta r\theta}$ is identical, we leave the derivation to…

Explicit Evaluation of Compatibility Equations in 3D Spherical Polars

The non-linear terms, quadratic in Γ, in Equ.(3) now play an essential part – even in small strain theory. The reason is that the Christoffel symbols (Γ) can be "large" due to the coordinate system, $(\eta_{\alpha\beta})$, even though the strain is small. However, when evaluating terms in the Riemann tensor, three orders of terms must be distinguished:-

[0] Zero'th order terms, which depend only upon the undeformed coordinates, $(\eta_{\alpha\beta})$, and not upon the strains, ε. Since the undeformed coordinate system also has zero intrinsic curvature, the sum of all such terms contributing to a given component of the Riemann tensor must be zero. We therefore just ignore them.

[1] First order terms: these are the terms linear in the strain, ε. These are the terms we wish to find.

[2] Terms quadratic in strain, ε: These will be neglected, assuming strains are small compared with unity.

The technique to explicitly evaluate the Riemann tensor in the small strain approximation can be illustrated by a couple of examples. In Equ.3, the first two terms, $\Gamma^\alpha{}_{\mu\nu,\beta} - \Gamma^\alpha{}_{\beta\mu,\nu}$, are linear in Γ and hence are required to first order in strain. An example is,

$$2\Gamma^r{}_{\theta\theta} = g^{r\tau}\left(g_{\theta\theta,\tau} - 2g_{\theta\tau,\theta}\right) \rightarrow \eta^{r\tau}\left(\lambda_{\theta\theta,\tau} - 2\lambda_{\theta\tau,\theta}\right) - \lambda^{r\tau}\left(\eta_{\theta\theta,\tau} - 2\eta_{\theta\tau,\theta}\right) \qquad (11a)$$

where we have used Equs.(7,8) to express the deformed metric, g, in terms of the undeformed metric, η, and the tensorial strain, λ. Note the use of the symbol "→" in (11). This is a reminder that (11) omits the largest (zero'th order) contribution to the Christoffel symbol, namely $2\Gamma^{r}{}_{\theta\theta}\left(0^{th}\text{ order}\right)=\eta^{r\tau}\left(\eta_{\theta\theta,\tau}-2\eta_{\theta\tau,\theta}\right)$. It also means that the second order terms have been ignored. Thus, "→" denotes "the first order part of". Appealing now to the explicit expressions for the undeformed metric, Equs.(9), Equ.(11) simplifies as follows,

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\begin{aligned}
2\Gamma^{r}{}_{\theta\theta} &\to \eta^{r\tau}\left(\lambda_{\theta\theta,\tau}-2\lambda_{\theta\tau,\theta}\right)-\lambda^{r\tau}\left(\eta_{\theta\theta,\tau}-2\eta_{\theta\tau,\theta}\right) \\
&\to \eta^{rr}\left(\lambda_{\theta\theta,r}-2\lambda_{\theta r,\theta}\right)-\lambda^{rr}\left(\eta_{\theta\theta,r}-2\eta_{\theta r,\theta}\right)-\lambda^{r\theta}\left(\eta_{\theta\theta,\theta}-2\eta_{\theta\theta,\theta}\right)-\lambda^{r\phi}\left(\eta_{\theta\theta,\phi}-2\eta_{\theta\phi,\theta}\right) \\
&\to \left(\lambda_{\theta\theta,r}-2\lambda_{\theta r,\theta}\right)-\lambda^{rr}\left(\eta_{\theta\theta,r}\right) \\
&\to \left(r^{2}\varepsilon_{rr}\right)_{,r}-2r\varepsilon_{r\theta,\theta}-2r\varepsilon_{rr}
\end{aligned}
\qquad (11b)
$$

where we have converted to conventional engineering strains using Equs.(10).

When evaluating the second pair of terms in the Riemann tensor, i.e. those quadratic in Γ, e.g. $\Gamma^{\alpha}{}_{\tau\nu}\Gamma^{\tau}{}_{\beta\mu}$, one of the Γ must be evaluated to zero'th order and the other to first order. So we also need the Chistoffel symbols to zero'th order. An example is,

$$2\Gamma^{r}{}_{\theta\theta}\left(0^{th}\text{ order}\right)=\eta^{r\tau}\left(\eta_{\theta\theta,\tau}-2\eta_{\theta\tau,\theta}\right)=\eta^{rr}\left(\eta_{\theta\theta,r}-2\eta_{\theta r,\theta}\right)=\eta^{rr}\left(\eta_{\theta\theta,r}\right)=2r \qquad (12)$$

A selection of other results (though not exhaustive) is given below,

<u>1st order parts</u>:- (13)

$$2\Gamma^{r}{}_{r\theta}\to 2\varepsilon_{r\theta}-\varepsilon_{rr,\theta}\,;\qquad 2\Gamma^{r}{}_{rr}\to-\varepsilon_{rr,r}\,;\qquad 2\Gamma^{\theta}{}_{r\theta}\to\frac{2}{r}\varepsilon_{\theta\theta}-\frac{1}{r^{2}}\left(r^{2}\varepsilon_{\theta\theta}\right)_{,r}$$

$$2\Gamma^{r}{}_{\phi\phi}\to\sin^{2}\theta\left(r^{2}\varepsilon_{\phi\phi}\right)_{,r}-2r\sin\theta\cdot\varepsilon_{r\phi,\phi}-2r\sin^{2}\theta\cdot\varepsilon_{rr}-2r\sin\theta\cos\theta\cdot\varepsilon_{r\theta}$$

$$2\Gamma^{r}{}_{r\phi}\to-\varepsilon_{rr,\phi}+2\sin\theta\cdot\varepsilon_{r\phi}\qquad 2\Gamma^{\phi}{}_{r\phi}=\frac{2}{r}\varepsilon_{\phi\phi}-\frac{1}{r^{2}}\left(r^{2}\varepsilon_{\phi\phi}\right)_{,r}$$

<u>0th order approximations</u>:- (14)

$$\Gamma^{r}{}_{r\theta}\approx\Gamma^{r}{}_{rr}\approx\Gamma^{\theta}{}_{\theta\theta}\approx\Gamma^{r}{}_{\phi r}\approx\Gamma^{\phi}{}_{\theta\theta}\approx\Gamma^{r}{}_{\phi\theta}\approx\Gamma^{\phi}{}_{r\theta}\approx\Gamma^{\phi}{}_{\phi\phi}\approx\Gamma^{\theta}{}_{r\phi}\approx 0$$

$$2\Gamma^{\theta}{}_{r\theta}\approx-\frac{2}{r}\,;\qquad 2\Gamma^{r}{}_{\phi\phi}\approx 2r\sin^{2}\theta\,;\qquad 2\Gamma^{\theta}{}_{\phi\phi}\approx 2\sin\theta\cos\theta\,;\qquad 2\Gamma^{\phi}{}_{r\phi}\approx-\frac{2}{r}$$

Expressions (11-14) are all that are required to evaluated the Riemann components $R^{r}{}_{\theta r\theta}$

and $R^r{}_{\phi r\phi}$ from Equ.(3). Equating these to zero yields Equs.(1) and (2) respectively. The remaining four compatibility equations would follow from equating the other four independent Riemann components to zero, i.e. $R^\phi{}_{\theta\phi\theta}$, $R^r{}_{\theta r\phi}$, $R^r{}_{\theta\phi\theta}$ and $R^r{}_{\phi\theta\phi}$ = 0. This is left as an exercise for the reader.