An Exceedingly Simple Climate Change Model
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1. Introduction

My intention is to devise a simple means of examining for myself whether the claims of the climate modellers is valid. The consensus is that a doubling of the current CO_2 concentration will cause an increase in the global average surface temperature of ~2.5^{o}C, for example see Ref.1. Is this true?

There is circumstantial evidence. Global average temperatures have apparently increased by ~0.74^{o}C in the century from 1906 to 2005 (Ref.1), and at a faster average rate of 0.13^{o}C per decade in the period 1956 to 2005 (Ref.1), an increase of ~0.65^{o}C in 50 years. At the same time the CO_2 concentration has increased from a pre-industrial level of ~280ppm to 315ppm in 1958 and to 385ppm in 2008 (Ref.2). The CO_2 concentration has thus increased at an average rate of ~0.4% per year over the last ~50 years. This is suggestive, but does not establish a causal link between the CO_2 concentration and global warming. To establish this more firmly, models of the climate are required in which the warming is predicted from known properties of CO_2.

2. The Model

2.1 The Basic Idea

The intention is to find the most direct way of estimating the change in global average surface temperature for (say) a doubling of the CO_2 concentration.

How is the earth's temperature established? The earth absorbs radiant energy from the sun at a rate $(1-a)S\pi r^2$, where the solar constant, S, is the energy flux from the sun, namely 1367 W/m^2 and r is the earth's radius. The albedo, a, is the fraction of this energy which is reflected and has an average value of around ~0.3. This incoming energy must balance with the radiated energy. If the effective temperature of the earth at ground level is T_g, the rate of radiating energy back into space is $4\pi r^2 \varepsilon \sigma T_g^4$, where $\sigma = 5.67$ x 10^{-8} W/m^{2}K^4 is Stefan's constant, and ε is the average emissivity. An average emissivity of ~0.57 produces an estimate for T_g of 293 K (20°C)[1].

However, the "effective ground temperature" and the "average emissivity" hide a multitude of complexities. Radiant energy does not propagate undisturbed from the earth's surface into outer space. The atmosphere is quite opaque at many infrared wavelengths. The average emissivity is a fudge which accounts for the effects of propagation through the atmosphere, amongst other things. Hence the effect of greenhouse gases is also implicit within this effective emissivity.

If we consider a position near the top of the atmosphere, the effects of propagation through the remaining tenuous gas will obviously diminish, ultimately to zero. However, it would be wrong to think of the radiant energy as being emitted only from the outermost layers of the atmosphere. Photons will propagate some considerable distance through the tenuous outer layers without being absorbed or scattered. Consequently the energy radiated back into space originates at a range of depths in the

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[1] This is the correct value for the average surface temperature, having fiddled the emissivity to get it right. The emissivity of the ground is far higher, >0.96, but that of clouds is low, perhaps ~0.5. Moreover the much lower temperature of the clouds brings the effective emissivity down further and rationalises a value of roughly around 0.6 or so.

atmosphere. It can be shown that the total radiated energy equals $4\pi r^2 \sigma T_1^4$ where T_1 is the temperature of the atmosphere at unit optical depth into the atmosphere. "Unit optical depth" simply means the position from which outgoing photons have a 50% chance of escaping into space without interaction with the atmospheric gases. Unit optical depth means a depth equal to the photon mean free path.

The significant feature of the expression $4\pi r^2 \sigma T_1^4$ for the total radiated energy is that it contains no emissivity factor. The surface at unit optical depth effectively acts as a radiator with unit emissivity. Equating this to $(1-a)S\pi r^2$ gives $T_1 = 254$ K (-19°C). Now the lapse rate of the troposphere is 6.5°C per km, so a fall of 39°C from the surface temperature of 20°C implies a height of $h_1 = 6$ km. Thus we conclude that the surface at 6 km altitude effectively acts as a perfect radiator for energy balance purposes.

How has this helped as regards estimating the effect of increased CO_2 concentration? Our simple model will be based upon assuming that the lapse rate of 6.5°C per km is unchanged, at least to first order, by modest changes in the CO_2 concentration. This is reasonable because, firstly, CO_2 is not the most significant greenhouse gas. Water vapour is. And secondly, in the troposphere, convection plays a major role in addition to radiation – and convection is unchanged by increases in the CO_2 concentration.

The effect of increased CO_2 concentration (if any) will be to push the surface of unit optical depth to a higher altitude, i.e., to increase h_1. This is because the increased opacity of the atmosphere means that a smaller physical depth is required to produce unit optical depth (in fact the optical depth is defined as the product of the opacity and the depth in metres). But the temperature of the radiant layer at unit optical depth is fixed at $T_1 = 254$ K by heat balance. Consequently, since the lapse rate is constant by assumption and the height of this layer is increased, it follows that the temperature of the ground must increase. Specifically, if h_1 increases to h_1', then the average ground level temperature must increase by $(h_1' - h_1) \times 6.5^o C$, where the heights are in km.

This is the basis of our extremely simple model. What remains is the hard bit – to find by how much an increased CO_2 concentration changes the height, h_1', of the layer at unit optical depth. Before tackling opacity, though, we first introduce a further simplification...

2.2 Physical Conditions Above h_1

We shall assume that the atmosphere above h_1 is of uniform pressure, density and temperature. Obviously this is not true. The pressure and density continue to reduce with increasing altitude. The temperature also reduces with height within the troposphere, and then increases again at higher altitudes (i.e. in the stratosphere and above). But what really matters most as regards the resistance to the passage of radiation is the number of molecules in the air column above h_1, rather than how these molecules are dispersed (to a first approximation).

Hence we assume the temperature is a constant 254 K above h_1. The pressure of the atmosphere is $\exp\{-h/7.64km\}$ Bar at a height of h km, and hence the pressure at $h_1 = 6$ km is 0.46 Bar. The pressure is assumed to remain at this value above h_1. Finally, the number density of any gas at STP is 2.68 x 10^{25} m^{-3}, and hence the number density

of the atmosphere at h_1 is (273/254)*(0.46/1)* 2.68 x 10^{25} m^{-3} = 1.31 x 10^{25} m^{-3}. So the density is 52% of that at ground level.

Half the mass of the atmosphere lies above 6km. In our simplified model the atmosphere therefore extends a distance 6km/0.52 = 11.4km above h_1, i.e. to a height of $h_T = 17.4$ km. (The real atmosphere extends much higher, of course, due to the diminishing density).

2.3 Opacity, Cross-Sections and Optical Depth

Strictly the opacity, κ, is defined such that,

$$\kappa\rho = \sum_i \sigma_i \rho_{Ni} = \frac{1}{\Lambda} \tag{1}$$

where ρ is the density of the gas and ρ_{Ni} is the number density of the molecules of the ith constituent (e.g. O_2 or CO_2, etc). The cross-section for photon absorption or scattering by a molecule of constituent i is σ_i. Finally, Λ is the mean free path of a photon. Equ.(1) follows from the definition of the scattering cross-section. The optical depth of a slab of atmosphere of thickness Δh is defined as the thickness in units of the mean free path, i.e.,

$$\tau = \Delta h / \Lambda = \Delta h \sum_i \sigma_i \rho_{Ni} \tag{2}$$

By definition, the atmosphere above h_1, i.e., between h_1 and h_T, has unit optical depth, so that,

$$\left(h_T - h_1\right) = \left[\sum_i \sigma_i \rho_{Ni}\right]^{-1} \tag{3}$$

We can define an average cross-section, $\langle\sigma\rangle$, over all constituents of the atmosphere by,

$$\langle\sigma\rangle\rho_N = \sum_i \sigma_i \rho_{Ni} \tag{4}$$

where $\rho_N = \sum_i \rho_{Ni}$ is the number density of all species. Since this is 1.31 x 10^{25} m^{-3} at h_1 and above, and since $h_T - h_1 = 11.4$ km, it follows that,
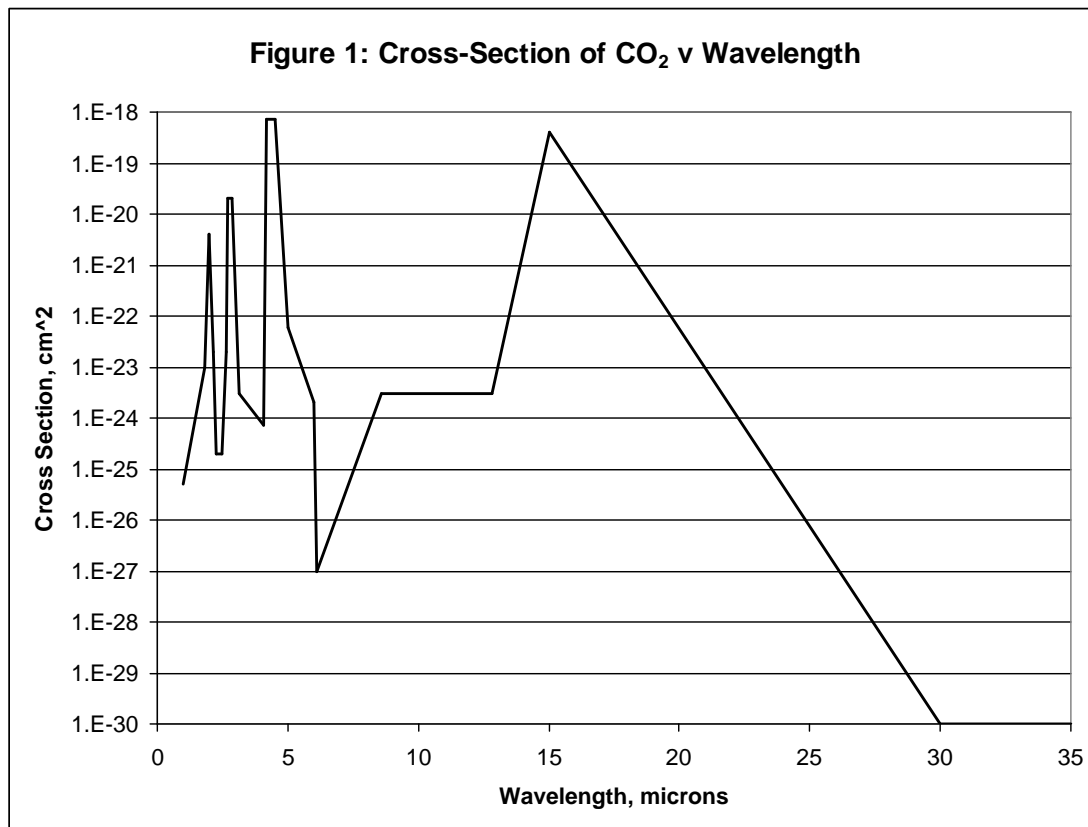
$$\langle\sigma\rangle = 6.65 \times 10^{-30} m^2 \tag{5}$$

This is an average not only over all gases, but also over all radiation wavelengths. In practice molecules will be millions of times more opaque at certain frequencies which resonate with molecular vibration frequencies. For CO_2, the gas of interest, it will be important to take the variation of cross-section with frequency into account. However another simplification of our crude model is that all other gases will be treated as having a frequency independent cross-section, equal to (5) after subtracting the average contribution due to CO_2.

2.4 CO_2 Cross-Section

The opacity of a given concentration of CO_2 can be found from its scattering cross-section for photons, as indicated in 2.3. The scattering cross-section has been taken from Freedman (Ref.3). These data were collated/derived for astrophysical purposes,

e.g. to model the atmospheres of hot planets like Venus, rather than for terrestrial climate modelling. I have not established whether this data is strictly independent of the CO_2 spectroscopic data normally used by climate modellers, but I hope that it is. For our purposes a simplified multi-linear representation of the cross-section has been used, as shown in Figure 1.

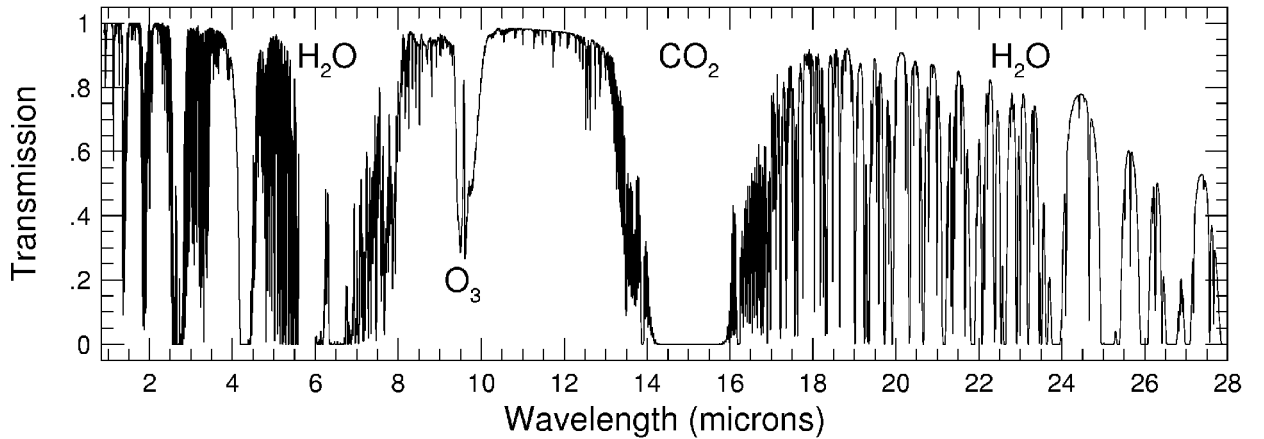**Figure 1: Cross-Section of CO_2 v Wavelength**

This spectrum covers the infrared region. The visible spectrum is just to the left of the leftmost point. Note that the black body spectrum at 254 K, the temperature at h_1, peaks at 22 microns. It is this which is relevant, not the spectrum of the incoming light from the sun (which peaks in the visible region due to the sun's surface temperature of ~5900 K). At 254 K there is negligible radiation below 1 micron. Above 30 microns CO_2 will be assumed effectively transparent (i.e. with a cross-section of 10^{-30} cm^2).

2.5 Rosseland Opacity and the Great Fallacy

There is a fallacy which can be found in many different forms in web discussions (see, for example, Refs.4). It runs like this. "CO_2 is a greenhouse gas because it absorbs infrared radiation. This absorption is overwhelmingly most significant at wavelengths where the CO_2 cross-section has resonance peaks, i.e. at around 2, 2.8, 4.3 and 15 microns. There is already sufficient CO_2 in the atmosphere to absorb all the infrared radiation in these wavelengths, as the transmission coefficient of Figure 2 shows. Consequently adding more CO_2 cannot make any difference."

This argument is fallacious. The reason relates to the way in which electromagnetic radiation transports energy through a gas. It is not sufficient to consider only the absorption of radiation. The greenhouse effect is not caused by CO_2 continually absorbing radiation. The more CO_2, or any other molecular species, absorbs radiation,

Figure 2: Transmission Coefficient for Infrared Radiation against Wavelength

the hotter it gets and hence the more it re-radiates the energy again. Over short periods, the atmosphere is in a state of dynamic equilibrium. This equilibrium is achieved by each layer of the atmosphere re-radiating as much energy as it absorbs. Thus any argument which ignores re-radiation and equilibrium is in error.

A nice analogy is to think of each band of wavelengths as representing a 'channel' through which radiant energy could potentially flow upwards towards space. Any wavelength with near zero transmission coefficient in Figure 2 is a channel with high resistance. Conversely, any wavelength with a transmission coefficient near unity is a channel with low resistance. It is clear, then, that most of the energy will flow through the low resistance channels. If we have a number of resistances, R_λ, in parallel the overall resistance is not the sum of these resistances but given by the familiar parallel resistors formula, $\dfrac{1}{R} = \sum_\lambda \dfrac{1}{R_\lambda}$. The channels of high resistance contribute very little to the overall resistance, R, which is dominated by the contribution of the low resistance channels. Hence we might guess that it is not the CO_2 absorption 'troughs' of Figure 2 that matter most, but the regions of high transmission.

In the case of radiative heat transport through a gas the above ideas result in an effective opacity known as Rosseland opacity. Rosseland opacity is familiar to astrophysicists modelling stellar atmospheres but perhaps less often used in terrestrial climate modelling. The theory goes like this. Suppose the flux of energy at frequency ω is F_ω at some height z where the temperature is T. Consider an increment of height, δz. The optical depth of this slice of atmosphere is $\delta z / \Lambda$ and this equals the proportion of photons absorbed by the slice. Hence the flux of energy emerging from the slice is reduced by an amount $-\delta F_\omega = \delta z F_\omega / \Lambda = \kappa \rho \delta z F_\omega$, i.e., $\dfrac{dF_\omega}{dz} = -\kappa \rho F_\omega$. But in thermal equilibrium F_ω is just the black body flux, i.e. $F_\omega = cB(\omega, T)/4$, so that $-\kappa \rho F_\omega = \dfrac{c}{4} \cdot \dfrac{dB}{dz} = \dfrac{c}{4} \cdot \dfrac{dB}{dT} \cdot \dfrac{dT}{dz}$. Recalling that the opacity and B depend upon frequency, the total flux integrated over all frequencies is thus,

$$F = \int F_\omega d\omega = \frac{c}{4} \cdot \frac{dT}{dz} \int \frac{1}{\kappa \rho} \cdot \frac{dB}{dT} \cdot d\omega \qquad (6)$$

It is therefore natural to define an effective opacity via,

$$\frac{1}{\kappa_R} = \frac{\int \frac{1}{\kappa} \cdot \frac{dB}{dT} \cdot d\omega}{\int \frac{dB}{dT} \cdot d\omega} \qquad (7)$$

This is the Rosseland opacity. In terms of this, (6) can be re-arranged to give,

$$\frac{dT}{dz} = C \frac{\kappa_R \rho F}{T^3} \qquad (8)$$

where C is a numerical constant. Thus, the Rosseland opacity can be seen to control (i.e. be proportional to) the temperature gradient through the gas – which is principally the quantity of interest to us. And Equ.(7) shows that, like the resistor analogy, the Rosseland opacity equals the reciprocal of the average of the reciprocals at each frequency. Hence, it will be the smaller opacities as a function of frequency which determine the effective (Rosseland) opacity, whilst the largest opacities contribute little.

2.6 Evaluating the Rosseland Opacity and Hence h_1'

Actually we shall calculate what amounts to the same thing, the equivalent Rosseland cross-section,

$$\frac{1}{\langle \sigma \rangle_R} = \frac{\int \frac{1}{\langle \sigma \rangle} \cdot \frac{dB}{dT} \cdot d\omega}{\int \frac{dB}{dT} \cdot d\omega} \qquad (9)$$

This cross-section is the average over all molecules in the atmosphere, as defined by Equ.(4). If the average cross-section over all molecules *except* CO_2 is σ_0 and the CO_2 cross-section per molecule is σ_{CO_2} then the average cross-section is,

$$\langle \sigma \rangle = (1 - f)\sigma_0 + f\sigma_{CO_2} \qquad (10)$$

where f is the fraction of the molecules in the atmosphere which are CO_2. This is the same as the volume fraction, currently standing at 385 ppm (2009). The factor of $1 - f$ will be dropped, being virtually unity. Hence the effective (Rosseland) cross-section is,

$$\frac{1}{\langle \sigma \rangle_R} = \frac{\int \frac{1}{(\sigma_0 + f\sigma_{CO_2})} \cdot \frac{dB}{dT} \cdot d\omega}{\int \frac{dB}{dT} \cdot d\omega} \qquad (11)$$

Equs.(3) and (4) become,

$$h_T - h_1 = \frac{1}{\langle \sigma \rangle_R \rho_N} \qquad (12)$$

Suppose Equs.(11) and (12) apply to the current atmosphere with $f = 385 \times 10^{-6}$. Now consider a possible future atmosphere with a different CO_2 fraction, f'. This will have a different height, h_1', for the layer at unit optical depth given by,

$$\frac{1}{\langle\sigma'\rangle_R} = \frac{\int \frac{1}{\left(\sigma_0 + f'\sigma_{CO_2}\right)} \cdot \frac{dB}{dT} \cdot d\omega}{\int \frac{dB}{dT} \cdot d\omega} \tag{13}$$

$$h_T - h_1' = \frac{1}{\langle\sigma'\rangle_R \rho_N} \tag{14}$$

Subtracting (14) from (12) gives the change in the height of the radiant layer,

$$h_1' - h_1 = \frac{\int \left[\frac{1}{\left(\sigma_0 + f\sigma_{CO_2}\right)} - \frac{1}{\left(\sigma_0 + f'\sigma_{CO_2}\right)}\right] \cdot \frac{dB}{dT} \cdot d\omega}{\int \frac{dB}{dT} \cdot d\omega} \tag{15}$$

Recall that the average ground level temperature increase is just $(h_1' - h_1) \times 6.5^o C$, so Equ.(15) is exactly what we require to find this quantity. The CO_2 cross-section is given by Figure 1. The derivative of the black body energy density is,

$$\frac{dB}{dT} = \frac{\hbar}{\pi^2 T}\left(\frac{\omega}{c}\right)^3 \frac{xe^x}{\left(e^x - 1\right)^2}, \qquad \text{where, } x = \frac{\hbar\omega}{kT} \tag{16}$$

The average cross-section excluding CO_2, σ_0, is found by requiring that (12) reproduces $h_1 = 6$ km, $h_T = 17.4$ km. We have already seen, (5), that this requires, $\langle\sigma\rangle_R = 6.65 \times 10^{-30} m^2$. To get this result from (11) with $f = 385 \times 10^{-6}$ and using Figure 1 for σ_{CO_2} turns out to require $\sigma_0 = 5.422 \times 10^{-30} m^2$.

We can now use (15) to find $(h_1' - h_1)$ for any other CO_2 concentration, f', of interest. Before doing so notice some important qualitative properties of (15). Firstly, note that any frequencies such that $f\sigma_{CO_2} \gg \sigma_0$ will contribute virtually nothing to the integral. This occurs if $\sigma_{CO_2} \gg \sigma_0 / f = 1.4 \times 10^{-26} m^2 = 1.4 \times 10^{-22} cm^2$. Comparing with Figure 1 we see that the peak levels of the four absorption bands in the CO_2 infrared spectrum are therefore unimportant, rather surprisingly. This resolves the fallacy discussed in Section 2.5. There are two other regimes:-

For $f\sigma_{CO_2} \ll \sigma_0$ the integrand is about $\dfrac{(f' - f)\sigma_{CO_2}}{\sigma_0^2}$, noting that this is positive for an increase in CO_2, i.e. the radiant layer moves to a higher altitude as anticipated.

For $f\sigma_{CO_2} \approx \sigma_0$ the integrand is about $\dfrac{(f' - f)}{2\sigma_0(f' + f)}$, again positive.

In the regime $f\sigma_{CO_2} << \sigma_0$ the integrand is therefore far less than $\dfrac{(f'-f)}{f\sigma_0}$, and hence far less than the typical contribution from the regime $f\sigma_{CO_2} \approx \sigma_0$. It follows that the integral in (15) is dominated by contributions from around $f\sigma_{CO_2} \approx \sigma_0$, i.e. from frequencies such that $\sigma_{CO_2} \approx 1.4 \times 10^{-22} cm^2$. Consequently it *is* important that the spectrum does have some absorption peaks, like those shown in Figure 1, otherwise there would be no regions in which the CO_2 cross-section took this value. If CO_2 absorbed radiation with a frequency independent cross-section of 10^{-25} cm^2, say, then there would be no greenhouse effect due to CO_2.

Evaluating (15) numerically, and converting to an average ground level temperature rise using $(h'_1 - h_1) \times 6.5^o C$, gives,

Scenario	CO2 ppm	f'	ΔT (^{o}C) IPCC 2007	ΔT (^{o}C) My Model
Pre-industrial	280	0.73	-0.74	-0.68
B1	600	1.55	1.8	1.00
A1T	700	1.82		1.34
B2	800	2.08		1.65
A1B	850	2.20	2.8	1.79
A2	1250	3.25	3.6	2.74
A1F1	1550	4.00		3.28

Apart form the pre-industrial case, the scenarios in the above Table are those considered in the 2007 IPCC report, Ref.1. The pre-industrial temperature change is the measured temperature change between 1906 and 2005. Running my model with the pre-industrial CO_2 concentration of 280 ppm gives a good estimate of this, i.e. 0.68^{o}C compared with the actual 0.74^{o}C.

For future increased CO_2 scenarios, my model produces rather smaller temperature rises than the IPCC consensus models. However, in view of the extreme simplicity of my model, accuracy could not be expected. All we can conclude from this exercise is that the IPCC consensus models do appear credible.

References

[1] IPCC, "Climate Change 2007: Synthesis Report", Intergovernmental Panel on Climate Change, adopted at IPCC Plenary XXVII (Valencia, Spain, 12-17 November 2007).

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[3] Richard Freedman and David Schwenke, Nasa Ames Research Center, http://www.iap.fr/molecules2008/PPT/molecules2008-freedman.pdf

[4] http://www.nov55.com/ntyg.html and http://www.john-daly.com/artifact.html