Classical Physics Needs Renormalisation Too!

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1. Introduction

I had always thought that renormalisation was required only in quantum mechanics, specifically in quantum field theory. In fact it is needed in classical field theory also.

2. Classical Electrostatics

Consider two point charges q_1 and q_2 separated by a distance r. We know very well what is their mutual potential energy, i.e.,

$$V = \frac{q_1 q_2}{r} \tag{1}$$

(using cgs units). This can be derived as the work done to bring the charges to distance r from infinity. Thus, we assume the Coulomb force expression,

$$\overline{F} = \overline{E}_1 q_2 \quad \text{where} \quad \overline{E}_1 = \frac{q_1}{r^2}\hat{r} \tag{2}$$

The work done is thus,

$$\text{W.D.} = \int_\infty^r \frac{q_1}{r'^2}\hat{r}' \cdot \left(-q_2 d\overline{r}'\right) = \frac{q_1 q_2}{r} \tag{3}$$

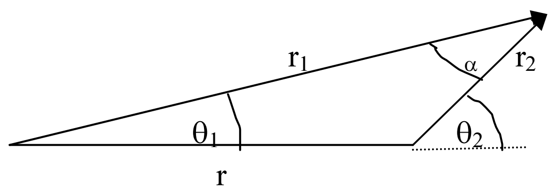
as per Equ.(1). Now we would like to understand this potential energy as residing within the electrostatic field. The expression for the field energy in terms of the field is known, i.e.,

$$\xi = \frac{1}{8\pi}\left|\overline{E}\right|^2 \tag{4}$$

In the case of two charges separated by a distance r, the field is the linear superposition of those due to the two charges separately,

$$\overline{E} = \frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2^2}\hat{r}_2 \tag{5}$$

where the two position vectors $\overline{r}_1$ and $\overline{r}_2$ run from the respective charge to the field point, thus,



$$\theta_2 = \theta_1 + \alpha$$

Substituting (5) into (4) gives the field energy density as,

$$\xi = \frac{1}{8\pi}\left[\frac{q_1^2}{r_1^4} + \frac{q_2^2}{r_2^4} + 2\frac{q_1 q_2}{r_1^2 r_2^2}\hat{r}_1 \cdot \hat{r}_2\right] \tag{6}$$

Now when we attempt to evaluate the total electrostatic field energy by integrating (6) over the whole of space, a shocking thing happens. It is infinite. The reason is the infinite self-energy term. If we tried to evaluate the field energy for just a single charge, say charge 1 from the field of Equ.(2), we would get,

$$\text{Energy} = \frac{1}{8\pi}\int_{\delta}^{\infty}\frac{q_1^2}{r_1^4}4\pi r_1^2 dr_1 = \frac{1}{2\delta} \tag{7}$$

Thus, if we let the radius of the charge, δ, become zero, the self-energy of the charge becomes infinite. Fortunately, the energy of the field for two charges, from (6), can be written,

$$\text{Energy} = \text{S.E.}_1 + \text{S.E.}_2 + \frac{1}{4\pi}\int_0^{\infty}\frac{q_1 q_2}{r_1^2 r_2^2}\hat{r}_1 \cdot \hat{r}_2 dV \tag{8}$$

where S.E. represents the self-energy of a charge as per Equ.(7). Now this self-energy exists even when the two charges reside at infinite separation. In providing an explanation for the mutual potential energy of the charges at a finite separation we clearly should not include this self-energy. Thus, the *total* field energy given by (8) must be renormalised before being equated with the potential energy by subtraction of the self-energy terms. Thus, we actually wish to demonstrate that,

$$V = \text{Energy} - \left(\text{S.E.}_1 + \text{S.E.}_2\right) = \frac{1}{4\pi}\int_0^{\infty}\frac{q_1 q_2}{r_1^2 r_2^2}\hat{r}_1 \cdot \hat{r}_2 dV \tag{9}$$

noting that this renormalisation may involve the difference between infinite quantities if the charges are point charges.

But is Equ.(9) correct? That is, does it reproduce the result of Equ.(1)? If so, we require to prove the identity,

$$\frac{1}{4\pi}\int_0^{\infty}\frac{1}{r_1^2 r_2^2}\hat{r}_1 \cdot \hat{r}_2 dV \equiv \frac{1}{r} \tag{10}$$

Proof:
The cosine theorem gives,

$$r_2^2 = r^2 + r_1^2 - 2rr_1\cos\theta_1 \tag{11}$$

Projecting onto r_1 gives,

$$r_2 \cos\alpha = r_1 - r\cos\theta_1 \tag{12}$$

We have,

$$\begin{aligned}\frac{1}{4\pi}\int_0^\infty \frac{1}{r_1^2 r_2^2}\hat{r}_1\cdot\hat{r}_2 dV &= \frac{1}{4\pi}\int_0^\infty \frac{\cos\alpha}{r_1^2 r_2^2}dV = \frac{1}{4\pi}\int_0^\infty \frac{\cos\alpha}{r_1^2 r_2^2}2\pi r_1^2 dr_1 d(\cos\theta_1)\\ &= \frac{1}{2}\int_0^\infty \frac{1}{r_2^2}\frac{r_1 - r\cos\theta_1}{r_2}dr_1 d(\cos\theta_1)\\ &= \frac{1}{2}\int_0^\infty \frac{r_1 - r\cos\theta_1}{\left(r^2 + r_1^2 - 2rr_1\cos\theta_1\right)^{3/2}}dr_1 d(\cos\theta_1)\end{aligned} \tag{13}$$

Note that we have reintroduced the zero lower limit of integration in (13). We have yet to show that the result is finite. By an elementary substitution the angular integral can be formed, namely,

$$\int_{-1}^{+1}\frac{a - bx}{(c - dx)^{3/2}}dx = \frac{2}{d}\left\{A\left[\frac{1}{\sqrt{c-d}} - \frac{1}{\sqrt{c+d}}\right] + B\left[\sqrt{c+d} - \sqrt{c-d}\right]\right\} \tag{14a}$$

$$\text{where, } A = a - \frac{bc}{d} \text{ and } B = \frac{b}{d} \tag{14b}$$

Hence, substituting: $a = r_1;\ b = r;\ c = r^2 + r_1^2;\ d = 2rr_1$ gives,

$$\sqrt{c+d} = r + r_1 \qquad \text{and} \qquad \sqrt{c-d} = |r - r_1| \tag{14c}$$

Hence we find…

For $r_1 > r$:-

$$\begin{aligned}\int_0^\infty \frac{(r_1 - r\cos\theta_1)d(\cos\theta_1)}{\left(r^2 + r_1^2 - 2rr_1\cos\theta_1\right)^{3/2}} &= \frac{1}{rr_1}\left\{\frac{r_1^2 - r^2}{2r_1}\left[\frac{1}{r_1 - r} - \frac{1}{r_1 + r}\right] + \frac{1}{2r_1}\left[r_1 + r - (r_1 - r)\right]\right\}\\ &= \frac{1}{rr_1}\left\{\frac{r_1^2 - r^2}{2r_1}\left[\frac{2r}{r_1^2 - r^2}\right] + \frac{1}{2r_1}[2r]\right\}\\ &= \frac{2}{r_1^2}\end{aligned} \tag{15}$$

For $r_1 < r$:-

$$\begin{aligned}\int_0^\infty \frac{(r_1 - r\cos\theta_1)d(\cos\theta_1)}{\left(r^2 + r_1^2 - 2rr_1\cos\theta_1\right)^{3/2}} &= \frac{1}{rr_1}\left\{\frac{r_1^2 - r^2}{2r_1}\left[-\frac{1}{r_1 - r} - \frac{1}{r_1 + r}\right] + \frac{1}{2r_1}\left[r_1 + r - (r - r_1)\right]\right\}\\ &= \frac{1}{rr_1}\left\{-\frac{r_1^2 - r^2}{2r_1}\left[\frac{2r_1}{r_1^2 - r^2}\right] + \frac{1}{2r_1}[2r_1]\right\}\\ &= \frac{1}{rr_1}\{-1 + 1\} \equiv 0\end{aligned} \tag{16}$$

Substituting (15) and (16) into (13) gives,

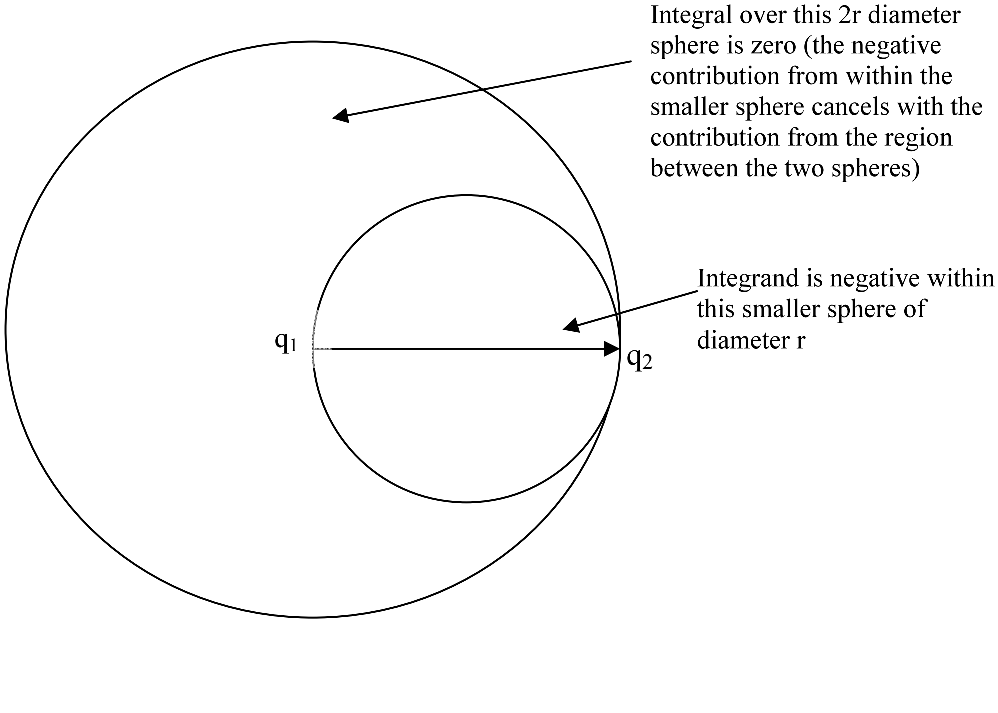
$$\begin{aligned}\frac{1}{4\pi}\int_0^\infty \frac{1}{r_1^2 r_2^2}\hat{r}_1\cdot\hat{r}_2 dV &= \frac{1}{2}\int_0^\infty \frac{r_1 - r\cos\theta_1}{\left(r^2 + r_1^2 - 2rr_1\cos\theta_1\right)^{3/2}} dr_1 d(\cos\theta_1) \\ &= \frac{1}{2}\int_r^\infty \frac{2}{r_1^2} dr_1 \qquad (17) \\ &= \frac{1}{r}\end{aligned}$$

thus proving Equ.(10) as required.

Note that the fact that the angular integral is zero within the circle $r_1 < r$, as given by Equ.(16), is essential to allow the lower limit of the r_1 integration to be replaced with 'r' instead of 0. If the integral of (17) were continued to 0 it would, of course, be divergent. The reason why the angular integration can be zero for $r_1 < r$ is that the numerator of (13), i.e. $r_1 - r\cos\theta_1$, changes sign over the range of integration. This is such an important point that it is worth considering a little further. Consider the original form of the integral, i.e.,

$$\frac{1}{4\pi}\int_0^\infty \frac{\cos\alpha}{r_1^2 r_2^2} dV$$

The integrand in this integral is negative when $\alpha > \pi/2$. Thus, the integrand is negative within the sphere given by $r_1^2 + r_2^2 = r^2$. This is the sphere centred on the mid-point between the two charges and passing through both charges, i.e. of diameter r. If we now consider the sphere of diameter 2r centred on one of the charges, we know from the above derivation that the integral is zero within this sphere. Note that this 2r diameter sphere contains the former sphere of diameter r. Pictorially,



3. The Difference Between Electrostatics and Gravity

I used to think that there was no essential difference between the (Newtonian) gravity of two point masses and the electrostatic case of two charges of different sign. Of course, gravity is always attractive. But for the case of two masses only, choosing charges of opposite signs surely makes the situations equivalent? This is not the case.

Whilst the (renormalised) potential energies in the two cases are the same (modulo replacing masses with charges), the total energy in the field is quite different. In the electrostatic case, the total field energy is always positive definite – since $|\overline{E}|^2$ is necessarily positive. The potential energy between two opposite charges is negative only as a result of the renormalisation process. In other words, the negative electrostatic potential energy is merely a result of the zero datum employed. We would nevertheless expect the energy in the electric field around two such opposite charges to have a net positive effect, e.g. it would produce an attractive gravitational field.

The situation with gravity is quite different. In this case the field energy is itself actually negative. This is true both when renormalised and unrenormalised. Thus, for gravity, the field energy is (in the non-relativistic Newtonian approximation),

$$\xi_{gravity} \propto -|\overline{E}_G|^2$$