Chapter 7

Reflections on Refraction

The laws of reflection and refraction can be derived from the appropriate boundary conditions applied to the wave solutions of Maxwell's equations

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The laws of reflection and refraction are generally learnt at school[1], but exposure to the electromagnetic description of light, e.g., Maxwell's equations, is usually confined to those studying physics or allied subjects at a further education level. Consequently it can be the case that physics students at university fail to appreciate that the laws of reflection and refraction are *consequences* of the electromagnetic properties of light. The theoretical derivation of these laws from the relevant boundary conditions applied to the plane wave solutions of Maxwell's equations are historically an important part of the triumph of Maxwell's theory.

Maxwell's equations in a medium of relative permittivity ε_r and relative permeability μ_r are,

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \nabla \cdot \overline{D} = \rho \qquad (1,2)$$

$$\nabla \times \overline{H} = \frac{\partial \overline{D}}{\partial t} + \overline{J} \qquad \nabla \cdot \overline{B} = 0 \qquad (3,4)$$

where $\overline{D} = \varepsilon_0 \varepsilon_r \overline{E}$ and $\overline{B} = \mu_0 \mu_r \overline{H}$. If there are no sources, $\rho = 0, \overline{J} = 0$, then these equations admit plane wave solutions of the form,

$$\overline{E} = \overline{E}_0 \cos(\overline{k} \cdot \overline{r} - \omega t) \qquad \overline{B} = \overline{B}_0 \cos(\overline{k} \cdot \overline{r} - \omega t) \qquad (5)$$

where the constant vectors $\overline{E}_0, \overline{B}_0, \overline{k}$ are mutually perpendicular and such that,

$$\hat{k} = \hat{E}_0 \times \hat{B}_0 \qquad E_0 = cB_0 \qquad n\omega = kc \qquad \overline{E}_0 \cdot \overline{B}_0 = 0 \qquad (6)$$

and where the caret denote unit vectors and n is the refractive index of the medium. Such a plane wave is uniquely defined by five degrees of freedom: the frequency (ω); the direction of propagation $[\hat{k}]$, which is two degrees of freedom; the magnitude of the electric field (E_0), which also specifies the magnitude of the magnetic field; and finally the orientation of the electric field $(\hat{E}_0)$, which is just one degree of freedom because it is required to be perpendicular to the direction of propagation (as is the magnetic field), $\hat{k} \cdot \hat{E}_0 = \hat{k} \cdot \hat{B}_0 = 0$.

For the present purposes the crucial issue is how the refractive index, n, enters the dispersion relation, i.e., the relation between frequency and wavelength, $n\omega = kc$. This is equivalent to writing $\omega = kv$ so that the velocity of light in a medium of refractive index n is $v = c/n$.

The behaviour of light beams incident on a boundary between two different media is a consequence of the boundary conditions imposed on the electric and magnetic field components. In the general case the boundary conditions are,

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[1] In the UK sense, "school" meaning where learning is done up to about the age of eighteen.

(i) The component E_p of the electric field parallel to the boundary must be continuous over the boundary;

(ii) The component B_n of the magnetic field normal to the boundary must be continuous over the boundary;

(iii) The component E_n of the electric field normal to the boundary is discontinuous over the boundary by an amount $\sigma_{SC} / \varepsilon_0 \varepsilon_r$ where σ_{SC} is the charge per unit area on the boundary;

(iv) The component B_p of the magnetic field parallel to the boundary is discontinuous over the boundary by an amount $\mu_0 \mu_r J_{SC}$ where J_{SC} is the current per unit length on the boundary.

However, we shall only need boundary condition (i) to derive the laws of reflection and refraction.

Consider a beam of light incident on a plane boundary between two media. The incident beam travels in a medium of refractive index n_1 and any transmitted (refracted) beam enters the other medium, of refractive index n_2. The plane boundary between the two will be taken as $z = 0$. The angle of incidence, θ_i, is the angle between the incoming wave's $\hat{k}_i$ vector and the normal to the boundary, i.e., $\hat{z}$, defined as oriented *into* the second medium. Hence $\cos\theta_i = \hat{k}_i \cdot \hat{z}$. The angles of reflection, θ_r, and refraction (transmission), θ_t, are similarly defined in terms of the directions of propagation $\hat{k}_r$ and $\hat{k}_t$ of these waves: $\cos\theta_t = \hat{k}_t \cdot \hat{z}$ and $\cos\theta_r = -\hat{k}_r \cdot \hat{z}$. Take the direction of propagation of the incident wave, $\hat{k}_i$, to lie in the (x, z) plane. Any such incident wave can be regarded as a superposition of waves which are polarised so that $\bar{E}_0$ is either parallel to $\hat{y}$ or lies in the plane of incidence, (x, z). In the latter case, boundary condition (i) requires that the x-components of the electric fields satisfy,

$$E_{i0x} \cos(k_{ix}x + k_{iy}y - \omega_i t) + E_{r0x} \cos(k_{rx}x + k_{ry}y - \omega_r t + \phi_r) = E_{t0x} \cos(k_{tx}x + k_{ty}y - \omega_t t + \phi_t) \tag{7a}$$

In the former case, boundary condition (i) requires that the y-components of the electric fields satisfy,

$$E_{i0y} \cos(k_{ix}x + k_{iy}y - \omega_i t) + E_{r0y} \cos(k_{rx}x + k_{ry}y - \omega_r t + \phi_r) = E_{t0y} \cos(k_{tx}x + k_{ty}y - \omega_t t + \phi_t) \tag{7b}$$

In Equs.(7a,7b) ϕ_r, ϕ_t are phases of the reflected and refracted waves relative to the incident wave. Either Equ.(7a) or Equ.(7b) must be true at all points (x, y) on the boundary. The relevant Equ.(7) must also be true at all times, t. This latter condition requires that the frequencies are all the same,

$$\omega_i = \omega_r = \omega_t \tag{8}$$

But the wave-numbers are given by,

$$k_i = n_1\omega_i / c \qquad k_r = n_1\omega_r / c \qquad k_t = n_2\omega_t / c \tag{9}$$

So the wavelengths of the incident and reflected waves are the same, but differ from that of the transmitted (refracted) wave if $n_2 \neq n_1$. Specifically $k_t = (n_2 / n_1)k_i$ so that the wavelength of the transmitted (refracted) wave is shorter if $n_2 > n_1$.

Because either Equ.(7a) or (7b) must be true at all points (x, y) on the boundary we must have,

$$k_{ix}x + k_{iy}y = k_{rx}x + k_{ry}y = k_{tx}x + k_{ty}y \tag{10}$$

But $k_{iy} = 0$ because the incident beam lies in the (x, z) plane. Hence it follows by equating coefficients of y that $k_{ry} = k_{ty} = 0$ and the reflected and refracted beams must also lie in the (x, z) plane. (Actually this follows simply from symmetry, since there is no more reason for the reflected or transmitted beams to be deflected towards the positive y-axis as the negative y-axis).

Equ.(10) thus reduces to $k_{ix}x = k_{rx}x = k_{tx}x$ which can also be written as,

$$k_i x \sin\theta_i = k_r x \sin\theta_r = k_t x \sin\theta_t \tag{11}$$

Equating coefficients of x and using (8,9) gives,

Law of Reflection: $\theta_r = \theta_i$ (12)

Law of Refraction: $n_1 \sin\theta_i = n_2 \sin\theta_t$ (13)

If the light is propagating from a less to a more optically dense medium, $n_2 > n_1$, then (13) indicates that the refracted beam is deflected nearer to the normal than the incident wave and (13) applies for all possible angles of incidence, $0 < \theta_i < \pi / 2$. However, if the light is propagating from a more to a less optically dense medium, $n_2 < n_1$, then (13) indicates that the refracted beam is deflected away from the normal compared to the incident wave. Equ.(13) then implies that there is a particular (critical) angle of incidence such that the angle of refraction reaches $\theta_t = \pi / 2$ and cannot increase further. The critical angle is given by,

For $n_2 < n_1$: $\sin\theta_c = n_2 / n_1$ (14)

For angles of incidence exceeding this, $\theta_i > \theta_c$ there is only a reflected wave, but no transmitted (refracted) wave. This is "total internal reflection".

The reader can continue this development, imposing the other boundary conditions and deriving the amplitudes of the reflected and transmitted (refracted) waves. An exercise is to show that, in the case of polarisation such that the incident electric field lies in the plane of incidence, and when the refractive indices are finite, there is a particular angle of incidence (Brewster's angle) such that there is no reflected beam. Moreover, this phenomenon does not occur when the incident polarisation is such that the electric field is normal to the plane of incidence $(\bar{E} \propto \bar{y})$. So reflection at Brewster's angle will filter out one polarisation component. More generally, reflection can produce a degree of polarisation which was not present in the incident beam. This effect is exploited in polarising sunglasses which reduce the glare from reflected light, e.g., off a wet road, by filtering out the horizontally polarised component.

A further exercise is to investigate the phase of the reflected wave with respect to the incident wave, ϕ_r. Except for the case $n_1 > n_2$ and at angles of incidence exceeding the critical angle, it turns out that we can take $\phi_r = 0$. (Sometimes $\phi_r = \pi$ might apply but this can equally be regarded as a change of sign of the reflected wave amplitude). However, for the case $n_1 > n_2$ and at angles of incidence exceeding the critical angle, $\theta > \theta_c$, there is a non-trivial phase change which differs according to the polarisation direction with respect to the plane of incidence. This can be exploited to design optical devices which can induce a relative phase change of $\pi / 2$ between the two plane polarised components, thus transforming plane polarised light into circularly polarised light.