Chapter 25: Entropy in the Universe

In which components of the universe does most entropy reside? By how much has structure formation reduced the entropy of baryonic matter? Three salient size scales of black holes which turn out to be the same.

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The estimates of entropy made below are not intended to be accurate. The intention is only to give the reader some feel for the orders of magnitude involved. In particular the message will be that the entropy of ordinary matter is insignificant compared with that of radiation, whilst the total entropy of matter and radiation is insignificant compared with that of black holes. Throughout we consider the entropy within the currently observable universe.

1. What is the Entropy of the CMB?

The current temperature of the cosmic microwave background (CMB) is 2.725°K. The photon density is thus $0.2436\left(\frac{k_B T}{\hbar c}\right)^3$ = 4.1 x 10^8 m^{-3}. The age of the universe is $t = 13.7$ Gyrs and, from Chapter Spare, the current radius to the horizon is $3.41ct$. Hence the volume of the observable universe is $\frac{4\pi}{3}(3.41ct)^3 = 3.6\times10^{80}$ m^3. So, the total number of CMB photons in the observable universe is 1.5×10^{89}. The average entropy per black body photon is $3.6k_B$, hence the total entropy of CMB photons in the universe is $5.3\times10^{89}k_B$. The entropy associated with the relic neutrinos can be shown to be of almost identical magnitude (accounting for all three types of neutrino).

2. What is the Entropy of Stellar Photons?

Are there more CMB photons, or more photons produced by stars? Actually the former outnumber the latter, as can be seen as follows.

To estimate the number of photons produced by stars, initially assume all stars to be of solar mass (2×10^{30} kg). If baryonic matter accounts of a fraction $\Omega_b = 0.045$ of the critical density, $\rho_{critical} = 9.6\times10^{-27}$ kg/m^3, then the total amount of baryonic matter in the observable universe is 1.6×10^{53} kg (or 10^{80} nucleons). However it is believed that only ~10% of baryonic matter is in the form of stars, so the number of stars in the observable universe if they were all of solar mass would be 8×10^{21}.

Assume that all these stars have been burning with the current brightness of the Sun for the whole age of the universe (13.7 Gyrs). This is not unreasonable since the lifetime of a solar mass star is indeed comparable with the age of the universe (perhaps closer to 10 Gyrs) and all but the first 0.5 Gyrs or so had populations of stars. The Sun's photospheric temperature is ~5890°K, so the photon flux from its surface is, from Stefan's law for a black body, $J_\gamma^N = \rho_\gamma^N c/4$ where $\rho_\gamma^N = 0.2436\left(\frac{kT}{\hbar c}\right)^3$ = 4.1×10^{18} m^{-3}, hence $J_\gamma^N = 3.1\times10^{26}$ $m^{-2}s^{-1}$. The radius of the Sun is 7×10^8 m, so its surface area is 6.16×10^{18} m^2. The total rate of emission of photons from the Sun is thus 1.9×10^{45} per second. Over the life of the universe (13.7 Gyrs =

4.3×10^{17} s) such a star will produce 8.2×10^{62} photons. Hence, the whole complement of 8×10^{21} stars in the observable universe will have produced $\sim 6\times10^{84}$ photons over the life of the universe (corresponding to an entropy of $\sim 2\times10^{85}k_B$).

One may be concerned that taking all stars to be of stellar mass might seriously skew the result. To examine the sensitivity of the result to this assumption assume an initial mass function $N(M)dM = M^{-2.35}dM$, in units of solar mass and where $N(M)$ is the number of stars relative to the number of stars of solar mass. The rate of photon emission can be estimated crudely as proportional to $M^{2.95}$, massive stars being far more luminous. The contribution of stars of mass M to the photon flux therefore varies as $M^{0.6}$. The average value of this weighting function up to some maximum stellar mass is $M_{\max}^{0.6}/1.6$. For example, considering stars up to 100 solar masses gives a factor of ~10. Hence the actual number of stellar photons might be about an order of magnitude greater than the estimate based upon solar mass stars alone. So let us say about 6×10^{85} photons over the life of the universe, corresponding to an entropy of $\sim 2\times10^{86}k_B$.

However, this is only the number of photons directly produced by stars. Thanks to absorption by interstellar dust and subsequent re-emission at lower temperatures the number of photons, and hence their entropy, will be increased. Bousso and Harnik (2010) suggest that each emitted photon is converted to 100 lower energy photons in this way. If so, the total entropy production due to stellar radiation may be perhaps $\sim 2\times10^{88}k_B$. However the CMB photons still dominate the total photon entropy by a factor of around 25 or so.

3. The Entropy of Baryonic Matter

Estimating the entropy of the baryonic component of the present universe presents a problem because both its density and its temperature vary by many orders of magnitude depending upon whether we consider the centre of stars or the cosmic voids, or a whole range of widely different interstellar gas clouds. So we adopt the cunning ploy of calculating the entropy at the time of recombination when the universe was still homogeneous (even on small scales, that is). We then claim that this is an upper bound to the baryonic entropy today because gravitational clumping will tend to have reduced the entropy of the baryonic component since then. Hence we consider the prevailing temperature to be ~3000K and employ the Sackur-Tetrode equation,

$$\frac{S_p}{k_B} = N_p\left\{\frac{5}{2} + \log\left[\left(\frac{M_p k_B T}{2\pi\hbar^2}\right)^{3/2}\frac{1}{\rho_p^N}\right]\right\} \tag{1}$$

This gives the entropy of N_p nucleons, of mass M_p and number density ρ_p^N. Ignoring for simplicity that some of the nucleons are tied up as alpha particles (which reduces their entropy) their number density is related to that of the photons by $\rho_p^N = \rho_\gamma^N / 2\times10^9$, assuming a photon:baryon ratio of 2×10^9. The photon number density is $\rho_\gamma^N = 0.2436\left(\frac{k_B T}{\hbar c}\right)^3 = 5.5\times10^{17}$ m^{-3} at a temperature of 3000K so that

$\rho_p^N = 2.7 \times 10^8$ m^{-3}. Hence, using $M_p = 1.67 \times 10^{-27}$ kg, equ.(1) gives $S_p = 55.6 N_p k_B$. We have seen already that there are $\sim 10^{80}$ nucleons in the observable universe, so the baryonic entropy immediately after recombination is $\sim 5.6 \times 10^{81} k_B$.

Immediately before recombination the free electrons would also have contributed to the entropy, by an amount also given by equ.(1) but using the electron mass and a number density ~87% of that of the nucleons, hence $S_e = 44.5 N_e k_B =\approx 4 \times 10^{81} k_B$. The total entropy of the nucleons and electrons just before recombination would therefore have been $\sim 10^{82} k_B$. Immediately after recombination the electrons' share of this entropy would be released as photonic entropy. However much of the baryonic matter today has been ionised again, so our approximate upper bound on the baryonic entropy is $\sim 10^{82} k_B$. This is six orders of magnitude smaller than the entropy of photons of stellar origin and eight orders of magnitude smaller than the CMB photons plus relic neutrinos.

4. Entropy of Relic Gravitons and Dark Matter

If relic gravitons exists, as presumably they should, their entropy is expected to be less than that of the CMB photons because they decoupled from matter much earlier, namely at around the Planck time. Egan and Lineweaver (2010) have estimated the graviton entropy to be perhaps $10^{85} - 10^{88} k_B$. As for dark matter, calculating its entropy is tricky given that we do not know what it is. Assuming that dark matter is WIMPs, Egan and Lineweaver (2010) have estimated its entropy to be in the range $10^{87} - 10^{89} k_B$. Consequently, the CMB photons and relic neutrinos appear to provide the dominant contribution to the universe's entropy, namely $\sim 10^{90} k_B$, at least as regards its matter/radiation content. But we shall now see that this pales into insignificance compared to the entropy of the gravitational degrees of freedom, i.e., black holes.

5. The Entropy of Black Holes

Non-rotating, uncharged black holes are assumed throughout for simplicity. Hawking (1971) proved that when black holes merge the surface area of the resulting black hole is greater than, or equal to, the sum of the surface areas of the original black holes. This is a rigorous result in general relativity providing that the weak energy condition holds (which is not very restrictive). This suggests an analogy with entropy which Bekenstein (1972) was the first to take seriously. However, the idea that the entropy of a black hole could be identified with a constant times its surface area became credible only when Hawking (1974) discovered that a black hole is not black. A black hole emits radiation at a well defined temperature. This temperature is,

$$k_B T = \frac{\hbar c}{8\pi m_G} \tag{2}$$

where m_G is the geometrical mass,

$$m_G = \frac{GM}{c^2} \tag{3}$$

and the radius of the black hole is,

$$R_{bh} = 2m_G \tag{4}$$

By the ‘radius’ of the black hole we mean the Schwarzschild radius, which is the event horizon. By its surface area we mean simply,

$$A = 4\pi R_{bh}^2 = 16\pi m_G^2 \tag{5}$$

Now we have an expression for temperature, the entropy follows from,

$$dS = \frac{dQ}{T} \tag{6}$$

where the ‘heat’ increment is identified with the energy equivalent of the in-falling mass, $dQ = c^2 dM$. This seems a most peculiar identification since mass is hardly regarded normally as being heat. But thanks to Hawking radiance it becomes more reasonable since, on evaporation, this mass-energy does indeed re-appear in the form of heat radiation.

Thus, using (2-5) in (6) and integrating gives,

$$\frac{S_{bh}}{k_B} = \int c^2 dM \cdot \frac{8\pi GM}{\hbar c^3} = \frac{4\pi GM^2}{\hbar c} = \frac{A}{(2L_{Planck})^2} \tag{7}$$

where the Planck length is given by,

$$L_{Planck} = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.61 \times 10^{-35} m \tag{8}$$

Thus, a solar mass black hole would have a radius of 3 km and an entropy of $\sim 10^{77} k_B$. A solar mass star contains $N_p \approx 10^{57}$ nucleons. Using a relevant temperature and particle density for such a star in the Sackur-Tetrode equation, (1), gives,

$$\frac{S_{star}}{k_B} \approx 18 N_p \approx 2 \times 10^{58} \tag{9}$$

Hence, the entropy of a solar mass black hole is a factor of $\sim 10^{19}$ greater than that of a star of the same mass. Note that the black hole’s entropy depends on the square of the mass, a result which is due to the Hawking temperature being inversely proportional to the mass, as can be seen from (7). Because a black hole’s entropy increases as the square of the mass it follows that it exceeds the entropy of the same mass of matter by an increasingly large factor at greater masses.

Entropy can usually be understood in terms of the (logarithm of the) number of available microstates consistent with the given macrostate. In this context the situation is not yet satisfactory for black holes because a completely general counting of microstates has not been accomplished. There are special cases in which the number of microstates, W , has been determined, e.g., via string theory, and the expected entropy, $S = k_B \log W$, has been found to be in agreement with the Hawking-Beckenstein formula, (7). This lends considerable credibility to the latter (or the former, depending upon your viewpoint). Further details are beyond my competence but one way into the literature is Myers (2003).

Why is black hole entropy so large? At a very heuristic level this may understood to be a consequence of the impossibility of gaining any details of the microstate prevailing within the black hole. Consequently W must count every physical microstate consistent with the black hole's macrostate, the latter being fully determined by its mass, charge and angular momentum. So the entropy of a black hole is the maximal entropy which can be achieved for a given mass, charge and angular momentum. The degrees of freedom of this given inventory of matter are chopped down to their finest level when counting W . Did we do this when calculating the entropy of baryonic matter in §3? No, we were content to count degrees of freedom at the level of nucleons and electrons. Similarly, the entropy of a gas under terrestrial conditions will generally relate to the molecular level, implicitly assuming that decomposition of its molecules into their component atoms is not relevant. From this perspective it may not be the entropy of black holes which is large, but our calculation of the entropy of matter which is small, being based on counting of microstates at only a coarse level.

5.1 The Total Entropy of Stellar Black Holes

Ignoring factors of order unity, the number of stars dN in the mass range dm, where $m = M / M_0$ is the mass in solar mass units, is,

$$dN \approx 1.35 N_{TOT} m_{\min}^{1.35} \cdot m^{-2.35} dm \tag{10}$$

where N_{TOT} is the total number of stars, and for the minimum stellar mass we can take $m_{\min} \approx 0.1$. The entropy of a black hole of mass m is $S(m) = S_0 m^2$, where $S_0 = 10^{77} k_B$ is the entropy of a solar mass black hole. Only stars with $m > 25$ or so will form black holes at the end of their life. Those in the mass range $25 < m < 42$ will go supernova, thus shedding most of their mass before leaving a much less massive black hole remnant. However stars with $m > 42$ will collapse to form black holes of similar mass. In truth it is doubtful that the initial mass function given by (10) is valid for such massive stars, but we shall assume it is to provide an upper bound on the entropy of stellar black holes. The entropy in the mass range dm is,

$$dS \sim 1.35 N_{TOT} S_0 m_{\min}^{1.35} \cdot m^{-0.35} dm \tag{11}$$

For the reasons mentioned above, this must be integrated from $m = 42$ to some maximum stellar mass, say $m_{\max} \sim 100$ for sake of argument. The total entropy of stellar black holes is thus,

$$S_{SBH} \approx \frac{1.35}{0.65} N_{TOT} S_0 m_{\min}^{1.35} \cdot \left(m_{\max}^{0.65} - 42^{0.65}\right) = 0.8 N_{TOT} S_0 \tag{12}$$

Integrating (10) to get the total mass of stars gives,

$$M_{TOT}^{stars} \approx 0.4 N_{TOT} M_0 \tag{13}$$

Substituting (13) into (12) gives,

$$S_{SBH} \approx 2 \frac{M_{TOT}^{stars}}{M_0} S_0 \tag{14}$$

But from §2 we have $M_{TOT}^{stars} / M_0 \approx 8 \times 10^{21}$ so we get,

$$S_{SBH} \approx 1.6 \times 10^{99} k_B \tag{15}$$

This is likely to be a rather generous estimate for the reasons discussed above. If the initial mass function, (10), is seriously awry for very massive stars, and the black hole entropy is actually dominated by stars with initial masses in the range $25 < m < 42$, and hence black holes with masses between 2.5 and 15, the entropy could be two or three orders of magnitude smaller than the upper bound given by (15), see Egan and Lineweaver (2010). This is quite possible since the largest observed stellar black holes at present are roughly 15 solar masses. However, even then, the stellar black hole entropy would be $\sim 10^{96}$ and hence exceed that of the dominant matter/radiation component, the CMB photons and relic neutrinos, by at least six orders of magnitude.

5.2 The Total Entropy of Supermassive Galactic Black Holes

Supermassive black holes inhabit the centre of most galaxies. There is a huge mass-gap in the black hole distribution. The largest stellar black holes cannot exceed about 140 solar masses, whereas the supermassive galactic black holes start at the order of hundreds of thousands of solar masses and extend to billions of solar masses. Perhaps the most common mass is $\sim 3\times 10^8 M_0$. Hence the entropy of one such galactic black hole is $(3\times 10^8)^2 (S_0 = 10^{77} k_B) \sim 10^{94} k_B$. So just one such black hole has $\sim 10^4$ times more entropy than the whole of the mass/radiation inventory of the observable universe (excluding black holes). It is believed that most galaxies harbour such a supermassive black hole, so since there are $\sim 10^{11}$ galaxies, the total entropy of supermassive black holes in the observable universe is $\sim 10^{105} k_B$. This exceeds the entropy of the stellar black holes by between six and nine orders of magnitude. In the entropy calculation of the ***contents*** of the universe, everything other than supermassive black holes is negligible. There is only one thing of greater entropy…

6. The Entropy of the Cosmic Event Horizon

When cosmologists refer to the "horizon" they generally mean what is more correctly called the "particle horizon", the greatest distance from which a signal can reach us. This defines the extent of the observable universe. However there is another cosmic horizon: the event horizon. This is at the greatest distance which a signal emitted by us at time t can eventually reach, given an arbitrary amount of time. It will vary with t. It is an event horizon because, by definition, no signal can escape its boundary. In a flat, static, Newtonian spacetime there is no event horizon: signals will propagate to infinity. But an expanding universe can have an event horizon, even if spatially flat. The current distance to the event horizon has been calculated by Egan and Lineweaver (2010) to be ~ 15.7 Glyr. Using this in the Hawking-Beckenstein formula, (7), implies an entropy of $2.6\times 10^{122} k_B$. This would appear to be the (logarithm of the) maximum number of possible microstates of the universe accessible from here and now. It is remarkably close to the maximum number of elementary computations which can have been made by the total mass-energy inventory of the observable universe to the present epoch (see Chapter 47). Is this coincidence?

This suggests that the Hawking-Beckenstein formula relates, not just to the number of spatial states at a given time, but to the number of state histories (phase space paths). Is this in accord with what is thought? Is this consistent with the usual statistical thermodynamics interpretation in terms of counting quantum states?

7. Three Salient Black Hole Masses and Why They Are equal

Since a star's entropy is proportional to its mass but a black hole's entropy varies as the mass-squared, (7), there is a mass at which the entropy is same whether in the form of a star or a black hole. In fact we can ask three interesting questions,

A) For what mass is a black hole's entropy equal to the original entropy of the matter of which it is composed? [*M* such that $S_{bh} = S_{star}$].

B) For what mass is a black hole's size equal to that of a fundamental particle (say, the Compton wavelength of a proton)? [*M* such that $R_{bh} = 2m_G = \hbar / M_p c$].

C) For what mass would a black hole formed in the Big Bang have just evaporated by now? [*M* such that $t_{bh} = t_{now}$?].

Curiously, the answer to all three questions is virtually the same. In the first two cases, this can be shown algebraically. In the last case, the age of the universe in the current epoch happens to make it true. This fact becomes slightly less mysterious when we recall that the age of the universe when observers exist is necessarily at least the order of the lifetime of solar mass stars (to be consistent with the evolution of life). These coincidences are demonstrated below.

(A) *M* such that $S_{bh} = S_{star}$?

Equating (7) and (9) gives,

$$M_{bh} \approx \left(\frac{18}{4\pi}\right)\left(\frac{\hbar c}{GM_p}\right) = \left(\frac{18}{4\pi}\right)\frac{M_{Planck}^2}{M_p} \tag{16}$$

where the Planck mass is,

$$M_{Planck} = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.18x10^{-8}\,kg \tag{17}$$

Hence $M_{bh} \approx 4 \times 10^{11}$ kg.

(B) *M* such that $R_{bh} = 2m_G = \hbar / M_p c$?

Using (3) gives,

$$M \approx \left(\frac{1}{2}\right)\left(\frac{\hbar c}{GM_p}\right) = \left(\frac{1}{2}\right)\frac{M_{Planck}^2}{M_p} \tag{18}$$

Thus, (16) and (18) are essentially the same, to within a factor of order unity so that our $M_{bh} \approx 4 \times 10^{11}$ kg black hole is the size of a proton (roughly).

(C) *M* such that $t_{bh} = t_{now}$?

To calculate the lifetime of a black hole we use the Stefan radiation formula applied its temperature as given by (2), together with the surface area from (5). Thus the rate of mass loss is,

$$\dot{M} = -\frac{Power}{c^2} = -\frac{\sigma T^4 A}{c^2} = -\frac{\pi^2 k^4}{60 c^4 \hbar^3} \cdot \left(\frac{\hbar c^3}{8\pi k GM}\right)^4 4\pi \left(\frac{2GM}{c^2}\right)^2 = \frac{1}{15360\pi} \cdot \frac{\hbar c^4}{G^2 M^2} \tag{19}$$

Integrating gives a lifetime of,

$$t_{bh} = 5120\pi \frac{G^2 M^3}{\hbar c^4} \quad \text{(photons only)} \tag{20}$$

Of course this is not really right because Stefan's law accounts only for electromagnetic radiation (photons). The other low mass quanta (neutrinos and gravitons) will also be radiated, reducing the lifetime by perhaps about a factor of 5. Moreover, for the mass envisaged here we will find $k_B T \sim 26$ MeV, so electron-positron pairs will also be radiated. Consequently the lifetime will be perhaps an order of magnitude less than given by (20), i.e.,

$$t_{bh} \sim 1600 \frac{G^2 M^3}{\hbar c^4} \quad \text{(all quanta)} \tag{21}$$

Or, the mass of a primordial black hole which would just have evaporated by now is,

$$M_{bh} \approx \left[\frac{\hbar c^4}{1600 G^2} t_{now} \right]^{1/3} \approx 4 \times 10^{11} \text{ kg} \tag{22}$$

Thus, the black hole which would just evaporate by now is the same black hole as considered in (A) and (B), above.

8. By How Much Has the Entropy of the Universe Increased?

To address this question we consider a fixed inventory of particles, specifically those particles and quanta within the currently observable universe. Hence the number of CMB photons considered is fixed at 1.5×10^{89} , even though at earlier epochs these would not have been within a causally connected region. It follows that the CMB photons and relic neutrinos have not changed their entropy, since their entropy ($\sim 10^{90} k_B$) is just a constant factor times the number of quanta. Since the CMB photons and relic neutrinos provide the dominant contribution to the total mass-energy entropy (excluding gravitational degrees of freedom, i.e., black holes) it follows that the entropy of the universe's mass-energy has changed by only a small fraction. Because the baryonic entropy is small, the entropy increase due to structure formation can be equated roughly with the entropy of the stellar photons, $\sim 2 \times 10^{88} k_B$. This would imply a mass-energy entropy increase due to structure formation of ~2%. However, it is not clear if dark matter might significantly alter this figure.

If we now include black hole entropy the picture changes dramatically. The entropy of the universe has increased by a large factor of around $10^{105} / 10^{90} = 10^{15}$.

9. By How Much Has the Baryonic Entropy Decreased?

We have a particular interest in baryonic matter because that is the component of the universe of which all visible structure is composed. Since this component has become more structured during the evolution of the universe, its entropy has decreased. This is evident in the above estimates since, at recombination, the entropy per nucleon was $\sim 100 k_B$ (§3) whereas stars have an entropy per nucleon of $\sim 18 k_B$ (§5). This is due to their far higher particle density and the impact of this on the Sackur-Tetrode equation, (1). Thus stars have managed to reduce their baryonic entropy by about a factor of five. In the case of rocky planets, the entropy per nucleon is substantially less than unity, due to their lower temperature and the fact that many nucleons are bound in nuclei and thence into atoms and molecular structures, reducing their degrees of freedom considerably. Rocky planets have thus achieved a reduction of the baryonic entropy by perhaps a factor of 1000. However these local effects belie the bigger picture. Only about 10% of the total baryonic mass is in the form of stars, and perhaps only ~0.01% in the form of planets. If we assume that the rest has changed its entropy little then overall the baryonic entropy has reduced by ~8%, or by a total of $\sim 10^{81} k_B$.

Hence, the amount by which the entropy of ordinary matter has decreased is nine orders of magnitude smaller than the total entropy of the universe's mass-energy (ignoring black holes) and yet it is this entropy decrease which has given rise to all the structure in the universe. It would seem that all structured matter in the universe is but a thin scum floating on an enormous ocean of chaos.

Table 1 Summary of Entropy Estimates

Component	**Entropy**
CMB photons	$\sim 5 \times 10^{89}$
Relic neutrinos	$\sim 5 \times 10^{89}$
Stellar photons + dust re-emission	$\sim 2 \times 10^{88}$
Baryonic matter	$\sim 10^{82}$
Dark matter and relic gravitons	$\sim 10^{87} - 10^{89}$
Stellar black holes	$\sim 10^{96} - 10^{99}$
Supermassive galactic black holes	$\sim 10^{105}$
Cosmic event horizon	2.6×10^{122}

The quantities refer to the currently observable universe, in k_B units. Accuracy is not intended.

10. References

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