Chapter ?

Where is the Charge in the Kerr-Newman Solution?

This chapter considers the case of the Kerr-Newman solution when the mass is small. The charge is zero or divergent or non-zero and finite depending how you measure it

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1. Summary – Where is the Charge?

In this Chapter we shall discuss the solution of the general relativistic field equations for a spinning, charged mass in the case that $m < \sqrt{a^2 + q^2}$, where we are working in geometrical units (explained below). This is the Kerr-Newman solution with a naked singularity and would be appropriate for (say) an electron or a proton if they were classical rather than quantum beasts. It was rather a surprise when Kerr (1963) discovered the exact solution to the Einstein equations for a spinning mass, of which the Kerr-Newman solution is an obvious extension with added charge. The topology of the Kerr-Newman spacetime in the regime $m < \sqrt{a^2 + q^2}$ is a ring which borders a disc, the latter acting as a bridge (or wormhole) between regions of positive and negative 'radius', see, for example, Hawking and Ellis (1973). The ring is a real curvature singularity and also a singularity of the electromagnetic field. The curvature and the electromagnetic field are regular on the disc away from its edge.

Confining attention to the physical spacetime with positive 'radius', we present plots of the electric field vector, Figures 1 and 2, which illustrate that the charge resides on both the ring and the disc, but with opposite signs. The charge seen at infinity (q) is the algebraic sum of these two opposing contributions. Integration shows that the magnitude of the charge on the disc is, in fact, divergent. Hence, the finite charge seen at infinity is effectively a result of an auto-renormalisation within the Kerr-Newman solution: the sum of the disc and ring contributions to the charge involves an infinite cancellation to leave a finite difference.

However, the more fundamental observation is that, with respect to the maximal analytical extension of the spacetime, i.e., including the region with negative 'radius', there actually is no charge at all. This follows by evaluating the 4-current density as the covariant derivative of the electromagnetic field tensor – which is identically zero. The same result can also be deduced by tracking the electric field lines. The charge which apparently resides on the disc disappears when the field is allowed to pass through the 'bridge' into the region with r < 0. Similarly, the charge on the ring disappears since the same number of lines of force emerge from it into the region with r > 0 as enter it from the region with r < 0. All lines of force are therefore continuous at all finite points, and hence there is no charge. The lines of force which are seen at $r = +\infty$ are balanced by those originating at $r = -\infty$. Adopting a topology in which the r > 0 points at infinity are joined to those in the r < 0 region would mean that the field lines would be continuous everywhere.

The rest of this Chapter presents the algebraic details underlying these conclusions.

2. The Kerr-Newman Solution

In Boyer-Lindquist coordinates, the Kerr-Newman metric is,

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta \cdot d\phi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left(\left(r^2 + a^2\right)d\phi - adt\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \quad (1)$$

where, $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2mr + a^2 + q^2$ (2)

In (1) the coordinates r, θ, ϕ, t can be interpreted as contravariant, $\{x^\mu\}$, and the coefficients are the covariant components of the metric tensor, $\{g_{\alpha\beta}\}$. The quantities m, a, q are the geometrical mass, spin and charge respectively. They are defined to have units of length. If the usual physical values of these quantities are written M,J,Q then we have,

$$q^2 = \frac{G}{\varepsilon_0 c^4}Q^2, \qquad m = \frac{G}{c^2}M, \qquad a = \frac{GJ}{mc^3} = \frac{J}{Mc} \quad (3)$$

In the case of an electron, say, which has $J = \hbar/2$, we find,

$$q = 4.88\times10^{-36}\,\text{m}, \qquad m = 6.75\times10^{-58}\,\text{m}, \qquad a = 1.93\times10^{-13}\,\text{m}$$

(in metres) and hence $m << q << a$ so that a "classical electron" would have a naked singularity[1].

The electromagnetic 4-vector potential consistent with the Kerr-Newman solution is,

$$\overline{A} = -\frac{qr}{\rho^2}\left(\overline{d}t - a\sin^2\theta \cdot \overline{d}\phi\right) \quad (4)$$

[see, for example, Misner, Thorne and Wheeler (1973)] where the bar denotes the 1-form $\overline{A}$ and the exterior derivative $\overline{d}$. Care is required in interpreting the electric and magnetic field components. In Boyer-Lindquist coordinates the covariant components of the field $F_{rt} = E_r$ and $F_{\theta t} = E_\theta$ are generally written,

$$E_r = q\left(r^2 - a^2\cos^2\theta\right)/\rho^4 \qquad \text{and} \qquad E_\theta^{BL} = -2qa^2 r\sin\theta\cos\theta/\rho^4 \quad (5)$$

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[1] Note that any 'point' particle with spin has a naked singularity if its mass is less than $\sqrt{\frac{\hbar c}{2G}}$ = 1.54 x 10^{-8} kg = 15 μg. This fate is avoided by normal matter since, at (say) the density of water, such a mass would be of macroscopic size, namely a cube of side 0.25mm. But, with a reasonable angular momentum, the geometric spin, a, will be much smaller than this. Since the source occupies a much greater spatial region than a, the 'point' source Kerr-Newman solution is not applicable. To make the geometric spin exceed 0.25mm for a mass of 15 μg requires an angular momentum exceeding 0.001 kgm^2s^{-1}. Since the moment of inertia of such a mass is ~2 x 10^{-16} kgm^2, a rotational rate of ~5 x 10^{12} cycles per second is needed, resulting in an edge velocity which would exceed the velocity of light. So a naked singularity is effectively vetoed for matter of familiar density.

However, it is immediately apparent that there is something funny about E_θ^{BL} because the dimensions are not as expected (the electric field usually has dimensions charge/length^2). This is because it has been defined with respect to $\bar{d}\theta$ rather than the more usual unit vector with unit length. In spherical polars in Euclidean space the appropriate unit vector would be $r \cdot \bar{d}\theta$. Here we interpret the spatial part of the Boyer-Lindquist coordinates as an ellipsoidal coordinate system which are related to a 'Cartesian' system as follows,

$$z = r\cos\theta, \qquad \eta = \sqrt{a^2 + r^2}\sin\theta, \qquad x = \eta\cos\phi, \qquad y = \eta\sin\phi \tag{6}$$

Of course, (x, y, z) is not really a Cartesian system, since the underlying spacetime is curved. The Boyer-Lindquist 'unit' vector in the θ-direction is thus,

$$\begin{aligned}\hat{\theta}_{BL} &\equiv \frac{\partial}{\partial\theta} = \frac{\partial\eta}{\partial\theta}\cdot\frac{\partial}{\partial\eta} + \frac{\partial z}{\partial\theta}\cdot\frac{\partial}{\partial z} = \frac{\partial\eta}{\partial\theta}\cdot\hat{\eta} + \frac{\partial z}{\partial\theta}\cdot\hat{z} \\ &= \sqrt{a^2 + r^2}\cos\theta\cdot\hat{\eta} - r\sin\theta\cdot\hat{z}\end{aligned} \tag{7}$$

If we interpret $\hat{\eta}, \hat{z}$ as orthogonal unit vectors, the Boyer-Lindquist $\hat{\theta}_{BL}$ therefore does not have unit length. A true unit vector is provided by normalising $\hat{\theta}_{BL}$ as follows,

$$\hat{\theta} \equiv \frac{\hat{\theta}_{BL}}{\left|\hat{\theta}_{BL}\right|} = \frac{\sqrt{a^2 + r^2}\cos\theta\cdot\hat{\eta} - r\sin\theta\cdot\hat{z}}{\rho} \tag{8}$$

The result is that we must divide E_θ^{BL} by ρ to get the correct component for our purposes, hence,

$$E_r = q\left(r^2 - a^2\cos^2\theta\right)/\rho^4 \qquad \text{and} \qquad E_\theta = -2qa^2 r\sin\theta\cos\theta/\rho^5 \tag{9}$$

3. Plots of the Electric Field Vector

Axisymmetry means that we need only consider the (r, θ) plane.

Figures 1 and 2 show depictions of the *directions* of the electric field. These plots used $a = q = 1$. The two plots are nominally the same, merely the result of using two different software packages to do the plotting. Attempts to plot the field showing both the direction and the magnitude, indicating the latter by the length of the indicative arrows, were unsatisfactory. The reason is that the strong singularity at the ring causes all the vectors away from the ring to appear as dots. Consequently Figures 1,2 show only the direction but not the magnitude of the electric field. This is sufficient for our purposes.

These Figures clearly illustrate,

- The field lines emanate from the ring, implying that there is a line-density of positive charge on the ring;
- The field lines terminate on both the upper and lower faces of the disc, implying that there is a surface density of negative charge on both disc surfaces;
- At large enough radii the field lines begin to approximate that of a positive point charge (turning towards the radial direction)..

The charge is therefore both on the ring *and* on the disc, but with opposing signs. It is clear from the Figures that the charge seen at infinity is less than the charge on the ring due to partial cancellation by the negative charge on the disc surfaces.

What is the ratio of the charge on the ring to that seen at infinity? This is addressed next.

4. The Charge on the Ring and the Disc Separately

The algebraic sum of the charges on the ring and the disc equals the charge seen at infinity, which is q. But what are the charges on the ring and the disc separately? To evaluate this we note that Gauss's theorem tells us that the charge on a surface is given in terms of the normal component of the electric field by,

$$\text{Charge} = \oiint \overline{E} \cdot d\overline{A} / 4\pi \qquad (10)$$

(where bars here denote 3-vectors). We are interested in integrating over the surface of the disc, at which the normal component of the electric field is the value of E_r with r = 0, i.e.,

$$E_r(r=0) = -\frac{q}{a^2 \cos^2\theta} \qquad (11)$$

The area element of the disc can be derived as follows. The volume element is given by,

$$dV = \sqrt{-g}\, dr d\theta d\phi \qquad (12)$$

where *g* is the determinant of the covariant metric tensor, (1). This gives,

$$\sqrt{-g} = \rho^2 \sin\theta \qquad (13)$$

Now the 'radial thickness' of this volume element can be obtained by setting all differentials other than dr to zero in (1), which gives,

$$\text{Radial thickness} = ds_r = \sqrt{g_{rr}} \cdot dr = \frac{\rho}{\sqrt{\Delta}} dr \qquad (14)$$

Dividing the volume element by the radial thickness gives the area element since $dV = ds_r \cdot dA$. Hence,

$$dA = \sqrt{\Delta} \cdot \rho \sin\theta \cdot d\theta d\phi \qquad (15)$$

On the disc, r = 0, this becomes,

$$dA = a\sqrt{a^2 + q^2} \cdot \cos\theta \sin\theta \cdot d\theta d\phi \qquad (16)$$

Since there are no ϕ dependencies we can consider just the annular area element,

$$dA = 2\pi a\sqrt{a^2 + q^2} \cdot \cos\theta \sin\theta \cdot d\theta \qquad (17)$$

Note that this differs from the Euclidean area element, $dA = 2\pi\eta d\eta = 2\pi a^2 \sin\theta \cos\theta \cdot d\theta$, only by a constant factor.

But substitution of (11) and (17) into (10) gives the charge on the disc to be,

$$\text{Charge on disc} = -\frac{q\sqrt{a^2+q^2}}{a}\int_0^{\pi/2}\tan\theta\cdot d\theta \tag{18}$$

(noting that the two surfaces of the disc introduces a factor of 2). But the integral in (18) is divergent at the ring, i.e., at $\theta = \pi/2$, since the integral of $\tan\theta$ is $-\log(\cos\theta)$.

We conclude that the total charge on the disc is $-\infty$ due to the divergent contribution from near the ring singularity.

Since the total charge is finite it follows that the charge on the ring must be $+\infty$. This may be confirmed as follows. Consider a torical region of small minor radius around the ring. Its area element is again given by (15). Consider the section of this toroid defined by $\eta = a$. By (6) this gives $r\sin\theta = a\cos\theta$. Hence, since we will be considering $r \to 0$ it suffices to consider $\theta \to \pi/2$. But using $\theta \to \pi/2$ in (15), and also for $r \to 0$, we get the area element $2\pi r\sqrt{a^2+q^2}\cdot d\theta$. This establishes that the area element if proportional to r for small r. But the electric field, (5), for $r \to 0$ is just $E_r \to q/r^2$. Consequently the total flux out of the small toroid is given by the integral of $2\pi r\sqrt{a^2+q^2}\times q/r^2\cdot d\theta$, which is divergent as $r \to 0$, but positive this time.

The finite charge seen at infinity is therefore due to a cancellation of two divergent quantities to leave a finite difference, namely *q*.

5. Analytic Expression for the Charge Density

The charge density is the time component of the current vector, $\Im^\mu$. The 4-vector current equals the covariant divergence of electromagnetic field tensor,

$$\Im^\mu = F^{\mu\nu}{}_{;\nu} \tag{19}$$

The covariant components of the electromagnetic field tensor are given in terms of Boyer-Lindquist coordinates by,

$$F_{rt} = E_r = q\left(r^2 - a^2\cos^2\theta\right)/\rho^4 \tag{20a}$$

$$F_{\theta t} = E_\theta^{BL} = -2qa^2 r\sin\theta\cos\theta/\rho^4 \tag{20b}$$

$$F_{\theta\phi} = B_r = 2qar\left(a^2 + r^2\right)\sin\theta\cos\theta/\rho^4 \tag{20c}$$

$$-F_{r\phi} = B_\theta = qa\sin^2\theta\left(r^2 - a^2\cos^2\theta\right)/\rho^4 \tag{20d}$$

Using (20a-d) together with the explicit expression, (1), for the metric tensor, it is ‘merely’ a matter of algebra to form the divergence of the contravariant electromagnetic field, (19), and hence to find the current. Performing this algebra by hand is tedious, to put it mildly. However, it is accomplished very easily using s computer algebra package which has GR facilities pre-coded. Hence it can be confirmed that $\Im^\mu$ is in fact identically zero.

Consequently, we conclude that the charge density is everywhere zero in the Kerr-Newman solution. This does not mean that there is no source, however. The same result would be found for the field of a static point source ignoring gravity, $E_r = q / r^2$, i.e., the (ordinary) divergence of this field appears to be identically zero. More correctly these fields may fail to have finite divergence at isolated points, lines or surfaces, and it is at these locations that the sources may be located. Nevertheless, in the case of the Kerr-Newman field there is a sense in which there really is no source.

6. The Topology of the Field Lines in the Maximally Extended Spacetime

Figures 1,2 show only half the complete spacetime, namely that half which has r > 0. There is another region with r < 0 in which the electric field looks identical to Figures 1,2 except that the direction of all the arrows is reversed. Thus, in the r < 0 region, the lines of force *emerge from* the disc, rather than appearing to end on the disc. In other words, the charge density on the disc in the r < 0 region appears to be positive. Similarly, the ring in the r < 0 region is a sink for the field lines, thus appearing to be a negative charge.

The r > 0 and r < 0 regions are joined together at the disc. The upper face of the disc in r > 0 is identified with the lower part of the disc in r < 0, and vice-versa. With this topology we see that, in reality, the field lines do not terminate on the disc at all but simply cross over continuously from the r> 0 region to the r < 0 region or the reverse. There is no charge on the disc because the field lines are continuous and do not terminate or emerge from it. The appearance of a surface charge density was an illusion brought about by confining attention to only part of the whole spacetime.

The same phenomenon occurs at the ring. It appears that field lines emerge from it when considering the r > 0 region in isolation. But in the r < 0 region, the same number of lines of force enter the ring. In fact, the lines of force are continuous and merely cross over from the r < 0 region into the r > 0 region.

An observer at $r \to +\infty$ 'sees' flux lines and concludes that there is a charge near $r = 0$. But actually the lines of force he sees have merely originated at the other spatial infinity, $r \to -\infty$, travelling by way of $r = 0$ and giving the illusion of a charge. If it made physical sense we could consider imbuing the maximally extended spacetime with a particular topology such that $r = +\infty$ and $r = -\infty$ are identified. All the lines of flux would then be continuous closed loops, with no source anywhere.

References

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Figure 1: Plot Showing Directions of Electric Field Vector Only (Not Magnitudes)

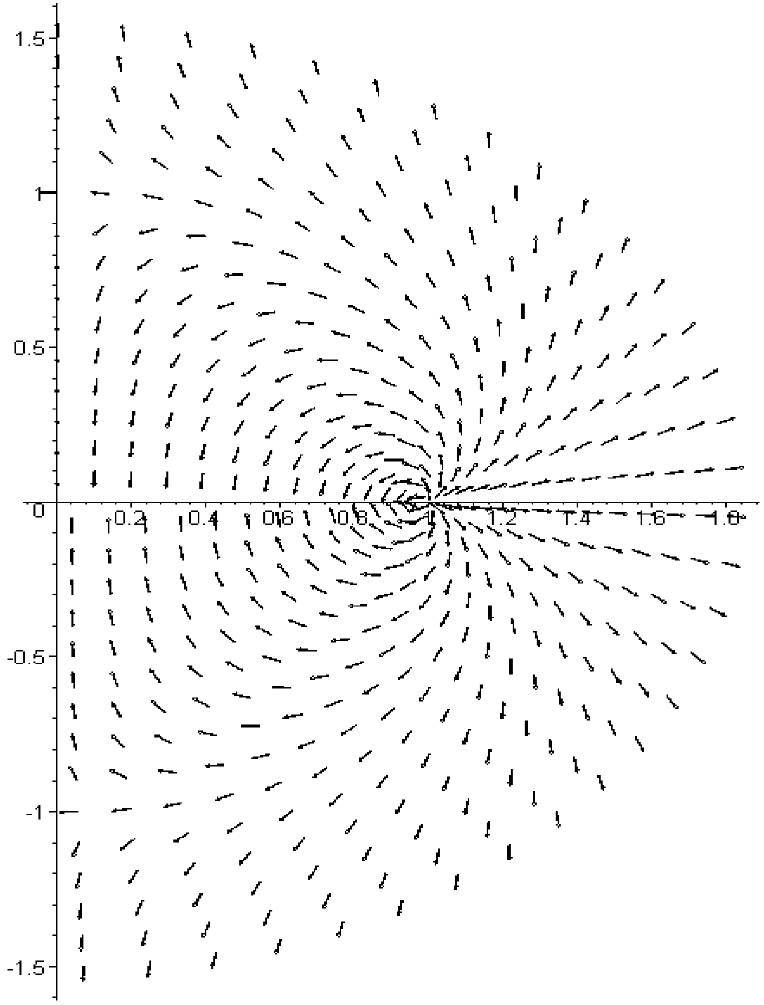

Figure 2: Alternative Plot Showing Directions of Electric Field Vector Only (Not Magnitudes)