Chapter ??

Cosmic Geometry: Times, Distances and Redshifts

The cosmological redshift and its relation to times and distances; how various measured of cosmological distances can be defined and evaluated; calculating the age of the Universe from the cosmological parameters

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1. Redshift Defined

From Chapter 23 the FLRW metrics can be written,

$$ds^2 = c^2 dt^2 - R^2(t)\left[d\psi^2 + f(\psi)^2\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right)\right] \qquad (1)$$

where $f(\psi) = \sin\psi$ with $0 \le \psi \le \pi$ for positive curvature, or $f(\psi) = \sinh\psi$ with $0 \le \psi < \infty$ for negative curvature, or $f(\psi) = \psi$ with $0 \le \psi < \infty$ for flat space. Taking our own position to be the origin of the spatial coordinates, $\psi = 0$, it follows that any light we receive from some distant galaxy must have travelled along a 'radial' line at constant θ, ϕ. Suppose a pulse of light is emitted from coordinate ψ at time t_1 and received by us at $\psi = 0$ at a later time t_2. Light travels on a null trajectory so that $cdt = R(t)d\psi$ for each increment of its path. Hence we must have,

$$\psi = \int_{t_1}^{t_2} \frac{cdt}{R(t)} \qquad (2)$$

Now consider another light pulse emitted from the same co-moving coordinate ψ at time $t_1 + \Delta t_1$, and received by us at $\psi = 0$ at time $t_2 + \Delta t_2$, then clearly,

$$\psi = \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} \frac{cdt}{R(t)} \qquad (3)$$

We may imagine the two pulses to be successive peaks in a sinusoidal wave train, and hence $f_1 = 1/\Delta t_1$ is the frequency which a co-moving observer would measure at the emission point, whereas $f_2 = 1/\Delta t_2$ is the frequency which we would observe. Subtracting (2) from (3) gives,

$$\int_{t_2}^{t_2+\Delta t_2} \frac{cdt}{R(t)} - \int_{t_1}^{t_1+\Delta t_1} \frac{cdt}{R(t)} = 0 \qquad (4)$$

But for time intervals small compared with the age of the universe (that is $f >> H$) this gives simply,

$$\frac{\Delta t_2}{R(t_2)} = \frac{\Delta t_1}{R(t_1)} \Rightarrow \frac{\Delta t_2}{\Delta t_1} = \frac{f_1}{f_2} = \frac{\lambda_2}{\lambda_1} = \frac{R(t_2)}{R(t_1)} \qquad (5)$$

The wavelength of the radiation simply expands in proportion to the size scale, $R(t)$. The redshift, z, of light emitted by a distant source and observed by us is defined as,

$$1 + z = \frac{f_{emitted}}{f_{observed}} = \frac{\lambda_{observed}}{\lambda_{emitted}} = \frac{R(t_{now})}{R(t_{emitted})} \qquad (6)$$

This is the fundamental relationship between redshift and size-scale. To convert this to a relationship between redshift and time (of emission) requires information about how the size scale varies with time, in other words the Friedmann equation. This is considered next.

2. The Friedmann Equation in terms of the Cosmological Parameters

From Chapter 23 Equ.(10b) the Friedmann equation is,

$$\dot{R}^2 = \frac{8\pi}{3} G\rho R^2 - kc^2 + \frac{\Lambda c^2}{3} R^2 \tag{7}$$

This can be re-written in terms of the cosmological parameters defined in Chapter 23, §8, as,

$$\frac{\dot{R}^2}{H_{now}^2} = \Omega_{m,now} \frac{R_{now}^3}{R} + \Omega_{k,now} R_{now}^2 + \Omega_{\Lambda,now} R^2 \tag{8}$$

where $\Omega_{m,now}$ is the density of matter (including dark matter) normalised by the critical density, $\Omega_{\Lambda,now}$ is the density of dark energy normalised by the critical density, $\Omega_{k,now}$ is a curvature parameter, and H_{now} is the Hubble parameter, all being evaluated at the present time. As written, (8) refers to the matter dominated era only, because we have assumed that $\rho R^3 = \rho_m R^3$ is a constant. The Friedmann equation, (7), is valid in any era, so a generalisation of (8) can be obtained by noting that, for radiation alone, it is $\rho_r R^4$ which is constant, where ρ_r is the density of the radiation. If we make the approximation that little energy is exchanged between the matter and radiation components then they are each separately conserved and hence both $\rho_m R^3$ and $\rho_r R^4$ are constants. Equ.(8) then generalises to,

$$\frac{\dot{R}^2}{H_{now}^2} = \Omega_{r,now} \frac{R_{now}^4}{R^2} + \Omega_{m,now} \frac{R_{now}^3}{R} + \Omega_{k,now} R_{now}^2 + \Omega_{\Lambda,now} R^2 \tag{9}$$

and we have introduced in obvious notation the density parameter of radiation, Ω_r. The numerical values of the parameters, from Chapter 23, are,

$$\Omega_{m,now} = 0.267 \pm 0.026 \tag{10a}$$

$$\Omega_{\Lambda,now} = 0.733 \pm 0.029 \tag{10b}$$

$$\Omega_{k,now} = 0 \pm 0.01 \tag{10c}$$

$$H_{now} = (2.30 \pm 0.08) \times 10^{-18} \text{ s}^{-1} \tag{10d}$$

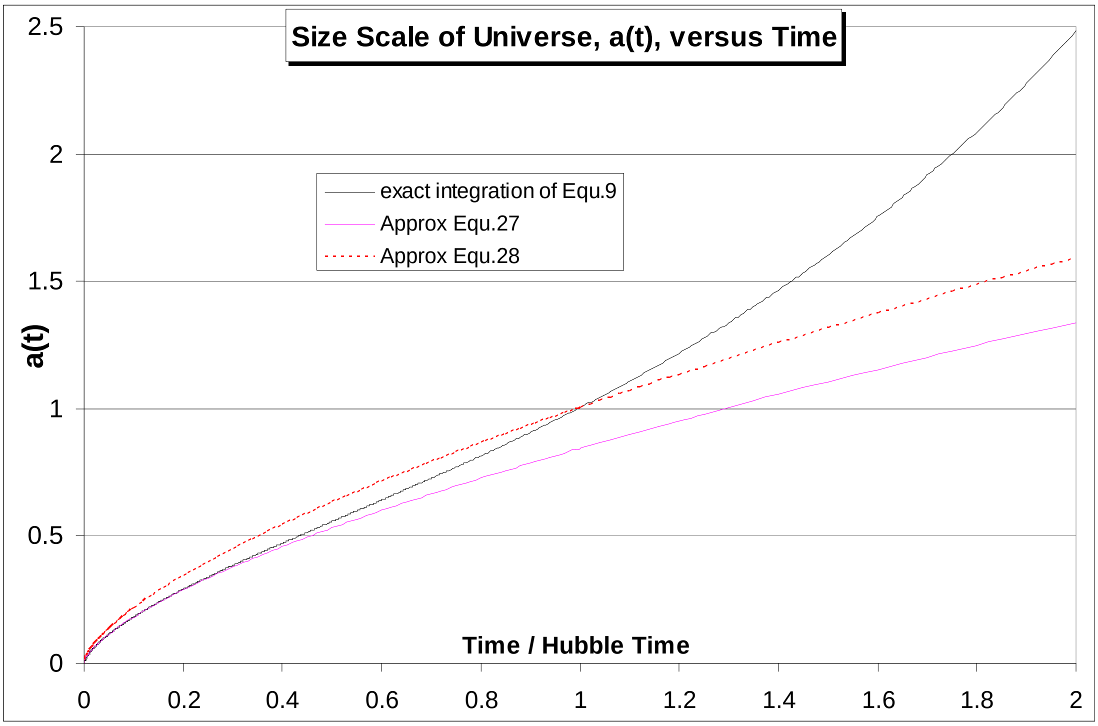
The radiation density parameter can be evaluated from the absolute radiation density given by $\rho_r = \frac{6.72\sigma T^4}{c^3}$ where $T = 2.725$ K is the CMB temperature (see Chapter 11), and the critical density of $\rho_{critical} = 9.6 \times 10^{-27}$ kg/m^3 (Chapter 11), giving,

$$\Omega_{r,now} = 8.1 \times 10^{-5} \tag{10e}$$

Equ.(9) can be integrated numerically to find the expansion function of the universe, $R(t)$, which can alternatively be expressed as the fraction $a(t) = R(t)/R_{now}$. The result is shown in Figure 1 where the time axis has been normalised by $\tau_{now} = 1/H_{now}$. We will see shortly that the present time is very close to unity on this axis. The graph also shows the simple closed form approximations Equs.27 and 28 (see §7). The changing sign of the curvature of the graph of $a(t)$ versus t illustrates how the gravity of radiation and matter is dominant over the first ~20% of the lifetime of the universe, but becomes increasingly influenced by dark energy (the cosmological constant, Λ) at later epochs. If correct, this implies that the future expansion of the universe will depart increasingly dramatically from the approximation of Equ.(28).

Figure 1: Size Scale of the Universe, a(t) = R(t)/R$_{now}$, versus Time

The black line is the result of numerically integrating Equ.9 using the currently favoured cosmological parameters given in §2. The pink and red lines are the closed form approximations of Equs.27 and 28 respectively based on ignoring dark energy (Λ). They form lower and upper bounds to the actual a(t) up to the present, but both become serious under-estimates in the future if Λ is as large as is currently believed.



3. Time of Matter-Radiation Equality

At the present time the density of radiation is negligible compared with that of matter. However, because $\rho_m \propto R^{-3}$ whereas $\rho_r \propto R^{-4}$, radiation must have been dominant at sufficiently early times, specifically because $\frac{\Omega_m}{\Omega_r} \propto R$. Hence $\frac{\Omega_{r,now}}{\Omega_{m,now}} \cdot \frac{\Omega_m}{\Omega_r} = \frac{R}{R_{now}}$.

The radiation era ends, and the era of matter dominance begins, when the two densities are equal, $\Omega_r = \Omega_m$. This occurs when,

$$\frac{\Omega_{m,now}}{\Omega_{r,now}} = \frac{R_{now}}{R} = 1 + z \qquad (11)$$

From (10a,e) the ratio in (11) is ~3294, so this our estimate of the redshift at which matter-radiation equality occurs. This is consistent with the 7-year WMAP result of 3196 ± 134, Jarosik *et al* (2011), not surprisingly because this is the source of the value of $\Omega_{m,now}$ which has been assumed. For a flat universe with $\Lambda = 0$, and ignoring radiation in the matter dominated era, we saw in Chapter 23 that $R \propto t^{2/3}$. This implies a time of equality equal to the present age of the universe divided by $3294^{3/2}$, giving 72,500 years. However it is clear from Figure 1 that Λ has a significant influence on $R(t)$. Based on Figure 1 the time of matter-radiation equality is estimated to be,

$$t_{eq} \approx 55{,}000 \text{ years} \qquad (z_{eq} \approx 3294) \tag{12}$$

4. The Age of the Universe

A quantity which is readily measured is redshift. Consequently it is appropriate to re-express (9) in terms of the redshift using the definition (6). This is done simply by dividing (9) by R^2 and noting that (6) implies that $a = R/R_{now} = (1+z)^{-1}$, where we understand R, and the time t, to relate to the emission of the radiation which we now see as redshifted by z. Differentiating gives $\dot{R}/R_{now} = -(1+z)^{-2}\dot{z}$ and hence $\dot{R}/R = -(1+z)^{-1}\dot{z}$, and so (9) becomes,

$$\tau_{now}\frac{\dot{z}}{1+z} = -\sqrt{\Omega_{r,now}(1+z)^4 + \Omega_{m,now}(1+z)^3 + \Omega_{k,now}(1+z)^2 + \Omega_{\Lambda,now}} \tag{13}$$

The redshift is zero at the current epoch, so integration of (13) over the period from t to t_{now} gives, (14)

$$\int_t^{t_{now}} dt = t_{now} - t = \tau_{now}\int_0^z \frac{dz'}{\sqrt{\Omega_{r,now}(1+z')^6 + \Omega_{m,now}(1+z')^5 + \Omega_{k,now}(1+z')^4 + \Omega_{\Lambda,now}(1+z')^2}}$$

The age of the universe is estimated by putting $t \to 0$ and $z \to \infty$, giving,

$$\frac{t_{now}}{\tau_{now}} = \int_0^\infty \frac{dz}{\sqrt{\Omega_{r,now}(1+z)^6 + \Omega_{m,now}(1+z)^5 + \Omega_{k,now}(1+z)^4 + \Omega_{\Lambda,now}(1+z)^2}} \tag{15}$$

Table 1 gives the result of carrying out the integral in (15) to find the age of the universe as a multiple of the Hubble time. In addition to the best estimates of the cosmological parameters, as given in §2, the result is also given for a few alternative assumptions. Using the best estimate parameters gives t_{now}/τ_0 very close to unity, 0.995.

5. Redshift-Time Relationship

The conversion between redshift and the time of emission follows immediately from the difference of (14) and (15), which is equivalent to integrating (13) from $t = 0$ to t,

$$\frac{t}{\tau_{now}} = \int_z^\infty \frac{dz'}{\sqrt{\Omega_{r,now}(1+z')^6 + \Omega_{m,now}(1+z')^5 + \Omega_{k,now}(1+z')^4 + \Omega_{\Lambda,now}(1+z')^2}} \tag{16}$$

This is plotted in Figure 2 for extreme redshifts back to the first hour, and in Figure 3 for redshifts in the range of observable bodies, i.e., up to ~10 or ~500Myrs.

Table 1: Current Age of the Universe Normalised by Hubble Time

The results of various combinations of cosmological parameters are given. The preferred parameter set is in bold. In all cases $\Omega_{r,now} = 8.1 \times 10^{-5}$ and $\Omega_m + \Omega_\Lambda + \Omega_k = 1$.

$\Omega_{m,now}$	$\Omega_{\Lambda,now}$	$\Omega_{k,now}$	t_{now} / τ_{now}
0.233	0.767	0	1.033
0.267	**0.733**	**0**	**0.995**
0.3	0.7	0	0.964
0.267	0.783	-0.05	1.013
0.267	0.683	0.05	0.979
0.267	0	0.733	0.820
0.045	0.733	0.222	1.267
0.045	0	0.955	0.938

(1)No dark energy; (2)No dark matter; (3)No dark energy nor dark matter

Figure 2: Time-Redshift Relationship to Extreme Redshifts

Time as fraction of the Hubble time for a given redshift, back as far as the first hour. Exact integration is compared with the approximate expression Equ.34 which is roughly indicative to redshifts of several thousand (e.g., to time of matter-radiation equality).

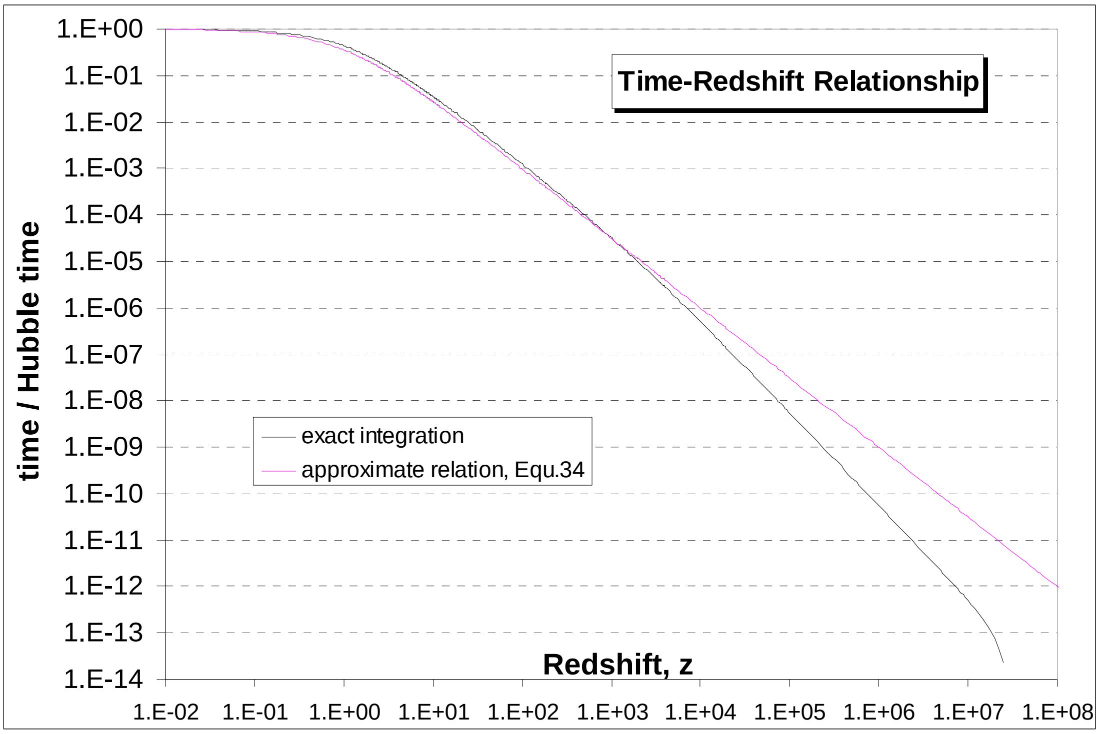
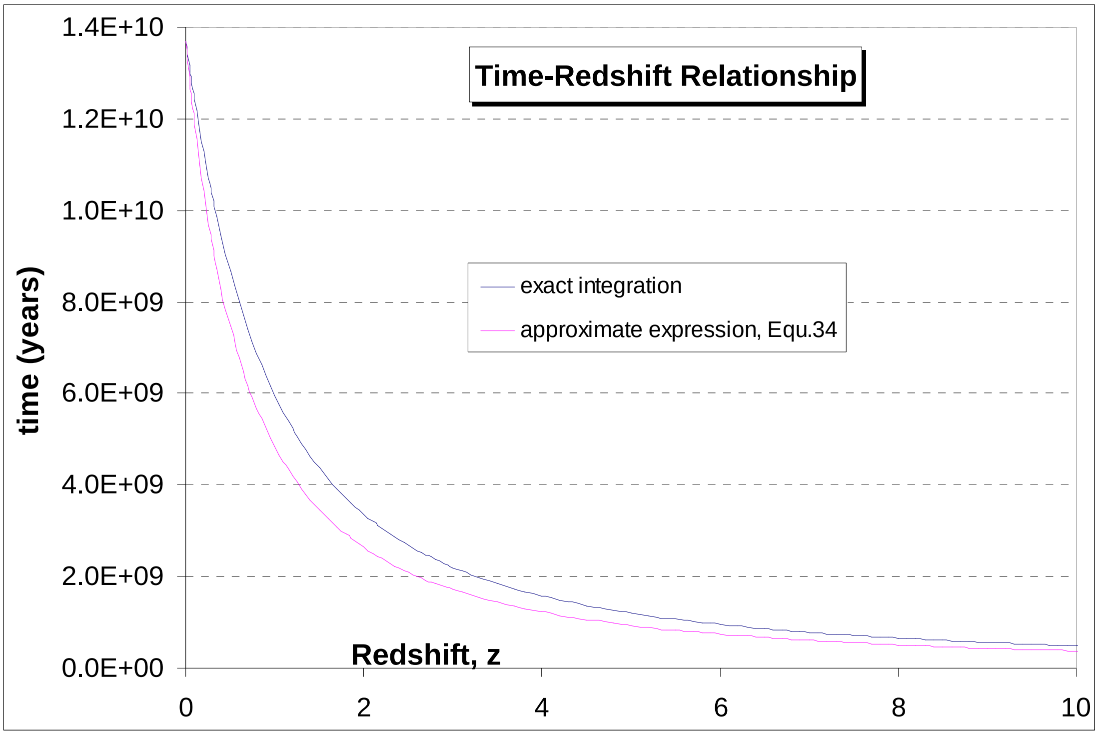


Figure 3: Time-Redshift Relationship at Observable Redshifts

Time in years from ~0.5Gyrs to present and up to redshifts comparable with the most distant astronomical bodies observed to date, ~10 (but note that the CMB dates from a redshift of ~1088).



6. Distance Measures D_{now} and D_{ltt}

Many distance measures are used in astronomy. Here we discuss only two: the distance now, D_{now}, and the light-travel time distance, $Dltt$. The latter is clearly,

$$D_{ltt} = \int_{t}^{t_{now}} c dt' = c(t_{now} - t) \qquad (17)$$

where t is the time at which a pulse of light must have been emitted from the body in question in order to reach us at time t_{now}. It is clear that the distance of the body now will be greater because it has been moving away from us (or, if you prefer, the universe has been expanding) during the time period from t to t_{now}.

It is important to realise that the distance now, D_{now}, can only be calculated based on the assumption that the body in question is co-moving with the Hubble flow. Otherwise, of course, its peculiar motion can result in it being displaced ‘randomly’ with respect to the Hubble flow. Consequently it is very distant objects (galaxies) which we have in mind.

Placing ourselves at the origin of the spatial coordinate system of the metric (1), the body in question can therefore be assumed to be at a fixed coordinate ψ. The physical distance, though, scales according to the universe’s expansion and the metric (1) shows that it is given at the present time by $s = D_{now} = R(t_{now})\psi$. A light beam is null and hence each element of its path has $R(t')d\psi' = cdt'$. Dividing these two equations gives,

$$\frac{R(t')d\psi'}{R(t_{now})\psi} = \frac{cdt'}{D_{now}} \tag{18}$$

Recall that ψ is just the constant coordinate of the galaxy being observed, so integrating the light path from $\psi' = 0$ to $\psi' = \psi$ gives the trivial $\int_0^{\psi} \frac{d\psi'}{\psi} = 1$, so that (18) can be re-arranged and integrated to give,

$$D_{now} = \int_t^{t_{now}} \frac{cdt'}{a(t')} \tag{19}$$

where,

$$a(t') = \frac{R(t')}{R(t_{now})} = \frac{1}{1+z'} \tag{20}$$

is just the dimensionless expansion factor, and z' is the redshift for time t'. Comparing (17) and (19) shows how the two distance measures differ. Given that $t' < t_{now}$, and assuming a universe which is monotonically expanding, we have $a < 1$ so that $D_{now} > D_{ltt}$. We can also define the reciprocal expansion factor,

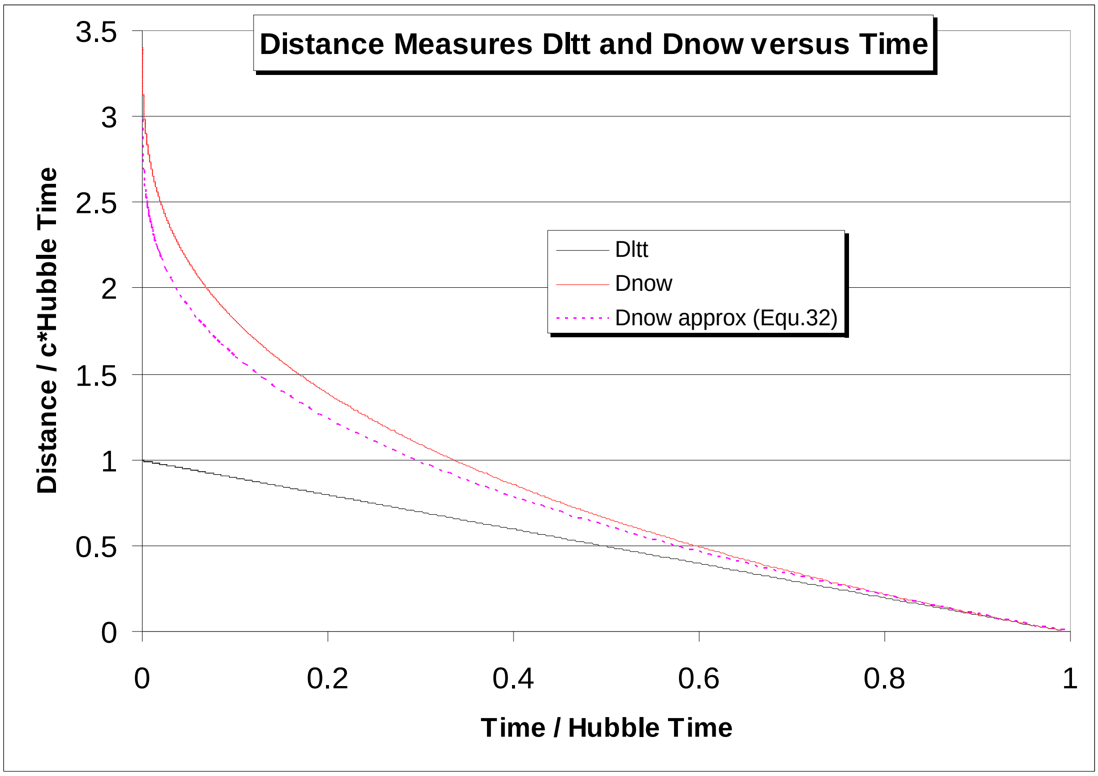
$$\tilde{a} = \frac{1}{a} = \frac{R(t_{now})}{R(t')} > 1 \tag{21}$$

The two distance measures differ by a factor which equals the average of $\tilde{a}$,

$$D_{now} = \langle \tilde{a} \rangle D_{ltt} \qquad \text{where,} \quad \langle \tilde{a} \rangle = \frac{1}{t_{now} - t} \int_t^{t_{now}} \tilde{a}(t')dt' \tag{22}$$

If we know the time of emission of the signal from the remote celestial body then (17) gives us the distance measure D_{ltt} immediately. The distance measure D_{now} could be found in terms of t by solving (9) for $R(t')$ and then performing the integral in (19). The two distance measures are compared as functions of time in Figure 4 assuming the best estimate cosmological parameters given in §2.

Figure 4: The Distance Measures D_{ltt} and D_{now} versus Time Compared.
Black line D_{ltt} from Equ.17 (exact). Red curve D_{now} from numerical evaluation of the integral in Equ.19 after integrating Equ.9 to find a(t), as in Fig.1. The pink dashed curve is the approximation of Equ.32 based on ignoring dark energy (Λ). As $t \rightarrow 0$ the latter approximation gives $D_{now} = 3ct_{now}$, whereas the exact integration with the cosmological parameters of §2 gives $D_{now} = 3.41ct_{now}$. The latter is the current size of the observable universe.



However, in general the time of emission is not known. What is observed directly is the redshift, z. Consequently it is more useful to re-express (17) and (19) in terms of z. To do so note that in both these expressions we can replace dt' with $\frac{dz'}{\dot{z}}$, giving,

$$D_{ltt} = c\int_0^z \frac{dz'}{|\dot{z}|} \quad (23)$$

$$D_{now} = c\int_0^z (1+z')\frac{dz'}{|\dot{z}|} \quad (24)$$

where we have used (20). Finally (13) is used to substitute for $\dot{z}$ giving,

$$D_{ltt} = c\tau_{now}\int_0^z \frac{dz'}{\sqrt{\Omega_{r,now}(1+z')^6 + \Omega_{m,now}(1+z')^5 + \Omega_{k,now}(1+z')^4 + \Omega_{\Lambda,now}(1+z')^2}} \quad (25)$$

$$D_{now} = c\tau_{now}\int_0^z \frac{dz'}{\sqrt{\Omega_{r,now}(1+z')^4 + \Omega_{m,now}(1+z')^3 + \Omega_{k,now}(1+z')^2 + \Omega_{\Lambda,now}}} \quad (26)$$

The two distance measures are compared as functions of redshift in Figure 5 assuming the best estimate cosmological parameters given in §2. As the horizon is approached

$D_{now} \to 3.41 D_{ltt} = 3.41 ct_{now}$. Consequently using $t_{now} = 13.7$ Gyrs the current size of the observable universe is estimated to be a radius of 46.7 Glyrs or 14.3 Gpc. Of course "observable" here means "observable in principle".

7. Matter-Dominated Closed Form Approximations

For comparison purposes, and also because it results in simple closed-form expressions which are often not too far wrong, it is of interest to derive the predictions for a, D_{ltt}, D_{now} as functions of either t or z if the only significant parameter were the matter density, $\Omega_{m,now}$, ignoring $\Omega_{r,now}, \Omega_{k,now}, \Omega_{\Lambda,now}$. The following are readily derived from the preceding equations and definitions,

$$a(t) \approx \left(\frac{3\sqrt{\Omega_{m,now}}}{2\tau_{now}} t \right)^{2/3} \tag{27}$$

$$a(t) \approx \left(\frac{t}{t_{now}} \right)^{2/3} \tag{28}$$

$$a(z) = \frac{1}{1+z} \qquad \text{(exact)} \tag{29}$$

$$D_{ltt}(t) = c(t_{now} - t) \qquad \text{(exact)} \tag{30}$$

$$D_{ltt}(z) = ct_{now} \left(1 - \frac{1}{(1+z)^{3/2}} \right) \tag{31}$$

$$D_{now}(t) = 3ct_{now} \left(1 - \left(\frac{t}{t_{now}} \right)^{1/3} \right) \tag{32}$$

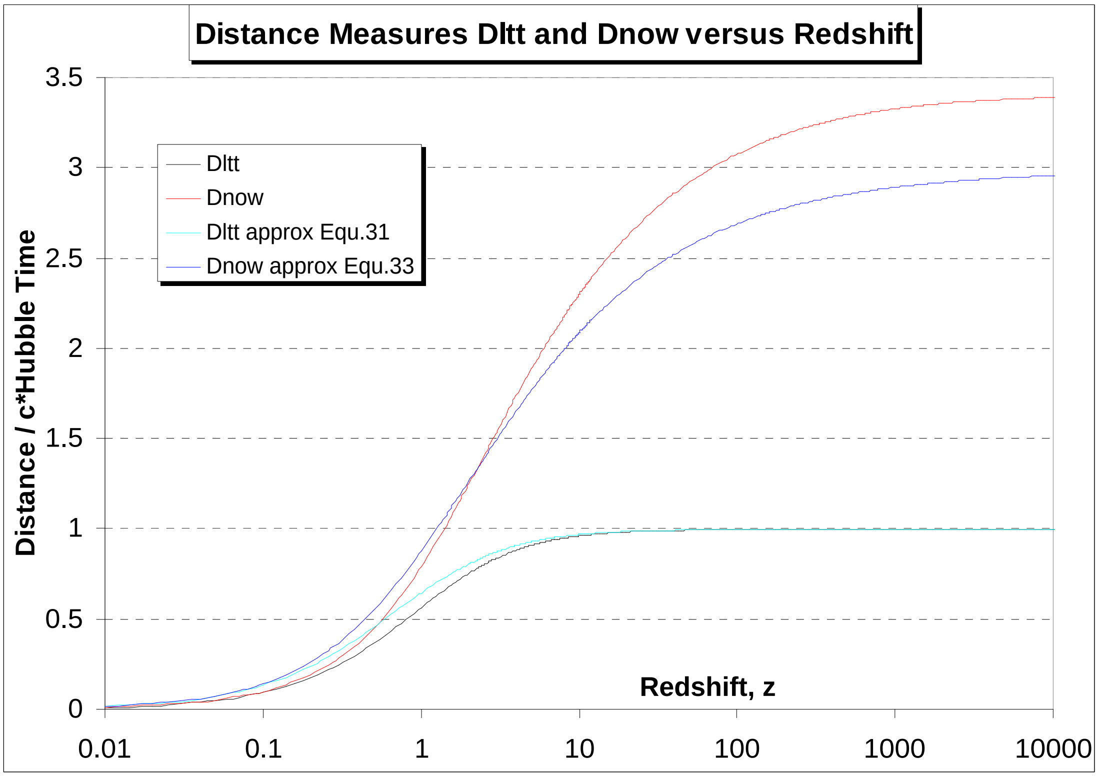
$$D_{now}(t) = 3ct_{now} \left(1 - \frac{1}{\sqrt{1+z}} \right) \tag{33}$$

$$t = \frac{t_{now}}{(1+z)^{3/2}} \tag{34}$$

Equ.(27) results directly from integration of (9) with $\Omega_{r,now} = \Omega_{k,now} = \Omega_{\Lambda,now} = 0$. It is the relevant expression if we regard $\Omega_{m,now}$ as a known quantity. However, (27) implies that t_{now} is given by $\frac{2\tau_{now}}{3\sqrt{\Omega_{m,now}}}$ which is not accurate. Equ.(28) is obtained by effectively tuning $\Omega_{m,now}$ to agree with the actual t_{now}. The two curves (27) and (28) are plotted in comparison with the exact integration of the Friedmann equation in Figure 1 and are seen to be lower and upper bounds respectively, but only up to the present epoch. Thereafter the cosmological constant drives an accelerating expansion exceeding both (27) and (28).

Equ.(32) is plotted in comparison with the exact integration of D_{now} as a function of time in Figure 4. Equs.(31) and (33) are plotted in comparison with the exact integration of D_{now} and D_{ltt} as functions of z in Figures 5 and 6. Equ.(34) is plotted in comparison with the exact time-redshift relation in Figures 2 and 3. Figure 2 shows that the useful approximation (34) is roughly indicative as far back as matter-radiation equality (z ~ 3200).

Figure 5: The Distance Measures D_{ltt} and D_{now} versus Redshift Compared.
Both distance measures are compared with approximate expressions based on ignoring the cosmological constant, Equs.31 and 33. As $z \to \infty$ and hence $t \to 0$, $D_{now} \to$ the current size of the observable universe. Integration using the consensus cosmological parameters of §2 gives this size as $3.41ct_{now}$.



8. The Hubble Parameter

The Hubble parameter has been defined as $H = \dot{R}/R$. To what distance measure does this relate? The answer is D_{now}. Specifically we have,

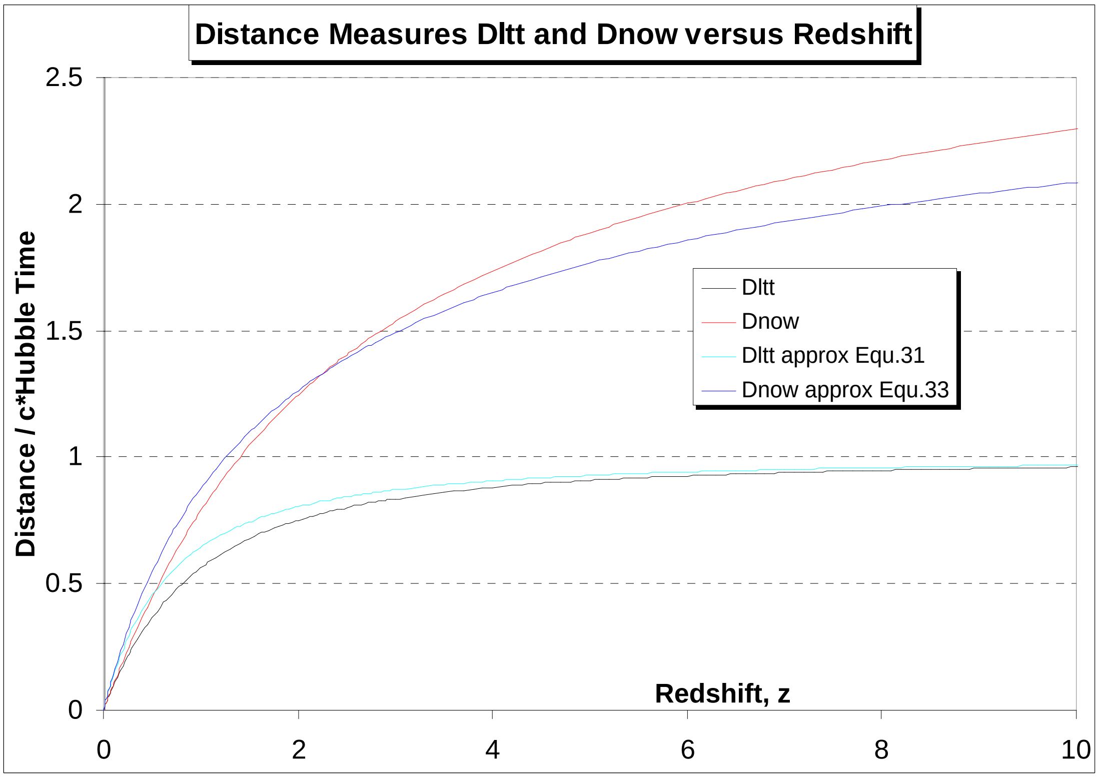
$$H = \frac{1}{D_{now}} \frac{dD_{now}}{dt_{now}} \tag{35}$$

This follows from $D_{now} = R(t_{now})\psi$ and the fact that a co-moving body is at a fixed ψ coordinate. So the 'velocity' appearing in the Hubble relation, $v = HD_{now}$, is just the natural definition,

$$v = \frac{dD_{now}}{dt_{now}} \tag{36}$$

This 'velocity' can exceed the speed of light because of the contribution of spatial expansion to D_{now}. In particular, because the horizon is at a distance of $D_{now} = 3.41ct_{now}$ it is receding with 'velocity' 3.41 times the speed of light. In fact any object observed at a redshift exceeding 1.4 is receding faster than the speed of light, as can be seen from Figure 6. This defines the Hubble radius, c/H, but it is of little physical significance. Objects which are further away, and receding from us faster, can be observed. The most distant objects observed to-date are at redshifts of around 10, and hence receding at ~2.3c (Figure 6). Incidentally this shows that the special relativistic formula for redshift $1+z=\sqrt{\frac{c+v}{c-v}}$ cannot be used with v interpreted as the recession velocity as given by (36), because this would imply a divergent redshift at the Hubble radius – which is not correct. The redshift is actually divergent at the cosmological horizon (also called the particle horizon), and is the greatest distance from which we can receive a signal here and now, given by $D_{now}=3.41ct_{now}$. Even the most celebrated of authors have been guilty in the past of confusion over these issues. Davis and Lineweaver (2004), Appendix B, lists 25 gaffes which have appeared in the literature.

Figure 6: The Distance Measures D_{ltt} and D_{now} versus Redshift Compared.
As Figure 5 but restricted to the redshifts of observable bodies



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