Chapter 47

The Physical Limits of Information and Computation

Information must have a physical manifestation, and computation is always a physical process, consequently physics imposes limits on the quantity of information and the speed of computation possible with finite resources

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Given a finite amount of mass-energy, what is the maximum amount of information that can be represented by it? What is the greatest number of elementary computational steps that can be carried out? These questions are answered below. It turns out that the answers depend upon either how much time you have available or upon how much space you have available.

1. Maximum Computation Rate and Power Requirements

An elementary computation is a single logic gate such as AND, NOT, OR or XOR, etc. One means of ensuring that the number of elementary computations is uniquely defined is to reduce a computation to a sequence of Toffoli (or Fredkin) gates, see for example Brown (2000) or Feynman (1996). Any computation can be carried out using only these gates, and hence must have a unique minimum. Consider the following statements,

- In principle, a computation can be carried out with no expenditure of energy*.
- The rate at which computations can be carried out is limited by the available energy*.

**Throughout this Chapter, both 'mass' and 'energy' can be read interchangeably as 'mass-energy'.*

Both the above statements are true. That they are not contradictory is due to the second statement referring to the computational *rate*, and the first referring to the *expenditure* of energy. The latter is distinct from energy being involved but in a conservative manner. It is trivially obvious that a computation cannot be performed unless some energy is available. This is because a computation is a physical process. Any physical process must involve some 'stuff' upon which the process is being carried out. According to the physical view of the world, this 'stuff' equates to mass-energy.

The first statement means that computation can be carried out, in principle, in a manner which does not ***dissipate*** any of the energy involved. That this is true derives from the observation that computation can be carried out in a reversible manner (Bennett, 1979 & 1982). Indeed, the meaning of 'reversible' in thermodynamics is 'without energy dissipation'. The content of Bennett's argument is essentially that logical reversibility implies the existence of a process which is also thermodynamically reversible.

This suggests that it should be possible, in principle, to solve the quantum-chromodynamic equations for the mass spectrum of the hadrons using just a handful of electrons and a few eV of energy. (Construction of a working device is left as an exercise for the reader). The snag is that, as well as being stupendously ingenious, you would have to wait a very long time for the answer. This is where the second statement comes in.

It is rather obvious that we will be able to compute faster with more mass-energy available – simply because we will now have enough material at our disposal to make a proper computer! However, for a given mass of material, how fast can we compute? The answer lies in recognising that the fundamental nature of a computation is to change 0 into 1 or vice-versa. Our computer must manifest these binary states physically. Thus, to compute quickly, we need to be able to change the binary physical states of the system as quickly as possible. Clearly, this will be facilitated by making the physical states which represent 0 and 1 very similar – so there is less changing to be done. But there is a limit to how similar two physical states can be whilst remaining distinguishable. This is the quantum limit. To be reliably distinguishable, the states must be orthogonal quantum states. Our problem therefore becomes, "how quickly can a quantum state be changed into an orthogonal quantum state?" The answer is provided by the uncertainty principle,

$$\Delta t \geq \frac{\pi\hbar}{2\Delta E} \tag{1a}$$

This defines the minimum time required to change from a state whose uncertainty in energy is ΔE to an orthogonal state. Note that a state with precisely defined energy, E, can never undergo change, since its time dependence is just a phase factor $e^{iEt/\hbar}$ with no overlap to any other state. So the possibility of change, and hence computation, requires a non-zero uncertainty in energy, $\Delta E \neq 0$.

The uncertainty in the energy of the state cannot exceed the expectation value of its energy, E, since energy cannot be negative[1]. Consequently, we can reduce the computation time given by (1a) to a minimum by taking $\Delta E = E$, and thus,

$$\Delta t_{Min} \geq \frac{\pi\hbar}{2E} \tag{1b}$$

These arguments follow Lloyd (2000, 2002). Since (1b) is the minimum time for an elementary computation, the maximum computation rate is,

$$\text{Maximum Computation Rate} = \frac{2E}{\pi\hbar} \tag{1c}$$

By summation over its parts, the same relation must also hold for the maximum computation rate of a macroscopic system in terms of its total energy.

1.1 Minimum Power Requirement for an Irreversible (Dissipative) Computer

1.1.1 At Absolute Zero

The minimum expended power requirement for a *reversible* computer of any given speed is zero. The associated maximum speed will depend upon how much energy is available for use (albeit not expended). However, we shall now assume instead that our computer is *irreversible* but dissipates energy at the minimum rate consistent with its speed. Equ.(1c) says that the minimum energy that must be available if the computer is to calculate at a speed C bits per second is $E = \pi\hbar C/2$. We are assuming that this amount of energy, i.e. the minimum required to support the stated computing speed, is dissipated at each

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[1] We are referring here to the total relativistic mass-energy. Of course the energy of , say, a bound atomic electron is generally regarded as negative, but this energy does not include the electron's rest mass energy.

computational step. But each such step takes a period of time $\tau = 1/C$. Hence, the power requirement is $P = E/\tau = \pi\hbar C/2\tau = \pi\hbar C^2/2$. Hence, the power requirement of such a computer is proportional to the computing speed squared. Unless we have taken special precautions to build a reversible computer, in practice this is likely to be the minimum power requirement of a dissipative computer.

1.1.2 At Finite Temperature

The preceding argument was based on the ultimate limit of distinguishability provided by the quantum uncertainty principle. But at finite temperature there is another restriction, known as the Landauer limit, Landauer (1961). This says that each irreversibly computed bit requires energy expenditure of $k_B T \log 2$, where k_B is Boltzmann's constant and T the absolute temperature. Thus, a computational rate C requires a dissipated power of $CkT \log 2$.

The above demands are modest. A modern home computer runs at a few GHz, which may be crudely interpreted as the computing speed. Hence, C^2 is ~ 10^{19} s^{-2}. Since Planck's constant is of order 10^{-34} Js, the minimum power requirement at absolute zero is ~10^{-15} W, whilst at room temperature it is ~10^{-11} W. The electrical power of a standard PC CPU chip is in the order of tens of W, so there is room for improvement by some 12 orders of magnitude before the room temperature Landauer limit is reached, and some 16 orders of magnitude before the ultimate quantum limit is achieved. And even these power requirements could be improved further, in principle, by going reversible.

1.2 Mass-Energy Limit on the Computational Capacity of the Universe

Using (1c) it is simple to calculate the maximum number of computations that can have been carried out within the observable universe since the Big Bang. The best estimate of the age of the universe is currently 13.7 Byrs = 4.3 x 10^{17} s. Including dark matter and dark energy as well as ordinary (baryonic) matter, the mean density of the universe is probably within ~1% of flatness (Komatsu 2009). So the mean density is very close to the critical density, $\frac{3}{8\pi G t^2}$ = 9.6 x 10^{-27} kg/m^3 = 5.7 H atoms/m^3. Multiplying by c^2 gives an energy density of 8.6 x 10^{-10} J/m^3. Hence, the upper limit to the physically possible computational rate per unit volume, using (1c), is 5.2 x $10^{24} s^{-1} m^{-3}$. A pretty impressive figure for just 5.7 hydrogen atoms!

To find the total number of computations over the whole observable universe requires the volume of the (observable) universe. For a flat geometry this is $V = \frac{4}{3}\pi R^3$ where R ~ 3.5ct ~ 4.5 x 10^{26} m, hence V = 4 x 10^{80} m^3. (The reason why the radius is 3.5ct is explained in Chapter 23). So, the maximum possible rate of computation by the whole observable universe is 2 x 10^{105} s^{-1}. Actually I think I use 3.41 later (Ch."Spare"), giving a volume of $\frac{4\pi}{3}(3.41ct)^3 = 3.6 \times 10^{80}$ (as used in Ch.25), so this needs changing. What happens if we use the radius of the event horizon instead, which Egan & Lineweaver give as 15.7 Glyr, hence a volume of 1.37 x 10^{79}. The maximum possible number of elementary computations within the event horizon is then 3.0 x 10^{121}, which is an order of magnitude less than the Hawking-Beckenstein entropy of the event horizon.

Multiplying the current computational rate of the universe by the age of the universe yields an upper bound for the total number of computations that can have been carried out over the life of the universe, i.e. ~0.8 x 10^{123}. These arguments follow Lloyd (2000, 2002).

This is an upper bound for the number of computations that can have been carried out by the mass-energy which is observable in the present epoch. This is not the same as the number of computations that could have observable consequences here and now. The latter is a smaller number. The reason is that only a fraction of the observable universe was previously in causal contact with us. At earlier times, the causally connected universe comprised smaller amounts of mass-energy, and hence would have had a smaller computational capacity. For material now appearing over the horizon, we can see the results of its computations carried out shortly after the Big Bang, but not more recent computations. A more restrictive bound on the causally relevant computational capacity of the universe can be found by integration over the varying size of the universe. However this reduces the number only by a factor of 2.

Ordinary (baryonic) matter comprises just 4.5% of the universe's density (Komatsu 2009). The rest is dark matter (~22.5%) and dark energy (~73%). Hence, the causally relevant computational capacity of the ordinary, baryonic matter alone is ~1.8 x 10^{121}.

1.3 Spacetime Limit on the Computational Capacity of the Universe

In addition to the computational limit due to the universe's mass-energy inventory, there is also a bound based purely on space-time limitations, without reference to the actual mass-energy content of the universe. This is derived as follows. One basis for quantum states is to use spatial localisation. States are distinguished because they are at different places. The limit to spatial resolution is provided by the Planck length,

$$L_P = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-35} m \qquad (2a)$$

Similarly, the limit to temporal resolution is provided by the Planck time,

$$t_P = \sqrt{\frac{\hbar G}{c^5}} = 5.4 \times 10^{-44} s \qquad (2b)$$

These imply that the maximum computational rate density would be achieved if one bit of information in every volume L_P^3 were changing every t_P seconds. The computational capacity of the universe would then be,

$$\frac{4\pi}{3}\left(\frac{R}{L_P}\right)^3\left(\frac{t}{t_P}\right) = \frac{4\pi}{3}\left(\frac{4.5\times10^{26}}{L_P}\right)^3\left(\frac{4.3\times10^{17}}{t_P}\right) = 0.74x10^{246} \qquad (3)$$

This is vastly greater than the true bound, 0.8 x 10^{123}, based on the mass-energy content of the universe (in fact it equals its square, to a good approximation). Thus, it is the mass-energy content of the universe which limits its computational capacity, not its size. This is a reasonable conclusion given the very low mean density of the universe. I think this Section needs to go. The geometrical (gravitational) degrees of freedom are not counted in this way but via the area of an event horizon. Delete?

1.4 Speculation Regarding Computational Restrictions on the Early Universe

Note that the space-time limit for the universe's computational capacity varies as t^4 whereas the mass-energy limit varies as t^2. This suggests that we can find a time at which the two limits become the same. Equating the algebraic expressions for the two computational capacities,

$$\frac{f^3 c^5 t^2}{\pi \hbar G} = \frac{4\pi}{3} \frac{f^3 c^{10} t^4}{\hbar^2 G^2} \tag{4}$$

gives the time of equality to be about,

$$t \sim t_P \tag{5}$$

Thus, at all times after about a Planck time the universe's spacetime is able to contain more information than it is required to contain to support its mass-energy content. However, at times earlier than about a Planck time there would not have been enough room in the universe to contain the information content of its mass-energy, an impossible situation presumably. Consequently times earlier than a Planck time would appear to be meaningless or impossible.

2. Maximum Amount of Information

2.1 Black Bodies

We now return to the question of how much information can be 'written' using a given finite amount of mass-energy. This is equivalent to asking what the maximum entropy of a given amount of mass-energy can be. Now the entropy of a gas of particles in thermodynamic equilibrium is essentially just the number of particles times a number of order between unity and 100 (providing that it is not so dense as to be nearing the liquid or solid state). For example, the entropy of a volume V of black body radiation (photons) is simply,

$$S / k = 3.6nV = 3.6N \tag{6}$$

where N is the number of photons in the volume V and $n = N/V$ is the number of photons per unit volume, which is,

$$n = 0.2436\left(\frac{k_B T}{\hbar c}\right)^3 \tag{7}$$

For a monotonic ideal gas, the Sackur-Tetrode equation gives the entropy as,

$$\frac{S}{k_B} = N\left\{\frac{5}{2} + \log\left[\left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2} \frac{V}{N}\right]\right\} \tag{8}$$

where N is the number of particles of mass m in volume V. Note that the *lower* the particle density, the *larger* is the second term in the {....}. This is because the greater amount of room available per particle leads to a greater number of accessible quantum states. For example, if we consider the mean number density of protons in the universe ($0.26/m^3$), and assume the temperature of the CMB, i.e. 2.7 K, then the second term in {…} is 63.4, and so the dimensionless entropy is ~66N. On the other hand if we consider hydrogen gas at STP, for which the molecular number density is 2.7 x 10^{25} m^{-3}, the second term in {…} is 11.4 and hence the entropy is ~14N. Consequently, even this extreme difference in conditions only makes a difference in the entropy per particle of around a factor of 5.

So, how can we contrive to maximize the entropy for a given amount of mass-energy? It is clear from the above discussion that we need simply to maximize the number of particles. Subject to conservation of mass-energy this obviously means producing more particles of smaller mass. Since photons have zero rest mass, we need to convert everything to photons (or possibly to gravitons, these being the only other quanta with zero rest mass). But the number of photons we get depends on their energy.

A spurious argument goes like this. We need very low energy photons to maximize their number. If conservation of mass-energy were the only limitation then we would conclude that there is no finite upper bound to the amount of information that may be represented by a finite amount of energy. We merely have to convert all our mass-energy to photons of sufficiently low energy. Thus, the amount of information obtainable from a mass M using photons of energy E_γ would be, from (6),

$$S/k = 3.6\frac{Mc^2}{E_\gamma} \qquad (9)$$

and this is unbounded as $E_\gamma \to 0$. However, this is an incorrect argument because the entropy of the photons is only given by (6) if we assume a black body spectrum of photon energies. In fact, for a given energy density, the black body spectrum of photon energies is exactly that which maximises the entropy, and hence is the answer to our question.

For a given mass (*M*) in a given volume *V*, the temperature, T_{bb}, of the equivalent black body radiation is that which equates with its energy density, i.e.,

$$\frac{Mc^2}{V} = 0.658\frac{(k_B T_{bb})^4}{(\hbar c)^3} \qquad (10)$$

Consequently, substituting (10) into (6,7) gives the maximum amount of information which can be represented by a mass M in a volume V to be,

$$\text{Maximum Information} = 1.2V^{1/4}\left(\frac{Mc}{\hbar}\right)^{3/4} \qquad (11)$$

Using the total mass and volume of the observable universe in the current epoch, as derived above (i.e., $V = 4$ x 10^{80} m^3 and $M = 3.8$ x 10^{54} kg, where this mass corresponds to the critical density and the currently observable universe) gives the maximum information content of the universe to be 10^{93}. If we restrict attention to the information possible with the normal baryonic matter alone, then this scales by $0.045^{3/4} = 0.10$ to 10^{92}.

To achieve the upper bound number of computations derived above (1.8 x 10^{121}) each ‘baryonic bit’ would have to have been involved in 1.8 x 10^{29} computations over the life of the universe, an average of ~4 x 10^{11} computations per baryonic bit per second. However, there are far more ‘baryonic bits’ than baryons (because the former relate to the maximum achievable information with the same mass). There are ~10^{80} baryons in the observable universe, so the maximum number of computations equates to 1.8 x 10^{41} computations per baryon, or an average of ~4 x 10^{23} computations per baryon per second. (Hence ~10^{12} times more baryonic bits than baryons). Hence, the mean computation time per baryon required to achieve the maximum possible number of computations is ~10^{-23} second.

This is a striking result because ~10^{-23} s is just the characteristic time of the strong nuclear interactions. Since we have used nothing in our derivation relating to the strength of the nuclear force, this appears at first sight to be a remarkable Cosmic Coincidence. But it is not. This characteristic time derives from the size of a typical nucleus (~3 fm), which, on dividing by c gives ~10^{-23} s. The size of a nucleus, or proton, also defines the order of magnitude of its mass through $M_n \sim \hbar / rc$. Hence a size of order 1 fm sets the mass scale of nucleons to be in the order of hundreds of MeV, quite correctly. Now, the computation rate per baryon can be derived algebraically using the above arguments to be,

$$\frac{M_n c^2}{\pi\hbar} \sim 5x10^{23} \tag{12}$$

i.e., the dependence on G and t cancel out. Since $M_n \sim \hbar / rc$ the characteristic computation time per baryon is inevitably ~r/c, i.e. the strong interaction time scale. There is no coincidence; it is just algebraic self-consistency. Nevertheless, it provides an interesting alternative interpretation of the maximum computation rate, namely that it is the maximum possible number of strong nuclear interactions.

2.2 Black Holes

Ch.25 “Entropy of the Universe” covers some of the same ground – review to avoid duplication.

There is, however, a loophole in the above derivation of the maximum information representable by a fixed amount of mass-energy, *M*. It is the assumption that the volume, *V*, is also specified and finite. This is explicit in Equ.(11). If the volume is allowed to diverge, then so does the available information. Physically this is because the temperature of the black body radiation reduces arbitrarily close to zero, with a divergent number of photons whose energy becomes arbitrarily small. The same is true of (8), though in that case the divergence in *V* is merely logarithmic, and arises due to the increasing number of quantum states available to each particle as *V* increases. However, this observation is not relevant to our *observable* universe *at the current epoch*, which has a finite size (due to its finite age and the finite speed of light).

A particular mechanism for generating large entropies for very large, but finite, volumes is as follows: Consider the whole mass-energy of the universe to collapse into a single enormous black hole. The entropy of a black hole is given by,

$$\frac{S_{bh}}{k_B} = \frac{4\pi GM^2}{\hbar c} = \frac{A}{(2L_{Planck})^2} \tag{13}$$

where *A* is the surface area of the black hole’s event horizon.

This is known as the Hawking-Beckenstein formula. It was first proposed by Beckenstein (1973), inspired by the theorem of Hawking (1971) in classical general relativity which states that the total surface area of a collection of black holes can never decrease, even if they merge. This property of black holes is necessary if (13) is to be a consistent formulation for entropy, since it is then required by the second law of thermodynamics, i.e., that the total entropy cannot decrease.

Equ.(13) compares with the entropy of the cloud of gas of the same mass from which the black hole was formed of order,

$$\frac{S_{gas}}{k_B} \approx 48 \frac{M}{M_n} \tag{14}$$

where M_n is the nucleon mass, derived from (8). The numerical factor depends insensitively on the density of the gas before collapse. We have used conditions typical of the giant molecular hydrogen clouds which form the precursors of stars (~10K and a molecular number density of ~3 x 10^8 m^{-3}). The black hole entropy may be expressed in a similar way via the Planck mass, defined as,

$$M_P = \sqrt{\frac{\hbar c}{G}} = 2.18 \times 10^{-8} \text{ kg} \tag{15}$$

Using this, (13) can be re-written as,

$$\frac{S_{bh}}{k_B} = 4\pi \left(\frac{M}{M_P} \right)^2 \tag{16}$$

Because the black hole entropy depends upon mass-squared, whereas the entropy of a gas is proportional to its mass, the black hole has far greater entropy. For example, the dimensionless entropy of a solar mass (2 x 10^{30} kg) black hole is, using (16), ~10^{77} compared with that of the gas cloud which formed it, using (14), <10^{59}, a difference of some 18 orders of magnitude. Considering the super-massive black holes which live at the centre of many galaxies, the difference becomes considerably great still.

If we consider a black hole with a mass equal to that of the observable universe, then the dimensionless entropy is ~3.8 x 10^{125}. This exceeds the maximum information 'writable' using the observable universe's mass-energy content calculated in §2.1 (i.e., ~10^{93}) by 32 orders of magnitude. It also exceeds the number of elementary computations that can have been carried out in the observable universe to-date, calculated in §1.2, but only by a couple of orders of magnitude. Probably need to review this in view of the entropy of the cosmic event horizon being only ~2.6 x 10122 (Egan & Lineweaver, 2010), see Chpater 25, "Entropy of the Universe". Probably delete this para.

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