Chapter 42

The M^cGlinn-O'Raifeartaigh Theorem

If an internal symmetry commutes with the homogeneous Lorentz group then it commutes with the whole Poincare group. The implication is that there is no easy, purely group-theoretical, route to deriving the hadronic mass spectrum.

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1. The Hadron Mass Spectrum and Symmetry

In the early 1960s the quark theory was emerging but non-abelian gauge field theories were yet to become predominant. It was clear that the hadrons were falling into regular taxonomic families which could be identified with the irreducible representations of SU(3). However, the masses of the hadrons within any one family (as well as between families) had widely differing masses. The great challenge was to calculate the hadronic mass spectrum (see Chapter 46 for the state of play now). Since 'internal' symmetries were topical, it was natural to ask whether they could assist with the hadron mass problem.

Any internal symmetry would presumably commute with the generators of the homogeneous Lorentz group, since rotational symmetry and invariance under Lorentz boosts are sacrosanct. But the mass (squared) of a particle state is obtained as the eigenvalue of the squared momentum operator, $P_\mu P^\mu$. For different particles within the same irreducible representation of the internal symmetry to have different masses, some of the internal symmetry generators must fail to commute with $P_\mu P^\mu$. Consequently, whilst the internal symmetry group and the Poincare group must be sub-groups, the overall symmetry group could not simply be their direct product. In other words, if the mass spectrum were to be consistent with an exact symmetry group, this group would need to have non-trivial structure connecting the internal and spacetime symmetries such that $\left[\Im, P_\mu P^\mu\right] \neq 0$ for some internal symmetry generators, $\Im$.

Lamentably it turns out that no such group exists. What we would now call flavour-SU(3) is not an exact symmetry. This route to calculating the hadronic mass spectrum does not work. This was proved by M^cGlinn (1964) and O'Raifeartaigh (1965). The precise statement of the theorem is,

Any Lie group whose algebra contains only the Poincare group and some other 'internal' algebra, and for which the Lie product of the (homogeneous) Lorentz group generators and the 'internal' generators are zero, is the direct product of the groups.

Denoting the internal generators as $\Im_i$, the generators of the homogeneous Lorentz group as $M_{\mu\nu}$, and the spacetime translation generator (momentum) as P_μ, the theorem can be written algebraically as,

$$\left\{\left[\Im_i, M_{\mu\nu}\right] = 0, \forall i, \mu, \nu\right\} \Rightarrow \left\{\left[\Im_i, P_\mu\right] = 0, \forall i, \mu\right\} \qquad (1)$$

This is proved below, following the M^cGlinn method, which is elementary though not very elegant. The proof of O'Raifeartaigh is more general but also more technically demanding.

2. Lie Algebras: A Brief Reminder

Lie groups are generated by exponentiating Lie algebras. A Lie algebra must be closed under the action of an antisymmetric binary operation, the Lie product: $a \otimes b \equiv -b \otimes a$. However the Lie product is not usually associative, $a \otimes (b \otimes c) \neq (a \otimes b) \otimes c$. In lieu of this the Lie product is required, as part of the definition of a Lie algebra, to obey,

$$a \otimes (b \otimes c) + b \otimes (c \otimes a) + c \otimes (a \otimes b) = 0 \qquad (2)$$

In most cases a Lie algebra is realised by interpreting the Lie product as a commutator,

$$a \otimes b = [a, b] \equiv ab - ba \qquad (3)$$

In this case (2) is an identity, which is why it is often referred to as the "Jacobi identity".

A (finite dimensional) Lie algebra is a vector space over a set of $\tilde{N}$ independent basis elements (or generators), $\{\Im_u, u \in [1, \tilde{N}]\}$. The Lie product between these generators suffices to determine the whole structure of the algebra. Hence,

$$\Im_u \otimes \Im_q = C_{uq}{}^w \Im_w \qquad (4)$$

where the RHS of (4) is summed over the repeated index, as usual. The $C_{uq}{}^w$ are known as the "structure coefficients". Due to the antisymmetry of the Lie product they are such that,

$$C_{uq}{}^w = -C_{qu}{}^w \qquad (5)$$

In the present application, the dimension, $\tilde{N}$, of the Lie algebra is $N + 10$ where N is the dimension of the sub-algebra of the internal symmetry, the additional 10 generators being those of the Poincare group algebra. Substituting (4) into the Jacobi identity, (2), yields the following constraints on the structure functions,

$$C_{uq}{}^w C_{xy}{}^q + C_{yq}{}^w C_{ux}{}^q + C_{xq}{}^w C_{yu}{}^q = 0 \qquad (6)$$

Again note that the convention of summing over repeated indices applies. The identity, (6), applies for all values of u, w, x, y from 1 to $\tilde{N} = N + 10$. This relation between the structure constants is central to the proof of the M^{c}Glinn-O'Raifeartaigh theorem as we shall see. It is worth noting that it is an identity which follows simply from the Lie product being the commutator.

The generators of the Lorentz algebra, the rotations and boosts, are written $M_{\mu\nu}$. These, together with the momentum operators, P_μ, comprise the Poincare algebra. The structure of the Poincare algebra is given by the commutators,

$$[M_{\theta\phi}, M_{\chi\psi}] = i(g_{\theta\psi} M_{\chi\phi} + g_{\theta\chi} M_{\phi\psi} + g_{\phi\psi} M_{\theta\chi} + g_{\chi\phi} M_{\psi\theta}) \qquad (7a)$$

$$[M_{\theta\phi}, P_\psi] = i(g_{\theta\psi} P_\phi - g_{\phi\psi} P_\theta) \qquad (7b)$$

$$[P_\theta, P_\phi] = 0 \qquad (7c)$$

where $g_{\theta\phi}$ is the Minkowski metric.

3. Index Conventions

To facilitate the proof we use the following conventions,

- The internal symmetry generators are denoted $\Im_1, \Im_2 ... \Im_N$. An arbitrary one of these is written with a small Latin index, such as $\Im_i$;
- The momentum operators are $\{P_\theta\} = \{\Im_{N+1}, \Im_{N+2}, \Im_{N+3}, \Im_{N+4}\}$. An arbitrary one of these is written with a capital Latin index, such as $\Im_I$;
- The Lorentz generators are $\{M_{\theta\phi}\} = \{\Im_{N+5}, \Im_{N+6}, ... \Im_{N+10}\}$. An arbitrary one of these is written using a Greek index, such as $\Im_\alpha$;
- An exception to the above rules are the specific letters *u*, *q*, *w*, *x*, *y*, *z* which are used to denoted any index from 1 through *N* + 10. Hence Equs.(4,5,6) are consistent with this convention;
- Another exception are the Greek letters θ, ϕ, χ, ψ which are used to denote the usual spacetime indices, $[0,3]$, as in Equs.(7a,b,c).

With these conventions the M^cGlinn-O'Raifeartaigh theorem can be written,

$$C_{i\alpha}{}^{w} = 0 \Rightarrow C_{iI}{}^{w} = 0 \qquad (8)$$

4. The Proof

Step 1

Consider firstly (6) with $u = i, x = \alpha, y = I, w = j$, i.e.,

$$C_{iq}{}^{j} C_{\alpha I}{}^{q} + C_{Iq}{}^{j} C_{i\alpha}{}^{q} + C_{\alpha q}{}^{j} C_{Ii}{}^{q} = 0 \qquad (9a)$$

But we are assuming $C_{i\alpha}{}^{w} = 0$ (i.e., that the internal symmetry commutes with the Lorentz group). Moreover $C_{\alpha q}{}^{j} = 0$ for any *q* because of $C_{\alpha i}{}^{w} = 0$ plus the fact that the Poincare algebra is a closed sub-algebra so that $C_{\alpha q}{}^{j} = 0$ when $q = \beta$ or $q = J$. Hence (9a) becomes,

$$C_{iq}{}^{j} C_{\alpha I}{}^{q} = 0 \qquad (9b)$$

Note that (7b) provides the values of $C_{\alpha I}{}^{q}$ and shows that there is at most only one q for which this is non-zero, and this is such that $q = K$. Moreover, for any K there is some α, I such that $C_{\alpha I}{}^{K} \neq 0$. Consequently (9b) gives,

$$C_{iK}{}^{j} = 0 \qquad (9c)$$

Recalling that our objective is to prove that $C_{iI}{}^{w} = 0$, (9c) establishes this for the case $w = j$. This leaves the cases $w = J$ and $w = \alpha$ to be proved.

Step 2

Now consider (6) with $w = I, u = i, x = \alpha, y = J$,

$$C_{iq}{}^{I} C_{\alpha J}{}^{q} + C_{Jq}{}^{I} C_{i\alpha}{}^{q} + C_{\alpha q}{}^{I} C_{Ji}{}^{q} = 0 \qquad (10a)$$

The second term in (10a) is zero due to the LHS of (8), $C_{i\alpha}{}^{w}=0$. In the first term $C_{\alpha J}{}^{q}$ is non-zero only for $q=K$ due to (7b). In the third term, $C_{\alpha q}{}^{I}$ is zero if $q=i$, due to (8), but it is also zero if $q=\beta$ because of (7a), i.e., there is no momentum term on the RHS of (7a). Hence the sums over *q* can be replaced by sums over *K*, giving,

$$C_{iK}{}^{I}C_{\alpha J}{}^{K}+C_{\alpha K}{}^{I}C_{Ji}{}^{K}=0 \tag{10b}$$

To deduce what (10b) implies for the constants $C_{iK}{}^{J}$ consider (7b) which specifies the constants $C_{\alpha K}{}^{u}$. It is clear that these are zero for $u=i$ or $u=\beta$ so they are non-zero only for $u=I$. To deduce an explicit algebraic expression for the $C_{\alpha K}{}^{I}$ it is convenient to adopt a notation $\alpha=(LM)$ with $L\neq M$ (because L,M are just equivalent to the usual spacetime indices). Hence, (7b) is equivalent to,

$$C_{(LM)K}{}^{I}=\delta_{IM}g_{LK}-\delta_{IL}g_{MK} \tag{11}$$

Substituting (11) in (10b) gives,

$$C_{iK}{}^{I}\left(\delta_{KM}g_{LJ}-\delta_{KL}g_{MJ}\right)+C_{Ji}{}^{K}\left(\delta_{IM}g_{LK}-\delta_{IL}g_{MK}\right)=0 \tag{10c}$$

Hence,

$$C_{iM}{}^{I}g_{LJ}-C_{iL}{}^{I}g_{MJ}-C_{iJ}{}^{L}\delta_{IM}g_{LL}+C_{iJ}{}^{M}\delta_{IL}g_{MM}=0 \tag{10d}$$

[NB: Summation does not apply in (10d)]. If we take I,J,L,M all different then the LHS of (10d) is identically zero. In fact, the LHS is zero unless one of I,J equals one of L,M.

If we take just one pair of these indices as equal, say $I=L$ but $J\neq M$ and $J\neq L$ (and recalling that we always have $L\neq M$), then we find,

$$C_{iJ}{}^{M}=0 \text{ (for } J\neq M\text{)} \tag{10e}$$

The same result follows if the equal pair is chosen to be $I=M$ or $J=L$ or $J=M$. Next consider $I=J=L$, or equivalently $I=J=M$. This is found to simply reproduce (10e). Finally consider $I=L, J=M$, or equivalently $I=M, J=L$. These give,

$$C_{iL}{}^{L}=C_{iM}{}^{M} \text{ (\textit{no summation})} \tag{10f}$$

Equs.(10e) and (10f) can be run together to read,

$$C_{iJ}{}^{K}=C_{i}\delta_{JK} \tag{10g}$$

for some constants C_i.

<u>Step 3</u>

Now consider (6) with $w=I, y=J, u=j, x=k$,

$$C_{jq}{}^{I}C_{kJ}{}^{q}+C_{Jq}{}^{I}C_{jk}{}^{q}+C_{kq}{}^{I}C_{Jj}{}^{q}=0 \tag{12a}$$

Consider the first and last terms in (12a). For $q=i$ they are zero because $C_{ji}{}^{I}=0$. For $q=\alpha$ they are zero because $C_{j\alpha}{}^{u}=0$. For $q=K$ the two terms cancel since, using (10g),

$$C_{jK}{}^{I}C_{kJ}{}^{K}+C_{kK}{}^{I}C_{Jj}{}^{K}=C_{j}\delta_{K}{}^{I}C_{k}\delta_{J}{}^{K}-C_{k}\delta_{K}{}^{I}C_{j}\delta_{J}{}^{K}\equiv 0 \tag{13}$$

Now considering the second term in (12a), $C_{jk}{}^{q}$ is non-zero only for $q=i$ due to closure of the internal sub-algebra. Consequently, (12a) becomes, using (10g) again,

$$C_{Ji}{}^{I}C_{jk}{}^{i} = -C_i \delta_{IJ} C_{jk}{}^{i} = 0 \tag{12b}$$

i.e.,

$$C_i C_{jk}{}^{i} = 0 \tag{12c}$$

and this applies for all j,k. But (12c) implies that $C_i = 0$ and hence (10g) becomes,

$$C_{iJ}{}^{K} = 0 \tag{14}$$

That (12c) implies $C_i = 0$ is essentially a matter of choosing the basis for the internal algebra appropriately. $C_{jk}{}^{i}$ are the components of the Lie product $\Im_j \otimes \Im_k$, considered as a vector, in direction i. So $C_i C_{jk}{}^{i} = 0$ says that the vector C_i is orthogonal to all possible vectors $\Im_j \otimes \Im_k$. Hence, so long as the vectors $\Im_j \otimes \Im_k$ span the whole *N*-dimensional space, the only possibility is that C_i is null. If they do not span the *N*-dimensional space then there is at least one $\Im_m$ which does not appear in the sum $C_{jk}{}^{i}\Im_i$ for any j,k and which does not lie in the closed sub-algebra, *S*, spanned by $\Im_j \otimes \Im_k$. It is then a matter of choosing a set of exceptional generators, $\{\Im_m\}$, which, together with S, spans the whole *N*-dimensional space, dropping any superfluous generators (i.e. using only *N* generators). $C_i = 0$ tehn follows from (12c) and hence so does (14). *Not sure I've got this para right*.

Step 4

Now consider (6) with $u=i, x=j, y=I, w=\alpha$,

$$C_{iq}{}^{\alpha}C_{jI}{}^{q} + C_{Iq}{}^{\alpha}C_{ij}{}^{q} + C_{jq}{}^{\alpha}C_{Ii}{}^{q} = 0 \tag{15a}$$

In the last term $C_{Ii}{}^{q}$ could be non-zero only for $q=\beta$ because of (9c) and (14), but then $C_{jq}{}^{\alpha} = C_{j\beta}{}^{\alpha} = 0$ by (8). In the first term $C_{iq}{}^{\alpha}$ could be non-zero only for $q=J$, but then $C_{jI}{}^{q} = C_{jI}{}^{J} = 0$ by (14). Finally, in the second term only the contributions from $q=k$ are non-zero, since $C_{ij}{}^{J} = C_{ij}{}^{\alpha} = 0$. Hence (15a) becomes,

$$C_{Ik}{}^{\alpha}C_{ij}{}^{k} = 0 \tag{15b}$$

Using the same argument as used following (12c) we conclude,

$$C_{Ik}{}^{\alpha} = -C_{kI}{}^{\alpha} = 0 \tag{15c}$$

This completes the proof since (9c), (14) and (15c) together give,

$$C_{iI}{}^{w} = 0 \tag{16}$$

which is the RHS of (8). **QED**.

References

W.D.McGlinn (1964), “Problem of Combining Interaction Symmetries and Relativistic Invariance”, Phys.Rev.Lett., **12**, 467-469.

L.O’Raifeartaigh (1965), “Lorentz Invariance and Internal Symmetry”, Phys.Rev.B **139**, 1052.