Chapter 41

Waves Making Tracks

Tracks in cloud chambers or bubble chambers give every impression that the agency causing them is particle-like. But actually the wavefunction description is also consistent with these linear tracks.

Last Update: 22/1/12

One gets used to the 'collapse of the wavefunction' despite its nature being a mystery. One becomes accustomed to a particle with well defined momentum, and hence approximating to a plane wave, nevertheless being detected as if it were particle-like. Thus, a plane wave quantum state normalised to unity is detected by a photographic screen as a single point (to within the resolving power of the emulsion). Or, impinging upon an array of detectors, a normalised plane wave state causes only one detector to click. Having accepted these instances of the collapse of the wavefunction, there is still another type of detection which is more unsettling. This is particle detection using a cloud chamber or bubble chamber. In these detectors the whole track of the particle is made visible. Just how can a plane wave give rise to a very particle-like straight track?

The resolution of this problem was first provided by Mott (1929) and it has been frequently discussed since, see for example Bell (1971) and Schiff (1968). The track arises because the passage of the particle nucleates droplets, or bubbles, in the supersaturated (or super-cooled) fluid in the chamber. We suppose the particle in question is energetic so that it looses little energy on each interaction, sufficient to nucleate droplets or bubbles, but without substantial alteration in its momentum. This, of course, is the classical description. It remains to be seen that a straight particle track also results from quantum mechanics. However the assumption that each nucleation event absorbs a negligible fraction of the particle's energy allows us to idealise the scattering between the particle and the fluid as elastic. Consider the first such scattering event. The incoming plane wave scatters to form an out-going spherical wave,

$$\psi_{scattered} \propto f(\theta)\frac{\exp\{ikr\}}{r} \tag{1}$$

where the radial coordinate *r* is centred on the scattering atom in the fluid. The scattering amplitude, *f*, will obviously depend upon the interaction between the particle and the atom. The Born approximation for an interaction represented by a potential function, $V(\bar{r})$, yields,

$$f(\theta) \propto \int V(\bar{r})\exp\{i\bar{q}\cdot\bar{r}\}\cdot d^3r \tag{2}$$

where $\bar{q} = \bar{k}_{in} - \bar{k}_{out}$ so that $q = 2k\sin\frac{\theta}{2}$, assuming $\bar{k}_{in}$ defines the polar axis from which θ is measured and $k \approx k_{in} \approx k_{out}$. The potential function will drop to virtually zero at distances larger than some size scale, *a*, roughly the "size of the atom". For particle energies of the order of keV or higher we will have $ak >> 1$ (for either electrons or nuclear particles). Consequently, so long as θ is not very small we also have $aq >> 1$. Consequently the exponential in (2) will oscillate rapidly over most of the volume integral, resulting in strong cancellation. The exceptional case is for very small scattering angles, θ, for which the exponential can be nearly unity. Consequently it is clear that the scattering amplitude will be large for near-forward scattering but diminish rapidly at larger angles. A specific illustration of this is provided by a Coulomb potential with an exponential shielding decay (due to the overall neutrality of the atom), $V \propto e^{-r/a}/r$. Explicit integration yields $f \propto 1/\left(1+a^2q^2\right)$, so the amplitude does indeed reduce steeply for $aq > 1$, i.e., away from the near-forward direction.

The occurrence of a straight track can be argued in several different ways. The simplest is by appeal to the strongly forward-focussed nature of $f(\theta)$. This is sufficient to produce a tightly collimated beam in typical cases. Taking $a \sim 1$ Angstrom and a particle energy of the order of MeV, the angular spread of the scattered beam is $\Delta\theta \sim 1/ak \sim 10^{-3}$ (electrons) or $\Delta\theta \sim 1/ak \sim 10^{-4}$ (protons). Note that this relationship of scattering angle to target size and particle energy (or de Broglie wavelength) is essentially the same as would apply if the particle-wave were diffracted from an aperture of size $\sim a$. This is not coincidence. Equ.(2) shows that the scattering amplitude is proportional to the spatial Fourier transform of the potential function. But diffraction patterns are the Fourier transform of the aperture function, so the potential is analogous to an aperture function.

This example may leave you feeling slightly uneasy because it appears to rely upon strongly biased forward scattering, which in turn appears to rest upon assumptions regarding the scattering potential. The unease is admittedly ameliorated by the fact that a strongly collimated $f(\theta)$ arises even from a spherically symmetric potential, the collimation therefore resulting purely from the initial plane wave, $\bar{k}_{in}$. Actually the only requirement is that the potential be confined to some spatially small region $\sim a$. Nevertheless a more convincing argument follows.

In reality, the wave resulting from the first scattering event does not then propagate freely. To form the particle track there will be a large number of nucleation events at a sufficiently close spacing to give the illusion of a continuous track. Consider the wave resulting from the second such event, assuming this occurs at a distance of b from the first,

$$\psi_{scattered} \propto f(\theta')\frac{\exp\{ikb\}}{b} f(\theta'')\frac{\exp\{ikr\}}{r} \qquad (3)$$

where the total deflected angle is now $\theta = \theta' + \theta''$. The probability amplitude for scattering through this total angle, due to just two scattering events separated by distance b, is the sum of all expressions like (3) with differing individual angles θ' and θ'' but the same sum $\theta = \theta' + \theta''$. Hence, ignoring the effect of the relative phases, the overall amplitude cannot exceed,

$$\tilde{f}(\theta) \propto \int f(\theta - \theta') f(\theta') d\theta' \qquad (4)$$

The initial momentum $\bar{k}_{in}$ becomes $\bar{k}'$ after the first scattering event and $\bar{k}''$ after the second. The Born approximation gives us,

$$f(\theta') f(\theta'') \propto \int V(\bar{r}')\exp\{i\bar{q}' \cdot \bar{r}'\} \cdot d^3r' \int V(\bar{r}'')\exp\{i\bar{q}'' \cdot \bar{r}''\} \cdot d^3r'' \qquad (6)$$

The two spatial integrals in (6) are over widely separated regions, because the atoms are small compared with the spacing between the two scattering events, $a << b$. The momentum transfers are given by,

$$\bar{q}' = \bar{k}_{in} - \bar{k}' \ ; \ q' \approx 2k \sin\frac{\theta'}{2} \tag{7}$$

$$\bar{q}'' = \bar{k}' - \bar{k}'' \ ; \ q'' \approx 2k \sin\frac{\theta''}{2} \tag{8}$$

The argument is now identical to that following Equ.(2). The potential function will drop to virtually zero at distances larger than ~*a*. Assuming this is of atomic dimensions, then for particle energies of the order of keV or higher we will have $ak >> 1$. Consequently, we also have $aq' >> 1$ and $aq'' >> 1$ providing that θ' and θ'' are not too small. The exponentials in (6) will then oscillate rapidly over most of the volume integrals, resulting in strong cancellation. The exceptional case is for very small scattering angles for which the exponentials can be nearly unity. Both θ' and θ'' must be small for the overall amplitude to be large. Consequently the integrand in (4) will be significant only for those cases in which $\bar{k}_{in}, \bar{k}'$ and $\bar{k}''$ are all nearly colinear.

If we suppose the particle undergoes just these two nucleation events, then $\bar{k}'' \equiv \bar{k}_{out}$. So the argument implies that the outgoing direction of motion is parallel to the incoming direction, to within some small tolerance, $\bar{k}_{in} \approx \bar{k}_{out}$. Moreover the two nucleation sites are displaced along the direction of $\bar{k}'$ which is also parallel to the incoming and outgoing directions of motion. So the direction of the particle track is faithful to the particle's actual velocity.

However, if we extend this argument to *N* successive scattering events we run into a problem. The momentum transfer between successive events $\bar{q}_i = \bar{k}_{i-1} - \bar{k}_i$ is again required to be small so that the scattering angle θ_i between $\bar{k}_{i-1}$ and $\bar{k}_i$ is generally within $\Delta\theta \sim 1/ak$. But over a large number, *N*, of uncorrelated scattering events the standard deviation of the overall angle of scattering would be $\sim \sqrt{N} \cdot \Delta\theta$, which could be a large angle if *N* is sufficiently large. This would suggest that interactions with very large numbers of atoms would cause the track to drift seriously away from the direction of the initial momentum. But provided that the particle's energy remains large, with relatively little energy being lost in the collisions, we know that this cannot be right. There is some wrong with the argument based solely on $f(\theta_i)$. What has been forgotten is the role of the phase of the wavefunction, and the importance of this when expressions like (3) are summed over all possible particle paths.

The generalisation of (3) to many scattering events is,

$$\psi_{scattered} \propto \prod_{j=in}^{out} f(\theta_j) \frac{\exp\{ikb_j\}}{b_j} = \left(\prod_{j=in}^{out} \frac{f(\theta_j)}{b_j} \right) \exp\left\{ ik \sum_{j=in}^{out} b_j \right\} \tag{9}$$

where b_j is the distance between the $j-1^{th}$ and j^{th} scatterings, and the product/sum extend over the whole track. The total amplitude will be the sum of (9) over both the intermediate scattering angles and the distances b_j. Both the direction of scattering and the distance to the next event can be represented by a vector $\bar{b}_i$, so the overall amplitude can be written,

$$\psi_{scattered} \propto \int \exp\left\{ ik \sum_{j=in}^{out} b_j \right\} \left(\prod_{j=in}^{out} \frac{f(\theta_j)}{b_j} d^3 b_j \right) \tag{10}$$

Now the distance between scattering events is large compared with the atomic size, *a*, so we have $bk >> ak >> 1$. So we can deploy the same rapid-phase-variation argument again in the context of the term $\exp\left\{ ik \sum_{j=in}^{out} b_j \right\}$ in (10). In this context the argument is just the method of stationary phase: the integral will be small unless $\sum_{j=in}^{out} b_j$ is stationary with respect to variations in the individual b_j - and subject to fixed starting and finishing points on the track. But this simply means that the track must be straight, because $\sum_{j=in}^{out} b_j$ is the total track length and this is minimum when the track is straight. The overall wavefunction in (10) is actually a path integral, and we see that the reason for the straight track is essentially the same as the reason why the classical limit of a free particle's trajectory is straight. This follows from applying the method of stationary phase to the Feynman path integral, Feynman (1948). The global term $\exp\left\{ ik \sum_{j=in}^{out} b_j \right\}$ prevents the particle drifting away from the straight track which would occur for a large number of uncorrelated classical scattering events.

References

N.F. Mott, Proc. Roy. Soc. Al26, 79 (1929).

J.S. Bell (1971) "On the Hypothesis that the Schrodinger Equation is Exact", Ref.TH.1424-CERN, *Contribution to the international colloquium on issues in contemporary physics and philosophy of science, and their relevance for our society, Penn State University, September 1971*.

L.I.Schiff (1968), "Quantum Mechanics", 3rd ed, McGraw-Hill (see §38).

R.P.Feynman (1948), "Space-Time Approach to Non-Relativistic Quantum Mechanics", Rev.Mod.Phys. **20**, 267.