Chapter 40

Pitfalls of Popularisation

Popularisers of physics have a tough job. We must allow them considerable latitude as regards strict accuracy. Rigour is often the enemy of comprehensibility. Nevertheless it is instructive to examine some sins that have been committed.

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Presenters of popular science programmes can sometimes get a little carried away. Quantum mechanics is particularly easy to misrepresent in the heat of the moment. Here we examine and moderate some of the claims which have been made.

1. Can a Macroscopic Object Spontaneously Escape from a Box?

The presenter shows us a 60g object which he places inside a box which is roughly a 50mm cube, complete with lid. This purports to be an illustration of Heisenberg's uncertainty principle. How long, he asks, will we have to wait before the object spontaneously appears outside the box? He supplies the answer as follows,

$$t \sim \frac{LSm}{\pi\hbar} \qquad (1)$$

where $L = 0.05$ m is the size of the box, $S \sim 0.04$ m is the size of the object, and $m =$ 0.06 kg is the mass of the object. As an illustration of the uncertainty principle this is fine. Neglecting factors of order unity, Equ.(1) may be derived as follows. Confining the object in the box implies a minimum momentum of $p \sim \pi\hbar / L$. If the object is to appear outside the box, the uncertainty in its position must be of the order of its size, so that the uncertainty in its momentum is $\Delta p \sim \hbar / 2S$. The length of time we must wait is related to the uncertainty in the object's energy by $t \sim \hbar / 2\Delta E$. Finally, the uncertainty in momentum and energy are related by $\Delta E = p\Delta p / m$. Putting these together does indeed produce (1).

Inserting the numerical values in (1) gives a comfortingly long time before the object (ostensibly) materialises outside the box: $\sim 10^{29}$ s $\sim 10^{12}$ times the age of the universe. So everyone is happy that the probability of the object spontaneously escaping is sufficiently small that, for all intents and purposes, it will just not happen, thus conforming to our experience.

Well, no. Actually I'm not very happy that a probability of $\sim 10^{-22}$ per year is small enough. After all there are $\sim 10^{24}$ stars in the observable universe, so there are an awful lot of places where such behaviour could occur. Recall that the first nuclear reaction in solar mass stars, $p + p \rightarrow D + e^{+} + \nu$, typically has a reaction time of $\sim 10^{10}$ years. The protons have a probability of only $\sim 10^{-17}$ of undergoing this reaction each time they collide. Yet this reaction is responsible for the sequence of events which leads to the formation of all the chemical elements in the universe (beyond lithium). My suspicion is that the probability for the object to escape the box is actually far smaller than 10^{-22} per year. It is, as we shall now see.

Something crucial has been missed in the estimate given by (1) – namely, the box! What has been calculated is the probability for spontaneous teleportation over a distance of 40mm *through empty space*. But really the object must get through the wall of the box. This makes the probability far, far smaller. Of course it does. If we make the box strong enough, the object is not going to get out – just as you'd expect. In fact, even a remarkably feeble box will prevent the object escaping for much, much longer times than implied by Equ.(1).

Heisenberg's uncertainty principle is no longer sufficient to calculate the confinement time when the object has a box wall to penetrate. We are now in the realms of quantum tunnelling. An order of magnitude estimate of the tunnelling probability is easily made. The two key parameters are the thickness of the box wall (*a*) and the energy which would be required to push the object into the wall (*V*). The latter is a measure of the strength of the wall. In reality this potential energy barrier is going to be very large. Two solids do not usually superimpose to occupy the same space. The attempt to force them to do so would usually result in one of them breaking up. This illustrates the very substantial energy required. Even if we assume there was enough interstitial space within the lattice of the two solids, the lattice disruption of both the object and the box wall would lead to a very large energy requirement. Fortunately, though, it is not necessary to attempt any realistic estimate of this barrier energy. Instead we shall assume a paltry 1 Joule, almost certainly several orders of magnitude smaller than the true potential barrier.

The relationship between momentum and non-relativistic kinetic energy is $E = p^2 / 2m$ and $p = \hbar k$. When within the wall, the object has a kinetic energy which is negative, namely $-V$, and hence impossible classically. In quantum mechanics this means that the wave-vector is imaginary, $k = i\sqrt{2mV} / \hbar$. The normally oscillatory wave $\exp\{ikx\}$ is therefore an exponential decay $\exp\left\{-\sqrt{2mV} \cdot x / \hbar\right\}$. Consequently the probability amplitude on the outer surface of the box is smaller than that on the inner surface of the box by a factor $\exp\left\{-\sqrt{2mV} \cdot a / \hbar\right\}$. But probabilities are the absolute squares of probability amplitudes, so the probability of the object tunnelling through the wall is roughly,

$$\text{Tunnelling probability} \sim \exp\left\{-\frac{2\sqrt{2mV}}{\hbar} \cdot a\right\} \tag{2}$$

Substituting *V* ~ 1 Joule and, say, *a* ~ 0.001 m, gives,

$$\text{Tunnelling probability} \sim \exp\left\{-6 \times 10^{30}\right\} \tag{3}$$

Now *that* is a small number! To drive home just how small, consider this. Really (3) gives the probability of tunnelling through the wall at every attempt to do so. But how frequently is the object making the attempt? A crude approximation is to assume that an attempt to penetrate the wall is being made on every 'cycle' where the cycles in question relate to the frequency derived from the object's energy. Let's give the object lots of energy by raising it to 3000K. Its typical thermal energy is then $k_B T =$ 4×10^{-20} J (where k_B is Boltzmann's constant). The associated frequency is ~6×10^{13} Hz. So we must multiply (3) by 6×10^{13} to get the probability of escape per second. Alternatively, we must multiply (3) by 2×10^{21} to get the probability of escape per year. So we get,

$$\text{Tunnelling probability} \sim 2 \times 10^{21} \times \exp\left\{-6 \times 10^{30}\right\} \text{ per year} \tag{4}$$

How much difference has it made to have multiplied (3) by 2×10^{21}? Effectively no difference at all because (4) is the same as,

$$\text{Tunnelling probability} \sim \exp\{49 - 6\times 10^{30}\} = \exp\{-6\times 10^{30}\} \text{ per year} \qquad (5)$$

which is identical to (3)! Note that the "=" sign in (5) is justified to an accuracy of 29 decimal places.

Unlike the comparatively huge probability of $\sim 10^{-22}$ per year given by (1), the probability given by (3) = (5) really is effectively zero. Even if there were 10^{500} universes in the multiverse (enough perhaps to realise every possible string theory) and even if they all survived for 10^{500} years, this would make negligible impact on the probability for the object to tunnel through the box wall within the life of any of these universes. It would remain as given by (3) to an accuracy of 26 decimal places.

So when people say, "in principle, the object could spontaneously appear outside the box if you waited long enough" I think it is best to simply regard this as untrue for macroscopic objects, for all practical purposes.

2. Is Everything Connected to Everything Else?

The presenter explains the uncertainty principle to the audience. "Two electrons", he says, "can never share the same quantum state". He takes up an object and says that the electrons in the object cannot share a quantum state with any other electrons in the universe. He warms the object in his hand, remarking that this will cause some electrons to move to a higher energy quantum state. He then confounds the audience by claiming that, as a result, every electron in the universe has had to react to this change of state within the object. What can he mean by this?

It would appear that he cannot mean literally every electron in the universe because, limited by the speed of light, after a time t only those electrons within a distance ct could possibly be affected – or so you might think. But we have already seen that entangled particles are no respecters of superluminal constraints – but only in a limited sense.

Suppose two hydrogen atoms are remote from each other and also remote from anything else. They are both in their ground state. (For this purpose we are ignoring the existence of electron and proton spins – or, equivalently, we are assuming the spin states of the two electrons, and the two protons, are the same). Are the electrons in the same quantum state? The answer is "no" by virtue of the trivial fact that they are at different locations. A semantic confusion can arise because, in many contexts, the two electrons might be described as being in the same state, i.e., the hydrogen atom ground state. But this really means the corresponding, or equivalent, state, not literally the same quantum state. Call the electron states corresponding to the atom on the left and the right $|1\rangle$ and $|2\rangle$ respectively. The state of the two electrons can be written as a direct product, adopting the convention that electron No.1 is written on the left. So $|1\rangle|2\rangle$ is the state of electron No.1 being in the ground state of atom 1 and electron No.2 being in the ground state of atom 2. Conversely $|2\rangle|1\rangle$ is the state of electron No.1 being in the ground state of atom 2 and electron No.2 being in the ground state of atom 1.

These situations are indistinguishable because the electrons are identical. But actually neither of these mathematical specifications is correct. The reason is the exclusion principle in its more general form, the spin-statistics theorem. This requires that the quantum state of many identical fermions must be completely antisymmetric with respect to interchange of any pair of the fermions. So the two particle state of the pair of ground state electrons is actually,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|1\rangle|2\rangle - |2\rangle|1\rangle\right] \tag{6}$$

Note that if we tried putting both electrons into the same state, i.e., on the same atom, the above expression would vanish – in compliance with the exclusion principle. Now the remarkable thing about the state (6) is that it shows that the two electrons are automatically entangled. (Recall that two particles are entangled if their state cannot be written as a direct product of one-particle states). One cannot say whether No.1 electron is on atom 1 or atom 2. It is delocalised. Suppose we excite atom 1 so that its electron state changes from $|1\rangle$ to $|1e\rangle$. The state of the two electrons is thus,

$$|\psi e\rangle = \frac{1}{\sqrt{2}}\left[|1e\rangle|2\rangle - |2\rangle|1e\rangle\right] \tag{7}$$

Which electron has been excited? It could be either. Of course it could, they are identical. In this sense the excitation of atom 1 does affect both electrons. But atom 2 is unaffected because whichever electron is excited the other electron is on atom 2. So the mystery is not that there is some strange action-at-a-distance going on, but that the electrons are delocalised. They have to be in order to respect the requirement for an antisymmetric wavefunction, which implies the exclusion principle and which in turn implies the solidity of matter.

Does the entangled, anti-symmetric wavefunction of a pair of electrons have any observable consequences? Yes, it certainly does in suitable circumstances. Consider the two electrons in the first excited state of a helium atom. Assume firstly that they are in a spin state which is symmetrical under interchange, i.e., one of the triplet states with unit spin. Their spatial state must therefore be anti-symmetric. In a first order perturbation approximation, this two-particle state can be considered as comprised of the single particle states u_1 and u_2. These are the hydrogenic spatial states but with a doubly charged nucleus (Z = 2), zero orbital angular momentum and with principal quantum number, n, equal to 1 and 2 respectively. The latter is the lowest single-particle excited state. To first order the spatial part of the two-electron wavefunction is thus,

$$\psi(\bar{r}_1, \bar{r}_2) = \frac{1}{\sqrt{2}}\left[u_1(\bar{r}_1)u_2(\bar{r}_2) - u_1(\bar{r}_2)u_2(\bar{r}_1)\right] \tag{8}$$

If there were no interaction between the electrons, (8) would be the exact wavefunction for the first excited state of helium and its energy would be $E_1 + E_2$, where $E_n = -\frac{2\alpha^2 mc^2}{n^2}$. But the Coulomb interaction between the two electrons will cause the energy to differ from simply $E_1 + E_2$. To first order the change in the energy level is given by the expectation value of the interaction potential in state (8), i.e.,

$$\Delta E = e^2 \int\int \frac{|\psi(\bar{r}_1, \bar{r}_2)|^2}{|\bar{r}_1 - \bar{r}_2|} d^3r_1 d^3r_2 \tag{9}$$

But due to the form of the wavefunction, (8), we can write this as,

$$\Delta E = \Delta E_C - \Delta E_E \tag{10a}$$

so that the energy level of the excited two-electron state becomes,

$$E_1 + E_2 + \Delta E_C - \Delta E_E \tag{10b}$$

where,

$$\Delta E_C = e^2 \int\int \frac{|u_1(\bar{r}_1)u_2(\bar{r}_2)|^2}{|\bar{r}_1 - \bar{r}_2|} d^3r_1 d^3r_2 \tag{10c}$$

and,

$$\Delta E_E = e^2 \int\int \frac{u_1^*(\bar{r}_1)u_2^*(\bar{r}_2)u_1(\bar{r}_2)u_2(\bar{r}_1)}{|\bar{r}_1 - \bar{r}_2|} d^3r_1 d^3r_2 \tag{10d}$$

Here ΔE_C is the direct Coulomb energy due to the electron-electron interaction. One might expect the energy shift to be simply this ΔE_C. But (10b) shows that the energy is also shifted *down* by ΔE_E, the so-called "exchange energy" – because it originates from the term which exchanges the two electrons. Note that (10d) evaluates to a positive quantity, so the exchange energy does result in (10b) being a lower energy level. Notice that this results from the minus sign in (8), i.e., the antisymmetry of the wavefunction as required by the exclusion principle.

Had we considered the antisymmetric spin state (the singlet state of spin zero which has the two electron spins oriented in opposing directions) then the spatial wavefunction would have to be symmetric under interchange so as to make a wavefunction which is overall antisymmetric. The minus sign in (8), and hence in (10a), would become a plus sign. Consequently it follows that the first excited state of helium will have a triplet electron spin state and an energy given to first order by (10b,c,d). The second excited energy level is $E_1 + E_2 + \Delta E_C + \Delta E_E$ and involves a singlet electron spin state. This simple prediction based on the required antisymmetry of the wavefunction is easily confirmed by spectroscopy.

However, one need hardly be impressed that the two electrons in a helium atom are "connected". The claim made above is that even two electrons which are remote from each other are somehow connected. Returning to our earlier example of two hydrogen atoms we can make them resemble the helium atom example by conceding a slight interaction between them. However, in this case any "connection" between the electrons can merely to be interpreted as a result of this interaction. Can some "connection" occur even when there is absolutely no interaction? There is a ready example for entangled particles in the EPR paradox. In contrast to (6), suppose we assume a spatially symmetric state but an antisymmetric spin singlet state,

$$|\psi\rangle = \frac{1}{2}\left[|1\rangle|2\rangle + |2\rangle|1\rangle\right]\left[|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right] \tag{11}$$

The "connection" between the two electrons consists of the spin of both electrons being indeterminate, but nevertheless the spin of one may be deduced by measuring the spin of the other – and without any physical signal being transferred between them. But this sort of entanglement "connection" is limited as regards what it may accomplish. In particular the connection does not provide a channel for the transfer of information. *What can it do…? Add.*

Consider *N* electrons, labelled 1, 2, 3… which are each in one of the distinct states labelled $u_\alpha, u_\beta, u_\gamma, \ldots$. These states should be understood to include both the spatial and spin parts. The state which is completely antisymmetric under interchange of any pair of electrons is given by the determinant,

$$U(1,2,3\ldots)=\frac{1}{\sqrt{N!}}\begin{pmatrix} u_\alpha(1) & u_\alpha(2) & u_\alpha(3) & etc \\ u_\beta(1) & u_\beta(2) & u_\beta(3) & \\ u_\gamma(1) & u_\gamma(2) & u_\gamma(3) & \\ etc & & & \end{pmatrix} \qquad (12)$$

There are $N!$ terms in (12), each of the form $u_\alpha(i)u_\beta(j)u_\gamma(k)\ldots$. One wonders at the meaning of this. Even for a quite modest number of electrons, say 20, the number of terms in (12) is greater than 2 x 10^{18} (more than the number of seconds since the Big Bang). If all the electrons in the universe share a single, totally antisymmetric state like (12), the number of terms the expression for the wavefunction would contain is monstrously large. Is this the sense in which "everything is connected"? It seems dubious.

It should be recalled that pure states are not the most general description of a quantum state. The density matrix provides a more general description. Pure entangled states will tend to decohere very quickly into classical mixtures under the influence of environmental interaction. Suppose we start with a two-particle spin singlet state. The environment first becomes entangled with it as a consequence of the interaction, thus,

$$\frac{1}{\sqrt{2}}\left[|\uparrow\rangle|\downarrow\rangle-|\downarrow\rangle|\uparrow\rangle\right]|env\rangle \rightarrow \frac{1}{\sqrt{2}}\left[|\uparrow\rangle|\downarrow\rangle|env1\rangle-|\downarrow\rangle|\uparrow\rangle|env2\rangle\right] \qquad (13)$$

The crucial thing about (13) is that the environment evolves differently according to whether the system is $|\uparrow\rangle|\downarrow\rangle$ or $|\downarrow\rangle|\uparrow\rangle$. This is perfectly reasonable. Suppose the environment consists of just one molecule which is closer to the first atom of the system than the second. Suppose also that the environment interaction is spin dependent. Then it is natural for the interaction strength to differ between $|\uparrow\rangle|\downarrow\rangle$ and $|\downarrow\rangle|\uparrow\rangle$. But tracing out the environment leaves the state as a classical mixture of $|\uparrow\rangle|\downarrow\rangle$ and $|\downarrow\rangle|\uparrow\rangle$. The initial pure entangled quantum state, which could have been used to drive teleportation or encryption, the hallmarks of "connectedness", has been made classical and is no longer useful in these respects. Large systems with many electrons are likely to decohere even more readily. So, in practice, decoherence will severely restrict the "connectivity" due to entanglement.

There is, however, a residual doubt. Decoherence consists of the system becoming entangled with the environment, as shown by (13). Choosing to ignore this – by tracing out the environment – does not mean that it does not exist. Does the entanglement and hence the spacelike-separated 'connectedness' really persist despite decoherence? Is everything connected to everything else? Shit, I don't know.

Can't be left like this.

Perhaps split §1 and §2 into separate Chapters. Could look for other examples of howlers.