Chapter 37

The Electron Ground State of the H-Atom has Orbital Angular Momentum Quantum Number Zero, So There is No Electric Current, Right? Wrong.

Even though the hydrogen atom ground state is conventionally assigned an l number of zero, the expectation value of angular momentum about the z axis is not zero, specifically it is $\sim \alpha^2/6$ ($\hbar$ units). The fact that this is not an integer is a clue to what is going on here. There is also a circulating electric current around the z axis, which is not small but remarkably large (about a milliamp). This current produces a magnetic dipole moment of one Bohr magneton. These statements apply assuming that the electron is in a state of definite z-component of total angular momentum, i.e., $j_z = \pm 1/2$. This provides the preferred axis for the small orbital angular momentum and the large current and magnetic dipole moment. This current is to be understood as a manifestation of the electron spin, and hence the corresponding magnetic moment is another way of looking at the magnetic moment due to the electron spin, rather than being additive to it.

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1. Specification of the Relativistic States of the H-Atom Electron

The ground state of the electron in a hydrogen atom can be found in the non-relativistic approximation by solving the Schrodinger equation with a Coulomb potential. The ground state has zero orbital angular momentum, $l = 0$. In the non-relativistic approximation the Schrodinger Hamiltonian, $\hat{H}_{\text{Sch}}$, and the operators $\hat{S}^2, \hat{L}^2, \hat{J}^2$ and $\hat{J}_z$ are all mutually commuting and hence states can be found which are simultaneous eigenstates of all these operators. This means that states can be defined with definite values for the energy, spin, orbital angular momentum, total angular momentum and z-component of total angular momentum. These are specified by the quantum numbers n, s, l, j, j_z respectively. All the one-electron states have $s = \frac{1}{2}$, of course. The ground state has $n = 1$, $l = 0$, $j = \frac{1}{2}$, $j_z = \pm\frac{1}{2}$. In the non-relativistic case the energy levels are determined by n alone, states with differing l, j, j_z for a given n being degenerate in the non-relativistic approximation.

It is, unfortunately, an all too common experience for me to discover that a subject of which I thought I had a firm grasp turns out to include a serious misunderstanding. Such is the case with the H-atom electron states calculated relativistically via the Dirac equation. Texts (e.g., Bjorken & Drell) will illustrate the energy levels by labelling the states with the numbers n, l, j, just as for the Schrodinger states. This misleads the reader since it gives the impression that l continues to be a well defined quantum number for the relativistic states. It does not, but this misconception is reinforced, at least for the careless reader, by statements within such texts like, "the 2-spinor components which comprise the Dirac 4-spinor are eigenfunctions of $\hat{S}^2, \hat{L}^2, \hat{J}^2$ and $\hat{J}_z$". It is easy to misunderstand such a statement as meaning that the Dirac 4-spinor itself is a simultaneous eigenstate of $\hat{H}_{\text{Dirac}}, \hat{S}^2, \hat{L}^2, \hat{J}^2$ and $\hat{J}_z$. It is not. It could not be because $\hat{L}^2$ does not commute with the Dirac Hamiltonian, $\hat{H}_{\text{Dirac}}$. This means that the orbital angular momentum is not a constant of the motion. To put it another way, it means that a state of definite energy does not have a definite orbital angular momentum: no state exists which simultaneously has a definite E and l. There must be some other operator which commutes with $\hat{H}_{\text{Dirac}}, \hat{S}^2, \hat{J}^2$ and $\hat{J}_z$ and which distinguishes the distinct states which, in the non-relativistic approximation, would be distinguished by their l values. This new "relativistic orbital angular momentum operator", is defined by $\hat{k} = \gamma^0 \left(\overline{\sigma}' \cdot \hat{\overline{L}} + 1\right)$, where $\overline{\sigma}' = \begin{pmatrix} \overline{\sigma} & 0 \\ 0 & \overline{\sigma} \end{pmatrix}$ but we shall have no need of it here. However, the energy levels of the Dirac Hamiltonian are defined by n and j alone. The dependence on j breaks (some of) the degeneracy of the Schrodinger energy levels and substantially improves the agreement with experimental (spectroscopic) energies.

2. The Ground State of the Dirac Equation for Hydrogen

To be more exact we are referring here to the ground state of the Dirac equation assuming a Coulomb potential, i.e., the lowest energy eigenstate of,

$$i\partial_t \psi = \hat{H}_{\text{Dirac}} \psi = \left[\overline{\alpha} \cdot \hat{\overline{p}} + \gamma^0 m - \frac{\tilde{\alpha}}{r} \right] \psi = E\psi \tag{2.1}$$

where $\hat{\overline{p}} = -i\overline{\nabla}$. We use the notation $\tilde{\alpha}$ for the fine structure constant rather than α to save confusion with the 4×4 spinor matrices $\alpha_i = \gamma^0 \gamma^i$. We shall employ the usual Pauli-Dirac representation of the Dirac gamma matrices in which $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$, $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\overline{\sigma}$ are the Pauli matrices. We shall not present the derivation of the energy eigenstates of (2.1) here. This may be found in standard texts. The ground state, assuming "spin up", takes the form,

$$\psi = e^{-iEt} \begin{pmatrix} g(r) \\ 0 \\ if(r)\cos\theta \\ if(r)e^{i\phi}\sin\theta \end{pmatrix} \tag{2.2}$$

Substitution of (2.2) into (2.1) shows the latter to be an identity in θ, ϕ and requires the following two radial equations,

$$\left(E + \frac{\tilde{\alpha}}{r} - m \right) g = \partial_r f + \frac{f}{r} \tag{2.3}$$

$$\left(E + \frac{\tilde{\alpha}}{r} + m \right) g = -\partial_r g \tag{2.4}$$

[These are equivalent to Bjorken & Drell's Eqs.(4.13) after the substitutions $G = -irg$ and $F = irf$ and noting that their ground state is specified by $\kappa = -1$]. In fact eqs.(2.3,2.4) apply for all the energy eigenstates and hence provide all the energy eigenvalues. The ground state, (2.2), is specified by $n=1$ and $j=1/2$, and a value $k=1$ for the operator $\hat{k}=\gamma^0\left(\overline{\sigma}'\cdot\hat{\overline{L}}+1\right)$. The ground state energy is,

$$E=m\sqrt{1-\tilde{\alpha}^2} \tag{2.5}$$

Note that the first two terms in the binomial expansion of (2.5) reproduce the non-relativistic result from the Schrodinger equation, $E-m=-\frac{\tilde{\alpha}^2}{2}m$. For later use we note that the explicit solution for the ground state is,

$$g=g_0 r^{-\zeta}\exp\{-m\tilde{\alpha}r\} \tag{2.6a}$$

$$f=f_0 g \tag{2.6b}$$

where,

$$\zeta=1-\sqrt{1-\tilde{\alpha}^2}\approx\frac{\tilde{\alpha}^2}{2}=\frac{1}{2\left(137^2\right)}=2.66\times10^{-5} \tag{2.6c}$$

$$f_0=\frac{\zeta}{\tilde{\alpha}}=0.00365 \tag{2.6d}$$

and g_0 is a normalisation constant determined from $\int\psi^+\psi dV=1$. Hence, the lower 2-spinor in (2.2) is small compared with the upper 2-spinor. The lower 2-spinor together with the factor $r^{-\zeta}$ may be considered to be the relativistic correction to the Schrodinger ground state, to which (2.2) reduces as $\zeta\to0$.

3.What is the Electron Current in the Dirac Ground State of Hydrogen?

The current is given by $J^\mu=q\overline{\psi}\gamma^\mu\psi$. Hence, the charge density is $-e\psi^+\psi$ which (2.2) gives as $-e\left(g^2+f^2\right)=-e\left(1+f_0^2\right)g^2$. We will now evaluate the components of the 3-current, $\overline{J}$, working in spherical polar coordinates. Writing $\psi=\begin{pmatrix}u\\v\end{pmatrix}$ we see that,

$$\begin{aligned}J_k&=q\overline{\psi}\gamma^k\psi=q\psi^+\alpha_k\psi=q\begin{pmatrix}u^+ & v^+\end{pmatrix}\begin{pmatrix}0&\sigma_k\\ \sigma_k&0\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}\\ &=q\left(u^+\sigma_k v+v^+\sigma_k u\right)=2q\Re\left(u^+\sigma_k v\right)\end{aligned} \tag{2.7}$$

In spherical polars the Pauli matrices are,

$$\sigma_r=\begin{pmatrix}c & se^{-i\phi}\\ se^{i\phi} & -c\end{pmatrix};\quad \sigma_\theta=\begin{pmatrix}-s & ce^{-i\phi}\\ ce^{i\phi} & s\end{pmatrix};\quad \sigma_\phi=\begin{pmatrix}0 & -ie^{-i\phi}\\ ie^{i\phi} & 0\end{pmatrix} \tag{2.8}$$

where $c=\cos\theta$, $s=\sin\theta$. Hence,

$$u^+\sigma_r v=\begin{pmatrix}g & 0\end{pmatrix}\begin{pmatrix}c & se^{-i\phi}\\ se^{i\phi} & -c\end{pmatrix}\begin{pmatrix}ifc\\ ifse^{i\phi}\end{pmatrix}=\begin{pmatrix}g & 0\end{pmatrix}\begin{pmatrix}if\\0\end{pmatrix}=igf \tag{2.9}$$

$$u^+\sigma_\theta v=\begin{pmatrix}g & 0\end{pmatrix}\begin{pmatrix}-s & ce^{-i\phi}\\ ce^{i\phi} & s\end{pmatrix}\begin{pmatrix}ifc\\ ifse^{i\phi}\end{pmatrix}=\begin{pmatrix}g & 0\end{pmatrix}\begin{pmatrix}0\\ ife^{i\phi}\end{pmatrix}=0 \tag{2.10}$$

Since (2.9) and (2.1) are purely imaginary and zero respectively we conclude, from (2.7), that $J_r = J_\theta = 0$, as we might have expected. However,

$$u^+\sigma_\phi v = \begin{pmatrix} g & 0 \end{pmatrix}\begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}\begin{pmatrix} ifc \\ ifse^{i\phi} \end{pmatrix} = \begin{pmatrix} g & 0 \end{pmatrix}\begin{pmatrix} fs \\ -fce^{i\phi} \end{pmatrix} = gfs \qquad (2.11)$$

Since this is real we have, from (2.7), a non-zero azimuthal circulating current,

$$J_\phi = 2qg(r)f(r)\sin\theta \qquad (2.12)$$

Moreover, since (2.12) has the same sign everywhere, there is a non-zero net circulating current given by integrating over r, θ,

$$\text{Total } \phi\text{-Current} = \int_0^\infty\int_0^\pi 2qg(r)f(r)\sin\theta \cdot r dr d\theta \qquad (2.13)$$

Using the explicit expressions (2.6) for the radial functions, but for simplicity ignoring the small power-of-r factor, i.e., $g = g_0 \exp\{-me^2 r\}$ and $f = f_0 g$, we get,

$$\text{Total } \phi\text{-Current} = \int_0^\infty\int_0^\pi 2qf_0 g_0^2 \exp\{-2m\tilde{\alpha}r\}\sin\theta \cdot r dr d\theta \qquad (2.14)$$

g_0 is found from normalisation, i.e.,

$$\int \psi^+\psi dV = 1 = 4\pi\int_0^\infty\int_0^\pi \left(1+f_0^2\right)g_0^2 \exp\{-2m\tilde{\alpha}r\}r^2 dr \qquad (2.15)$$

Writing,

$$I_n(\lambda) = \int_0^\infty r^n e^{-\lambda r} dr \qquad (2.16)$$

Integrating by parts gives,

$$I_0(\lambda) = \frac{1}{\lambda}; \quad I_1(\lambda) = \frac{1}{\lambda^2}; \quad I_2(\lambda) = \frac{2}{\lambda^3}; \quad I_3(\lambda) = \frac{6}{\lambda^4} \qquad (2.17)$$

The angular integral in (2.14) gives a factor of 2 and hence we get,

$$\text{Total } \phi\text{-Current} = 4qf_0 \frac{I_1(2m\tilde{\alpha})}{4\pi\left(1+f_0^2\right)I_2(2m\tilde{\alpha})} = \frac{-ef_0}{\pi\left(1+f_0^2\right)}\frac{2m\tilde{\alpha}}{2} \qquad (2.18)$$

i.e., inserting the universal constants to convert from $\hbar = c = 1$ units to MKSA, and retaining only the leading term, and using $f_0 \approx \tilde{\alpha}/2$,

$$\text{Total } \phi\text{-Current} = -\frac{\alpha^2}{2\pi}\cdot\frac{mc^2 e}{\hbar} \qquad (2.19)$$

which evaluates to 1.05 milliamps, a prodigiously large current for one electron! Note that (2.19) can be written as $\frac{E_{GS}}{\pi\hbar}e$, where E_{GS} is the first order approximation to the ground state energy, so that $\frac{\pi\hbar}{E_{GS}}$ is of the order of the time associated via the uncertainty principle with this energy. The current is equivalent to that which would be generated by the whole electron cloud, of total charge $-e$, rotating with this period (though, of course, it isn't't!).

3.1 The Associated Magnetic Dipole Moment

The circular loops of current at each r, θ position cause a magnetic dipole whose magnitude is given by the product of the current and the loop area, $\pi(r\sin\theta)^2$. Hence the total magnetic moment due to the Dirac current is,

$$M_{\text{dipole}} = \int_0^{\infty}\int_0^{\pi} 2qf_0 g_0^2 \exp\{-2m\tilde{\alpha}r\}\sin\theta \cdot \pi(r\sin\theta)^2 \cdot r dr d\theta \qquad (2.20)$$

The angular integral gives a factor of $4/3$, hence,

$$\begin{aligned} M_{\text{dipole}} &= \frac{8\pi}{3} qf_0 g_0^2 \int_0^{\infty} \exp\{-2m\tilde{\alpha}r\} r^3 dr \\ &= \frac{8\pi}{3} qf_0 \frac{I_3(2m\tilde{\alpha})}{4\pi(1+f_0^2)I_2(2m\tilde{\alpha})} = \frac{2}{3}\frac{qf_0}{(1+f_0^2)} \cdot \frac{3}{2m\tilde{\alpha}} \\ &= \frac{qf_0}{(1+f_0^2)m\tilde{\alpha}} \approx \frac{qf_0}{m\tilde{\alpha}} \approx -\frac{e}{2m} = -\mu_{\mathbf{B}} \end{aligned} \qquad (2.20)$$

because $f_0 \approx \tilde{\alpha}/2$. Hence, the magnetic dipole moment due to the Dirac current is one Bohr magneton (with a minus sign due to the electron being negatively charged). This is the dipole moment that would normally be associated with an orbiting charge of $-e$ with an angular momentum of one $\hbar$ unit. This is not a possible interpretation, though, since the orbital angular momentum of the ground state is nominally zero (and, as we shall see, though not actually zero is very small, i.e., $<<\hbar$).

However, one Bohr magneton is also the magnetic dipole moment that we would normally associate with the electron spin, whose angular momentum is $\hbar/2$ but whose gyromagnetic ratio, g, takes the anomalous value 2 rather than 1. This suggests that the Dirac current and its associated magnetic moment should be regarded as another way of looking at the effects of the electron spin. This is examined next...to be added at some point...

4. Does the Electron Spin Distribution Reproduce the Dirac Current?

I have not yet been able to produce a model to demonstrate this.

5. The Orbital Angular Momentum of the H-Atom Ground State

Note carefully that we are assuming that by "ground state" we mean, not only an $n=1$ state, but a state of definite j and j_z, specifically the state given by (2.2), for which $j = j_z = 1/2$. Why should we expect the orbital angular momentum to have a non-zero expectation value in this state? It is because, whilst the upper 2-spinor in (2.2) has $l=0$, the lower 2-spinor has $l=1$. Since the lower 2-spinor is the small relativistic correction (i.e., $f << g$) we expect that the orbital angular momentum will be very small (<<1) but non-zero.

I explored there are three methods to evaluate the expectation value of the orbital angular momentum. Two of them I believe to be correct, and agree with each other. The third is fallacious and is discussed here as a warning to others lest they fall into the same trap as myself. These three methods are discussed in turn below.

5.1 Evaluate Directly the Expectation Value of the $\hat{L}_z$ Operator

The orbital angular momentum operator is, in spherical polars,

$$\hat{\overline{L}} = -i\overline{r} \times \overline{\nabla} = -i\left(\hat{\phi}\partial_\theta - \frac{\hat{\theta}}{\sin\theta}\partial_\phi\right) \tag{5.1}$$

Acting upon the ground state, (2.2), gives.

$$\hat{\overline{L}}\psi = \begin{pmatrix} 0 \\ 0 \\ -f\sin\theta \\ fe^{i\phi}\cos\theta \end{pmatrix}\hat{\phi} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ ife^{i\phi} \end{pmatrix}\hat{\theta} \tag{5.2}$$

Since the polar unit vectors can be written in terms of the Cartesian directions as,

$$\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta \tag{5.3a}$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi \tag{5.3b}$$

$$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta \tag{5.3c}$$

the z-projection of (5.2) gives,

$$\hat{L}_z\psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ ife^{i\phi}\sin\theta \end{pmatrix} \tag{5.4}$$

Hence, the expectation value of $\hat{L}_z$ is,

$$\left\langle \hat{L}_z \right\rangle = \int \psi^+ \hat{L}_z \psi \cdot dV = \int f^2 \sin^2\theta \cdot dV = \int f^2 \sin^2\theta \cdot 2\pi r^2 dr d(\cos\theta) = \frac{8\pi}{3}\int f^2 r^2 dr \tag{5.5}$$

Substituting from (2.6) for the radial function, but for simplicity ignoring the small power-of-r factor, i.e., using $g = g_0 \exp\{-me^2 r\}$ and $f = f_0 g$, and using (2.6c,d), $f_0 = \tilde{\alpha}/2$, we get,

$$\left\langle \hat{L}_z \right\rangle = \frac{8\pi}{3}\int f_0^2 g_0^2 r^2 \exp\{-2m\tilde{\alpha}r\}dr = \frac{2\pi}{3}\tilde{\alpha}^2 g_0^2 I_2(2m,\tilde{\alpha}) \tag{5.6}$$

where I_n is defined by (2.16). Using the value of g_0^2 from (2.15) gives,

$$\left\langle \hat{L}_z \right\rangle = \frac{2\pi}{3}\tilde{\alpha}^2 \frac{I_2(2m\tilde{\alpha})}{4\pi(1+f_0^2)I_2(2m\tilde{\alpha})} = \frac{\tilde{\alpha}^2/6}{1+\tilde{\alpha}^2/4} = 8.9\times10^{-6} \tag{5.7}$$

(this in units of $\hbar$, of course). As expected $\langle L_z \rangle$ is <<1.

5.2 Evaluate $\langle L_z \rangle$ from the Moment of the Momentum

Since $\overline{L} = \overline{r} \times \overline{p}$ we can evaluate $\langle L_z \rangle$ from $\langle \hat{z} \cdot (\overline{r} \times \overline{p}) \rangle$. The radial component of $\overline{p}$ does not contribute . Using (2.2) we find that the θ component of $\overline{p}$, i.e., $-\frac{i}{r}\psi^+ \partial_\theta \psi$, is zero. This leaves the ϕ component of $\overline{p}$, which is,

$$\begin{aligned} p_\phi &= -\frac{i}{r\sin\theta}\psi^+ \partial_\phi \psi = -\frac{i}{r\sin\theta}\begin{pmatrix} u^+ & v^+ \end{pmatrix}\begin{pmatrix} 0 \\ \partial_\phi v \end{pmatrix} = -\frac{i}{r\sin\theta} v^+ \partial_\phi v \\ &= -\frac{i}{r\sin\theta}\begin{pmatrix} -ifc & -ifse^{-i\phi} \end{pmatrix}\begin{pmatrix} 0 \\ -fse^{i\phi} \end{pmatrix} = \frac{f^2 \sin\theta}{r} \end{aligned} \tag{5.8}$$

Since $\hat{z} \cdot (\overline{r} \times \hat{\phi}) = r\sin\theta$ (or, equivalently, the distance from the z axis is $r\sin\theta$) we get,

$$\langle L_z \rangle = \int f^2 \sin^2\theta \cdot dV \tag{5.9}$$

This is identical to (5.5) and hence reproduces (5.7) when integrated. Hence the methods of §5.1 and §5.2 agree.

5.3 (Incorrectly) from the expression for the current $j^\mu = q\overline{\psi}\gamma^\mu\psi$

The reader may, like me, be tempted to argue that if $j^\mu = q\overline{\psi}\gamma^\mu\psi$ is the current flux, then surely $\rho^\mu = m\overline{\psi}\gamma^\mu\psi$ must be the mass flux, and all one must do to get the momentum flux in physical units is to multiply by c^2. Thus, I supposed (incorrectly) that the momentum flux must be $p^\mu = mc^2\overline{\psi}\gamma^\mu\psi$, the units being correct (momentum per unit area per unit time). It follows immediately from (2.12) that the ϕ component of this supposed momentum flux would be $p_\phi = 2mc^2 g(r) f(r)\sin\theta$. Multiplying this by the distance from the z axis, $r\sin\theta$, and integrating would therefore give an angular momentum of,

$$\begin{aligned} L_z(\text{spurious}) &= \int 2mg(r)f(r)r\sin^2\theta \cdot dV \\ &= \int 2mg(r)f(r)\sin^2\theta \cdot 2\pi r^3 dr d(\cos\theta) \\ &= \frac{16\pi}{3} m \int f_0 g_0^2 \exp\{-2m\tilde{\alpha}r\} \cdot r^3 dr \\ &= \frac{16\pi}{3} m \frac{\tilde{\alpha}}{2} \frac{I_3(2m\tilde{\alpha})}{4\pi(1+f_0^2)I_2(2m\tilde{\alpha})} \\ &= \frac{16\pi}{3} m \frac{\tilde{\alpha}}{2} \frac{1}{4\pi(1+f_0^2)} \frac{3}{2m\tilde{\alpha}} \\ &= \frac{1}{(1+f_0^2)} \\ &\approx 1 \end{aligned} \tag{5.10}$$

This is most unexpected since the orbital angular momentum must be <<1, the non-zero value merely being a relativistic correction to the non-relativistic result that $l = 0$, an expectation that has been confirmed by (5.7), i.e., that L_z is ~10^{-5} in $\hbar$ units.

Moreover, (5.10) is not even the correct result for the total angular momentum, which is actually $j_z = 1/2$, essentially just the spin.

On the other hand, (5.10) is indeed the angular momentum which would be associated with the magnetic moment of one Bohr magneton for an orbital current as evaluated in (2.2) from the Dirac current, $j^\mu = q\overline{\psi}\gamma^\mu\psi$, i.e., for a gyromagnetic ratio appropriate for an orbital current (i.e., g = 1). However, we conclude that it is spurious to regard $p^\mu = mc^2\overline{\psi}\gamma^\mu\psi$ as a momentum flux and (5.10) is not a physically real angular momentum.

6. What magnetic vector potential, $\overline{A}$, corresponds to ψ?

It should always be borne in mind that the use of (2.1) as a description of the hydrogen atom electron is mathematically inconsistent. Whilst the Coulomb potential is the correct solution of Maxwell's equation for the proton, it ignores the effect of the electron's own charge and current in modifying the electromagnetic field. The Dirac equation for an arbitrary e/m potential is,

$$(i\partial\!\!\!/ - qA\!\!\!/ - m)\psi = 0 \tag{6.1}$$

where $\partial\!\!\!/ = \gamma^\mu \partial_\mu = \gamma^0\partial_0 + \overline{\gamma}\cdot\overline{\nabla}$ and $A\!\!\!/ = \gamma^\mu A_\mu = \gamma^0 V - \overline{\gamma}\cdot\overline{A}$. Assuming the e/m potential A_μ is real, this Dirac equation can be inverted to find A_μ explicitly in terms of the Dirac field, thus,

$$A^\nu = \frac{i\left[\overline{\psi}\gamma^\nu \partial\!\!\!/\psi - \overline{\psi}\overleftarrow{\partial\!\!\!/}\gamma^\nu\psi\right] - 2m\overline{\psi}\gamma^\nu\psi}{2q\overline{\psi}\psi} = \frac{\Re\left(i\overline{\psi}\gamma^\nu\partial\!\!\!/\psi\right) - 2m\overline{\psi}\gamma^\nu\psi}{2q\overline{\psi}\psi} \tag{6.2}$$

For the ground state Dirac field given by (2.2), we find by substitution of (2.2) into (6.2) that $A_r = A_\theta = 0$ but that,

$$qA_\phi = \left(\frac{\frac{2f^2}{r} + g\partial_r g + f\partial_r f + 2mgf}{g^2 - f^2}\right)\sin\theta \tag{6.3}$$

Using the explicit expressions (2.6) for the radial functions, but for simplicity ignoring the small power-of-r factor, i.e., $g = g_0\exp\{-me^2 r\}$ and $f = f_0 g$, $f_0 = \widetilde{\alpha}/2$, we get,

$$\frac{\partial_r g}{g} + \frac{2mf}{g} = -m\widetilde{\alpha} + 2m\frac{\widetilde{\alpha}}{2} = 0 \tag{6.4}$$

Hence, the leading order terms in (6.3) cancel and we are left with,

$$\begin{aligned} qA_\phi &\approx \left(\frac{\frac{2f^2}{r} + f\partial_r f}{g^2}\right)\sin\theta \\ &\approx \left(\frac{2}{r} - m\widetilde{\alpha}\right)\frac{\widetilde{\alpha}^2}{4}\sin\theta \end{aligned} \tag{6.5}$$

But recall that the magnetic vector potential (times q) acts like an additional momentum. The total angular momentum due to this additional linear momentum is obtained by multiplying by the distance from the z axis ($r\sin\theta$) and integrating over volume, remembering to weight by $\psi^+\psi$, thus,

$$\begin{aligned}
L_z(\text{e/m}) &= \int (2 - m\tilde{\alpha}r)\frac{\tilde{\alpha}^2}{4}\sin^2\theta \cdot \psi^+\psi \cdot 2\pi r^2 dr d(\cos\theta) \\
&= \frac{2\pi}{3}\tilde{\alpha}^2 \int (2 - m\tilde{\alpha}r)\cdot r^2 \cdot (1 + f_0^2) g_0^2 \exp\{-2m\tilde{\alpha}r\} dr \\
&= \frac{2\pi}{3}\tilde{\alpha}^2 (1 + f_0^2) g_0^2 [2I_2(2m\tilde{\alpha}) - m\tilde{\alpha}I_3(2m\tilde{\alpha})] \\
&= \frac{2\pi}{3}\tilde{\alpha}^2 \frac{[2I_2(2m\tilde{\alpha}) - m\tilde{\alpha}I_3(2m\tilde{\alpha})]}{4\pi I_2(2m\tilde{\alpha})} \qquad (6.6) \\
&= \frac{1}{6}\tilde{\alpha}^2 \left[2 - m\tilde{\alpha}\frac{3}{(2m\tilde{\alpha})}\right] \\
&= \frac{\tilde{\alpha}^2}{12}
\end{aligned}$$

This is just half as big as the "$-i\bar{r} \times \overline{\nabla}$" angular momentum, (5.7), so is not negligible in comparison - though both are small. (6.6) is additive to the moment from (5.7) because the Lagrange conjugate momentum is $\bar{\pi} = \bar{p} - q\bar{A} = \bar{p} + e\bar{A}$.