Chapter 32

Angular Momentum Eigenstates

How the angular momentum eigenvalues and the corresponding states are derived by purely algebraic methods

Last update: 10/1/12

We start from the commutation relations of the three angular momentum operators,

$$[L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y \tag{1}$$

Note that these same commutation relations would apply for a representation of SO(3) applying to real vectors in 3D space,

$$\overline{L} = \hbar\left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right] \tag{2}$$

or to the self-representation of SU(2) as,

$$\overline{\sigma} = \frac{\hbar}{2}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \tag{3}$$

or to the infinite dimensional representation as differential operators, $\overline{L} = \overline{r} \times \overline{p}$ where $\overline{p} = -i\hbar\overline{\nabla}$. We wish to find the eigenvalues of L_z and $L^2 = L_x^2 + L_y^2 + L_z^2$ by using the commutator properties, (1), only, without appeal to any particular representation.

Note that L_z and L^2 commute,

$$\begin{aligned}[L_z, L^2] &= [L_z, L_x^2 + L_y^2] = L_z L_x^2 - L_x^2 L_z + L_z L_y^2 - L_y^2 L_z \\ &= (L_x L_z + i\hbar L_y)L_x - L_x(L_z L_x - i\hbar L_y) + (L_y L_z - i\hbar L_x)L_y - L_y(L_z L_y + i\hbar L_x) = 0\end{aligned} \tag{4}$$

Consequently there are states which are eigenstates simultaneously of both L_z and L^2. Suppose $|a,b\rangle$ is such as state, where,

$$L^2|a,b\rangle = a|a,b\rangle \qquad L_z|a,b\rangle = b|a,b\rangle \tag{5}$$

Define

$$L_\pm = L_x \pm iL_y \tag{6}$$

Then,

$$[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar L_y \pm i(-i\hbar L_x) = \pm\hbar L_x + i\hbar L_y = \pm\hbar(L_x \pm iL_y) = \pm\hbar L_\pm \tag{7}$$

These operators $L_\pm$ act as "ladder operators" which change one eigenstate into another. This can be seen using (5,7) as follows,

$$L_z L_+|a,b\rangle = (L_+ L_z + \hbar L_+)|a,b\rangle = (bL_+ + \hbar L_+)|a,b\rangle = (b+\hbar)L_+|a,b\rangle \tag{8}$$

So, given an eigenstate $|a,b\rangle$ of L_z with eigenvalue *b*, we can find another eigenstate of L_z namely $L_+|a,b\rangle$ with eigenvalue $(b+\hbar)$. In the same way the operator L_- forms a new eigenstate with eigenvalue $b-\hbar$. A sequence of eigenstates with eigenvalues spaced by $\hbar$ is therefore constructed by repeating these operations, apparently giving,

$$L_+{}^n|a,b\rangle \text{ has eigenvalue } (b+n\hbar); \quad L_-{}^n|a,b\rangle \text{ has eigenvalue } (b-n\hbar) \qquad (9)$$

It follows from $L^2 = L_x^2 + L_y^2 + L_z^2$ that $\langle L^2 \rangle = a \geq \langle L_z^2 \rangle = b^2$. Hence, for a given value of *a* there is an upper bound for *b* such that $b^2 \leq a$. It follows that the process defined by (9) cannot go on for ever because eventually we get to the largest possible value for the eigenvalue. So where does the reasoning leading to (9) break down? The only possibility is that, for some largest possible *b* we get $L_+|a,b_{\max}\rangle = 0$ and the process terminates. Now consider,

$$L_-L_+ = (L_x - iL_y)(L_x + iL_y) = L_x^2 + L_y^2 + i[L_x, L_y] = L^2 - L_z^2 - \hbar L_z \qquad (10)$$

Hence, $$L_-L_+|a,b\rangle = (L^2 - L_z^2 - \hbar L_z)|a,b\rangle = (a - b^2 - \hbar b)|a,b\rangle \qquad (11)$$

So, if we choose $b = b_{\max}$ in (11), we must have $a - b_{\max}^2 - \hbar b_{\max} = 0$, i.e.,

$$a = b_{\max}(b_{\max} + \hbar) \qquad (12)$$

By the same reasoning there is a minimum *b* so that $L_-|a,b_{\min}\rangle = 0$ and so consider,

$$L_+L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 + L_y^2 - i[L_x, L_y] = L^2 - L_z^2 + \hbar L_z \qquad (13)$$

This gives, $$a = b_{\min}(b_{\min} - \hbar) \qquad (14)$$

But we also know that the eigenvalues are spaced apart by $\hbar$, so that the maximum and minimum values for *b* must differ by an integral multiple of $\hbar$, i.e., there must be a positive integer *n* such that $b_{\max} - b_{\min} = n\hbar$. Equating (12) and (14) and using this relation between $b_{\max}$ and $b_{\min}$ gives,

$$a = b_{\max}(b_{\max} + \hbar) = (b_{\max} - n\hbar)(b_{\max} - n\hbar - \hbar) \qquad (15)$$

Re-arranging and simplifying this gives simply,

$$b_{\max} = \frac{n}{2}\hbar \qquad \text{and} \qquad b_{\min} = -\frac{n}{2}\hbar \qquad (16)$$

Substitution in (12) gives $$a = \frac{n}{2}\left(\frac{n}{2} + 1\right)\hbar^2 \qquad (17)$$

Hence, we find that any system of operators obeying the commutator algebra (1) has total angular momentum given by $L^2 \to l(l+1)\hbar^2$, where $l = n/2$ can only take half-integral values, 0, ½ , 1, $^3/_2$, 2,…etc.

The possible azimuthal quantum numbers are $m = b/\hbar = -l, -l+1, \ldots 0, \ldots (l-1), +l$
Hence, there are $(2l+1)$ azimuthal states for a given *l*. For example, for *l* = 2 there are 5 azimuthal states.

Spin states can take any of these possible values, i.e., *l* = 0, ½ , 1, $^3/_2$, 2,…etc.

Orbital angular momentum states can only take integral values, *l* = 0, 1, 2,…etc.