Chapter 31

If the Universe is Expanding Why Can I Never Find a Parking Space?

No, it's not because your car is getting bigger too.

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It is likely that anyone who has thought about universal expansion at all has, at some time, been puzzled by this: why is the solar system not expanding also? Indeed, why is the Earth itself and objects on a still smaller scale, including your own body, not expanding? It would be silly to imagine that everything was expanding in proportion. Such an expansion would not be detectable, and hence would be inconsistent with the observed cosmic expansion. Cosmic expansion means that the ratio of the size of the observable universe to the length of the metre rule in my tool box is increasing. But the ratio of the size of the solar system (or the Earth) to my metre rule is not increasing – not due to cosmic expansion, anyway. (Of course the orbits of the planets are no doubt varying slowly for purely local reasons).

My first thought on this problem was that the Jeans mass (or Jeans length) would be relevant. Actually it is not. Masses or regions exceeding the Jeans scale may gravitationally collapse without benefit of a cooling mechanism. However, gravitational collapse can occur on smaller scales provided that there is a cooling mechanism. This is what leads to star formation. At mass/size scales either greater or less than the Jeans scale gravitationally bound systems can occur, and it is their bound nature which is significant in neutralising the effects of cosmic expansion locally.

In fact the binding need not be gravitational. Take a single atom, for example. The size of a hydrogen atom is set by the Bohr radius,

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} \tag{1}$$

This is derived by solving the Schrodinger equation. If it were true that cosmic expansion was causing atoms to become larger, and if we retained our local dynamics (i.e., the usual Schrodinger equation), we would be forced to conclude that at least one of the universal 'constants' $m_e, e, \hbar$ was changing (noting that ε_0 just determines the units used for charge). It is physically fairly obvious that the cosmic expansion will not affect atomic size simply because the electrostatic force which binds atoms is so very much stronger than the gravitational force (by a factor of $\sim 10^{39}$). The same applies to rocky planets, for the same reason: solids are bound by chemical bonds which are electrostatic in nature.

However if we consider a sufficiently large piece of the universe we must eventually get to a scale which is expanding. To investigate where the boundary between local non-expansion and global expansion lies we shall follow the simple approach of Price and Romano (2012). These authors considered a "classical atom" under the influence of cosmic expansion. That is they considered a charge in the attractive electrostatic field of an opposite charge together with a force equivalent to the influence of cosmic expansion, treating the problem classically. In the non-relativistic case we might just as well regard the attractive electrostatic force as replaced by a gravitational force. Ignoring the cosmic expansion for a moment, the equation of motion is,

$$\ddot{\vec{r}} = -\frac{C}{r^2}\hat{r} \tag{2}$$

where $C = GM$ and M is the mass at the origin causing the gravitational attraction experienced by our trajectory particle. Being a central force the angular momentum is a constant of the motion (see Chapter 15),

$$L = r^2\dot{\phi} \qquad \text{and} \qquad \dot{L} = 0 \tag{3}$$

where L is the angular momentum per unit trajectory mass and ϕ is the angular coordinate subtended at the origin in the plane of motion. The radial part of the equation of motion reduces to,

$$\ddot{r} = -\frac{C}{r^2} + \frac{L^2}{r^3} \tag{4}$$

The second term on the RHS being simply the usual centrifugal acceleration often denoted $r\omega^2$, where $\omega \equiv \dot{\phi}$.

How are we to incorporate the effects of cosmic expansion into the equation of motion? The expansion can be written,

$$r = a(t)R \tag{5}$$

Here R is the radial coordinate embedded in the Hubble flow. A particle which was co-moving with the cosmic expansion would have a constant *R* coordinate. Its *r* coordinate, on the other hand, would be increasing, $\dot{a} > 0$. The acceleration of a co-moving particle as seen in the local *r*-coordinate frame would be,

$$\ddot{r} = \ddot{a}R = \frac{\ddot{a}}{a}r \tag{6}$$

In order to reproduce this acceleration in the *r*-coordinate when distant from the source of gravity at the origin, Price and Romano suggest that it is plausible to replace the equation of motion (4) with,

$$\ddot{r} = -\frac{C}{r^2} + \frac{L^2}{r^3} + \frac{\ddot{a}}{a}r \tag{7}$$

The last term can be interpreted as the outward tug on the particle due to the cosmic expansion. You might reasonably be concerned about this *ad hoc* adjustment to the equation of motion. However Price and Romano show via a fully relativistic treatment (i.e., the Friedman equation) that it is valid in the non-relativistic limit.

We are not particularly concerned about what the actual cosmic expansion, i.e., the function $a(t)$, might be. The principle regarding whether or not our test particle shares the expansion is presumably not strongly dependent upon the specific expansion function. A simplification is achieved by assuming an exponential expansion rate,

$$a(t) = \exp\left\{\frac{t}{T_u}\right\} \tag{8}$$

Such a de Sitter cosmology might actually apply during an inflationary epoch, or at times such that dark energy has become dominant. However its appropriateness for our universe is not really the issue. Substituting (8) into (7) gives an equation of motion with no explicit time dependence,

$$\ddot{r} = -\frac{C}{r^2} + \frac{L^2}{r^3} + \frac{r}{T_u^2} \tag{9}$$

This is analogous to motion in a potential which depends upon the radial coordinate *r* alone. The "energy" (per unit trajectory mass) is,

$$E = \frac{\dot{r}^2}{2} + V(r) \quad \text{where, } V(r) = \frac{L^2}{2r^2} - \frac{C}{r} - \frac{r^2}{2T_u^2} \tag{10}$$

where V is an effective potential including the influence of cosmic expansion. The equation of motion, (9), is readily seen to imply that this "energy" is a constant of the motion, and *vice-versa*.

Ignoring the cosmic term for a moment, and considering circular motion of radius a, Equ.(3) relates the period of the motion, T_p, to the angular momentum $L = 2\pi a^2 / T_p$. But for circular motion $\dot{r} = \ddot{r} = 0$ so that (4) gives us $a = L^2 / C$. Hence we get the period of the motion, unperturbed by any cosmic influence, in terms of L and C,

$$T_p = 2\pi \frac{L^3}{C^2} \tag{11}$$

Define a dimensionless coordinate and dimensionless potential by,

$$\tilde{r} = \frac{C}{L^2} r \qquad \text{and} \qquad \tilde{V} = \frac{L^2}{C^2} V \tag{12}$$

The dimensionless potential is thus,

$$\tilde{V}(r) = \frac{1}{2\tilde{r}^2} - \frac{1}{\tilde{r}} - \xi^2 \frac{\tilde{r}^2}{2} \tag{13}$$

where,

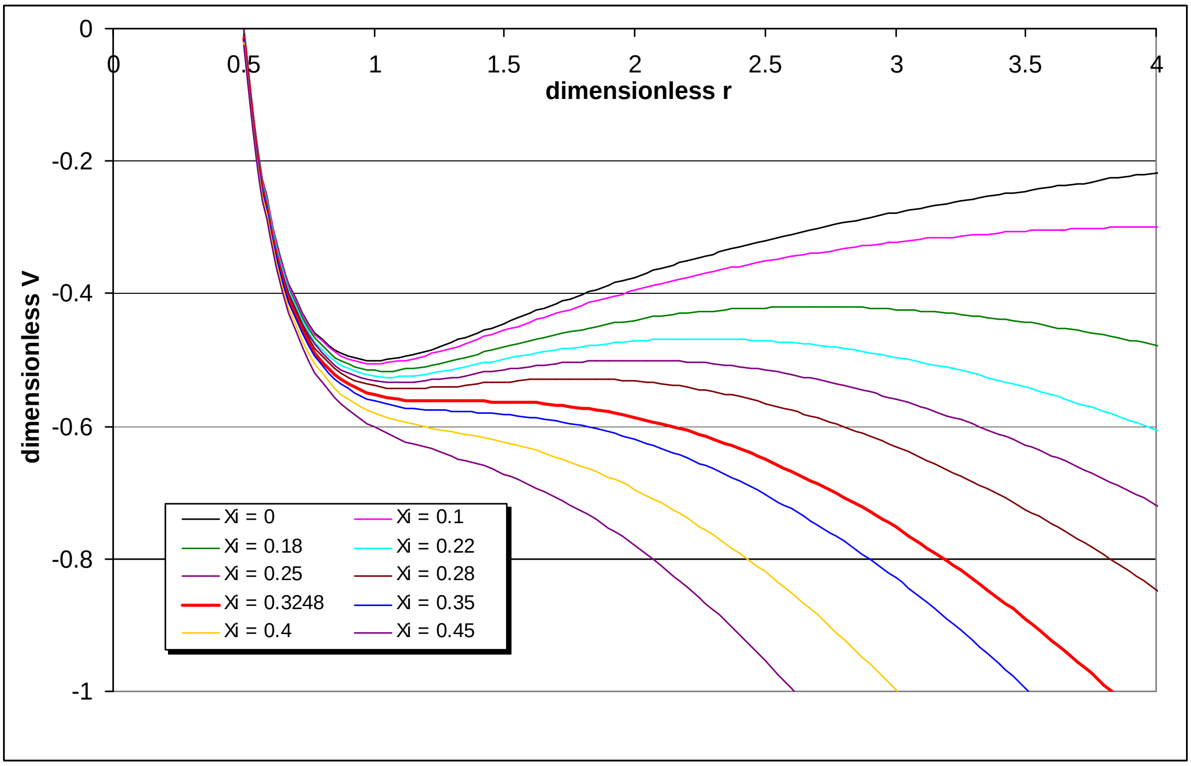
$$\xi = \frac{T_p}{2\pi T_u} \tag{14}$$

Since the dimensionless coordinate $\tilde{r}$ is of order unity, it is clear from (13) that the cosmic expansion will have little influence on the local motion if $\xi << 1$. From (14) it follows that this condition is met so long as the local dynamical timescale is short compared with the age of the universe (which also determines the universal expansion rate). This is not an onerous requirement! So without further detailed analysis it is fairly clear that universal expansion will affect dynamics only at the largest scales, for which evolution is on a similar timescale to that of the universe as a whole.

This can be examined a little more closely by considering the qualitative form of the potential function, (13). It is plotted for several values of ξ in Figure 1. The turning points can be found by equating the derivative of (13) to zero, giving,

$$-1+\tilde{r}-\xi^2\tilde{r}^4=0 \qquad (15)$$

Figure 1 Plot of the dimensionless potential function, Equ.(13), against dimensionless radial coordinate for various values of ξ



Hence, for $\xi=0$, when there is no cosmic influence, the minimum is at $\tilde{r}=1$ and has the value $\tilde{V}=-0.5$.

For $\xi=0.25$ the minimum is at $\tilde{r}=1.087$ and has the value $\tilde{V}=-0.534$. This case also has a maximum at $\tilde{r}=2$ and has the value $\tilde{V}=-0.5$.

The minimum and maximum of $\tilde{V}$ move closer together as ξ is increased, ultimately merging into a single point of inflection. This point can be solved by equating the second derivative of $\tilde{V}$ to zero, i.e., the derivative of (15), $1-4\xi^2\tilde{r}^3=0$, simultaneously with (15). This gives $\tilde{r}=4/3$ and,

$$\xi_{crit}=\frac{3\sqrt{3}}{16}=0.3248... \qquad (16)$$

For $\xi > \xi_{crit}$ there is no turning point of $\tilde{V}$.

These qualitative features of the potential function allow us to discern simply the dynamic behaviour of our test particle. For any case with $\xi < \xi_{crit}$, and hence with a minimum in $\tilde{V}$, suppose the initial condition is that the particle has zero radial velocity and is at a point r_0 such that $\tilde{V}(r_0) < \tilde{V}_{\max}$ where $\tilde{V}_{\max}$ is the maximum turning point. The subsequent motion can be visualised as akin to a marble released from r_0 in a container with the shape of $\tilde{V}(\tilde{r})$. The marble rolls backwards and forwards between points r_0 and r_1 where the latter is such that $\tilde{V}(r_1) = \tilde{V}(r_0)$. The actual motion is in a roughly elliptical orbit, where r_0 and r_1 are the semi-minor and semi-major axes. In the case that the starting point is at the minimum of the potential, the radial coordinate remains constant and the orbit is a circle.

The detailed trajectory of the particle will inevitably be affected to some degree by the cosmic expansion, i.e., by the ξ -dependent term in the equation of motion. However for $\xi < \xi_{crit}$, and for the starting conditions described above, the important conclusion is that there is a fixed constant orbit which does not expand over time. The particle trajectory does not share the cosmic expansion ***at all***. The local gravity wins over the cosmic term.

However, if $\xi > \xi_{crit}$ there is no maximum or minimum of the potential. In this case the cosmic term wins over the local gravity and there is no orbital behaviour. The marble drops down the infinite slope in Figure 1 and disappears to infinity, i.e., our particle expands with the universe.

This simple illustration shows that what does *not* happen is that we get an almost closed stable orbit which slowly expands under the influence of the cosmic expansion. Instead its all-or-nothing: there is either a fixed, constant orbit which does not expand, or the particle becomes entrained in the Hubble flow and does not orbit at all.

The parameter ξ can also be expressed as a ratio of densities. For a uniform density of material, ρ , the period under the action of the local gravity alone is,

$$T_p = \sqrt{\frac{3\pi}{G\rho}} \tag{17}$$

The age of the universe in terms of the critical density, ρ_c , is,

$$T_u = \sqrt{\frac{3}{8\pi G\rho_c}} \tag{18}$$

Hence (14) becomes,

$$\xi = \sqrt{\frac{2\rho_c}{\rho}} \tag{19}$$

The critical density is equivalent to less than 6 hydrogen atoms per cubic metre. Consequently for all but the largest structures in the universe it is clear that we will easily be within the regime $\xi < \xi_{crit} = 0.3248$. For example, despite the remarkably low mean density of a typical galaxy (perhaps only ~10^{-20} kgm^{-3}, due to the sparsity of stars within the galactic volume) it is still about 6 orders of magnitude greater than the critical density. We conclude that galaxies will not be expanding with the universe.

However, at the level of galactic superclusters this changes. At this scale it is no longer clear that we are in the $\xi < \xi_{crit}$ regime. This is as it should be because superclusters must share the cosmic expansion. The largest scale structures in the observable universe are vast "bubbles" of almost completely empty space. Cosmic expansion consists of all these voided "bubbles" inflating further. The superclusters are the "bubble boundaries". Consequently, cosmic expansion can be envisaged as these bubble walls, the superclusters, being stretched more and more. At this scale, matter is affected by the cosmic expansion. The overwhelming majority of the mass of superclusters resides in the galactic clusters and groups of which they are composed. These are generally taken to be the largest gravitationally bound structures in the universe. What is not clear (to me) is whether galactic clusters are affected by the cosmic expansion, since their mean density is also very low. However their gravitationally bound nature suggests not.

In conclusion, cosmic expansion would help you with your parking problem only if your car park were comparable to a supercluster in size, ~100 Mpc.

References

R.H.Price and J.D.Romano (2012) "In an expanding universe, what doesn't expand?", arXiv:gr-qc/0508052v2 submitted to American Journal of Physics 2012.