Chapter 30

The Unavoidable Weirdness

EPR, Hidden Variables and Bell's Inequality

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1. The EPR Paradox

Einstein famously did not like the indeterminacy of quantum mechanics. He felt the theory was incomplete. The paper by Einstein, Podolski and Rosen (EPR, 1935) purported to show that either quantum mechanics was incomplete or that relativistic causality was violated. The essence of the argument, though not expressed in this way by EPR, is captured as follows.

Suppose an atom or particle with zero angular momentum decays into a pair of particles which have two spin states, such as spin ½ particles with mass, or alternatively massless spin 1 particles. These particles then move away from each other at great speed. Suppose also that due to some selection rule, or by some other means, we know that the pair of particles are created in an S-wave state, i.e., without orbital angular momentum. It follows that the particles' spin state must be the singlet state of spin zero, which we can write as $\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2-|\downarrow\rangle_1|\uparrow\rangle_2\right)$. This means that, as we would have expected, if we measure particle 1 to be "spin up" then we will certainly measure particle 2 to be "spin down", and vice-versa. But we have no way to tell in advance of an individual measurement which of these two outcomes will be found.

So far there is no problem apparent. The difficulty occurs when we note that the measurements on the two particles can be arranged to be at a spacelike separation, so that no causal connection between them is possible. But quantum mechanics would have us believe that before the measurement on particle 1, the spin state of particle 2 is a superposition of both spin up and spin down – as indicated by the Hilbert state $\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2-|\downarrow\rangle_1|\uparrow\rangle_2\right)$. And yet, after the measurement on particle 1 has been carried out, which could result in either an up or down spin, the outcome of a measurement on particle 2 becomes determined despite no causal connection between them being possible. EPR argued that either a faster-than-light interaction occurs or quantum mechanics must be incomplete. The implication appears to be that there must be some "hidden variable" carried by each particle which determines the spin that will be registered for each particle separately. Otherwise how could the second particle contrive to always have the opposite spin to the first, if the spin of the first were not decided until it is measured and that measurement is at a space-like separation from the measurement on the second particle? On the face of it the argument's a good 'un – an Einstein special.

2. Hidden Variables – The Algebraic Years

The counter to EPR from the likes of Bohr and Born was that, in fact, there *is* no violation of causality and hence that quantum mechanics is unobjectionable. The reason is that the correlation between measurement outcomes at the two spacelike separated events cannot be used to send any information. So nothing acausal happens. If one were able to influence the outcome of the measurement of particle 1, then this *would* constitute an acausal connection with particle 2, since it would provide a means of faster-than-light communication. But the point is that the very indeterminacy of the measurement of particle 1 prevents any such signalling to particle 2.

If you are not comfortable with this counter-argument, you are in good company. The EPR paradox has been the subject of intense debate for over 75 years now. However, the existence of "hidden variables" is even less credible as the resolution of the paradox now than in 1935.

For several decades after the dawn of quantum theory, the Copenhagen interpretation was dominant. In part this was due to the patriarchal influence of Bohr. But the Copenhagen interpretation was also buttressed by von Neumann's purported proof that hidden variables could not exist. This proof was published in 1932, before the EPR paper, so either EPR were unimpressed by it, or else they were unaware of it. The latter seems unlikely, although Einstein had other things on his mind in the period 1932-35, such as fleeing from Nazi Germany and his wife's ultimately fatal illness. Von Neumann's 'proof' is fatally flawed, as pointed out by Bell (1966), as well as by other people much earlier. It is worth examining what went wrong with it. Rather than reproducing von Neumann's original 'proof' we shall use the much simpler argument of Bell (1966) based on the same false premise.

The false premise is that the expectation value of a linear combination of observables equals the same linear combination of their individual expectation values. Expressed algebraically, $\langle aP + bQ \rangle = a\langle P \rangle + b\langle Q \rangle$. In other words, "expectation value" is a homomorphism. Quantum mechanics has this property, of course, because it is simply a re-writing of the linearity of the Hilbert space operators. But von Neumann imposed this condition on the hypothetical hidden variable theories as well. Now it is easy to show via a counter-example that this eliminates deterministic (i.e., hidden variable) theories. Consider a spin ½ particle and the observable $\hat{Q} = \overline{\beta} \cdot \overline{\sigma}$, where $\overline{\sigma}$ are the Pauli matrices and $\overline{\beta}$ is an arbitrary real 3-vector. Upon measurement, this observable[1] can only take the value $|\overline{\beta}|$ or $-|\overline{\beta}|$, where $|\overline{\beta}| = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2}$. For a given fully specified state, if a deterministic theory existed then the expectation value would either be $|\overline{\beta}|$ or $-|\overline{\beta}|$ depending on the state, since one or other of these results would be definite. But this contradicts the homomorphism requirement, since this requires that the expectation value

[1] This observable must be understood to mean, "measure the spin in direction $\hat{\beta}$". If you insist on expanding $\overline{\beta} \cdot \overline{\sigma}$ in terms of a sum over its x, y and z components then you will have trouble seeing what the possible measurement outcomes might be. This is because the eigenvectors of the sum are not eigenvectors of any of the three terms (in general), because the three terms do not commute. Actually, this is precisely the point being made. The possible measurement outcomes are the eigenvalues of $\overline{\beta} \cdot \overline{\sigma}$ which are given by $\left\| \begin{matrix} \beta_z - \lambda & \beta_x - i\beta_y \\ \beta_x + i\beta_y & -\beta_z - \lambda \end{matrix} \right\| = 0$ which readily yields $\lambda = \pm|\overline{\beta}|$

be linear in each component of $\overline{\beta}$, namely $\langle \hat{Q} \rangle = \overline{\beta} \cdot \langle \overline{\sigma} \rangle$. Von Neumann concluded that deterministic states, i.e. hidden variables, could not exist. This was an unjustified claim. Whilst it follows rather trivially from $\langle aP + bQ \rangle = a\langle P \rangle + b\langle Q \rangle$, this is an unreasonable restriction. Actually what von Neumann demonstrated was that, if deterministic, hidden variable, theories existed, then their expectation values for fully defined states could not respect this homomorphism property.

Bell (1966) demonstrated that there was no call to assume that deterministic states with specified hidden variables would obey the homomorphism property. He constructed a simple deterministic model in terms of a hidden variable λ which always produced one of the outcomes $|\overline{\beta}|$ or $-|\overline{\beta}|$ on measurement, and hence clearly violated the homomorphism condition as regards expectation values for fully specified states, i.e. for a given value of λ. But, on averaging over all possible values for λ, results for the expectation values in agreement with quantum mechanics were obtained. Thus, the homomorphism property holds once the hidden variables have been averaged-out. Writing the quantum state as $|\psi\rangle$, and the deterministic state in the hidden variable theory as $|\psi,\lambda\rangle$, then,

Quantum mechanics: $\langle \psi | \overline{\beta} \cdot \overline{\sigma} | \psi \rangle = \overline{\beta} \cdot \overline{s}$, where $\overline{s}$ is the real vector $\overline{s} = \langle \psi | \overline{\sigma} | \psi \rangle$ (1)

Hidden variables: $\langle \psi,\lambda | \overline{\beta} \cdot \overline{\sigma} | \psi,\lambda \rangle = |\overline{\beta}|$ or $-|\overline{\beta}|$ depending on λ (2)

but, $\int \langle \psi,\lambda | \overline{\beta} \cdot \overline{\sigma} | \psi,\lambda \rangle d\lambda = \overline{\beta} \cdot \overline{s}$ (3)

Bell (1966) gives an explicit deterministic rule for choosing the sign on the RHS of (2) for a given state, including the value of λ. From this the result (3) is shown to follow. Hence the hidden variable theory agrees with quantum mechanics, in the limited sense of reproducing (3), despite having deterministic outcomes, (2), for individual measurements. [NB: It may look odd that the RHS of (3) is in terms of the spin direction, $\overline{s}$, because this information seems to have been lost on the RHS of (2). However, $\overline{s}$ is actually codified in the choice of sign on the RHS of (2)].

Do not get the wrong idea. This is not a viable hidden variable theory. We shall see below that there is no local hidden variable theory which can reproduce all the predictions of quantum mechanics. What Bell achieved with the above model was a firm refutation of von Neumann's *argument* for the non-existence of hidden variables.

In the 1966 paper Bell also discusses alternative algebraic 'proofs' that hidden variable theories cannot exist, due to Jauch and Piron (1963) and a corollary of a result by Gleason (1957). He shows that both of these suffer from a similar defect. Innocent looking algebraic conditions are assumed to hold for the candidate, deterministic, hidden variable theories which are unduly restrictive. These conditions hold in quantum theory, and hence are required to hold for averages over the hidden variables, but there is no reason to assume they hold for fully specified deterministic states (i.e., for specified λ). On the contrary, there are physical reasons why this should not be expected, as Bell discusses.

In 1967, Kochen and Specker proved a truly remarkable theorem. Assume that observables can be represented by self-adjoint operators in Hilbert space. Assume that a unique number, the 'possessed value', can be assigned to every observable, *Q*. Call it *Value(Q)*. So this is effectively saying that there is a deterministic underlying theory. And finally assume that taking the "*Value*" commutes with functions, $Value(f(Q)) = f(Value(Q))$. Kochen and Specker then showed that, for Hilbert spaces of dimension greater than two, these assumptions would result in a contradiction. In other words, it is not possible for self-adjoint operators on Hilbert space to be assigned unique numerical values which also respect the functional composition property $Value(f(Q)) = f(Value(Q))$. For the proof see Kochen and Specker (1967) or alternatively Redhead (1987).

3. Bell's Inequality and the Experimental Refutation of Hidden Variables

Curiously, Bell's demolition of the algebraic 'proofs' against hidden variables paved the way for what became a far more convincing refutation. He derived an inequality involving measurement outcomes which any local, deterministic hidden variable theory must obey but which is not obeyed by quantum mechanics. Thus the experimental confirmation of the quantum mechanical prediction provides a refutation of all local deterministic hidden variable theories. Ultimately hidden variables have been killed (or at least fatally wounded) by experiment.

Bell (1964) considered the EPR situation, as described above. A pair of spin ½ particles emerge from the decay a spinless precursor, and hence in the singlet state. Bell envisages the spin of one particle being measured in a direction given by unit vector $\hat{a}$, and the spin of the other particle being measured in direction $\hat{b}$. These vectors can be oriented arbitrarily in 3D space. At issue is the correlation between the two spin measurements, for which it suffices to consider the expectation value of their product. The quantum mechanical expectation value of the product is,

$$\langle (\bar{\sigma}_1 \cdot \hat{a})(\bar{\sigma}_2 \cdot \hat{b}) \rangle = -\hat{a} \cdot \hat{b} = -\cos\theta_{ab} \tag{4}$$

In the particular case that $\hat{a} = \hat{b}$ this produces an expectation value for the product of -1, i.e. the spins are always opposed. (NB: we are measuring spins in units of $\hbar/2$).

Bell argued that if the outcome of an individual spin measurement is determined by a hidden variable λ, then the first particle could be predicted with certainty to have a spin of $A(\hat{a}, \lambda) = \pm 1$ if λ were known. Similarly, the second particle also has determinate spin, $B(\hat{b}, \lambda) = \pm 1$. The expectation value of the product of spins, averaged over many measurements, is thus, according to the hidden variable theory,

$$E(\hat{a}, \hat{b}) = \int A(\hat{a}, \lambda) B(\hat{b}, \lambda) \rho(\lambda) d\lambda \tag{5}$$

where $\rho(\lambda)$ is some probability density of the hidden variable so that $\int \rho(\lambda) d\lambda = 1$.Bell (1964) shows that (5) is inconsistent with (4). More generally, he derives an inequality which must be respected by any such hidden variable theory, i.e.,

$$1 + E\left(\hat{b}, \hat{c}\right) \geq \left|E\left(\hat{a}, \hat{b}\right) - E(\hat{a}, \hat{c})\right| \tag{6}$$

But the quantum mechanical expectation value, (4), does not obey (6). For example, consider $\hat{a}$ and $\hat{c}$ to be perpendicular with $\hat{b}$ at 45^o to both. Then, assuming the quantum mechanical result (4), the inequality would require,

$$1 - \frac{1}{\sqrt{2}} \geq \left| -\frac{1}{\sqrt{2}} - 0 \right| = \frac{1}{\sqrt{2}}$$

But the LHS = 0.292 whereas the RHS = 0.707, so the inequality is clearly false. Note that it is not even a close miss. The quantum expectation value disrespects the Bell inequality quite radically. This is important because it means that experiments to discriminate between the two need not necessarily be of very great precision.

The proof of Bell's inequality, (6), is remarkably simple. Any theory must predict that the two particles have opposite spins when measured in the same direction since this is required by the conservation of angular momentum. Hence we have,

$$B(\hat{a}, \lambda) = -A(\hat{a}, \lambda) \tag{7}$$

So that (5) becomes, $E\left(\hat{a}, \hat{b}\right) = -\int A(\hat{a}, \lambda) A\left(\hat{b}, \lambda\right) \rho(\lambda) d\lambda$ (8)

Hence,

$$\begin{aligned} E\left(\hat{a}, \hat{b}\right) - E(\hat{a}, \hat{c}) &= -\int A(\hat{a}, \lambda) A\left(\hat{b}, \lambda\right) \rho(\lambda) d\lambda + \int A(\hat{a}, \lambda) A(\hat{c}, \lambda) \rho(\lambda) d\lambda \\ &= \int A(\hat{a}, \lambda) A\left(\hat{b}, \lambda\right) \left[A\left(\hat{b}, \lambda\right) A(\hat{c}, \lambda) - 1\right] \rho(\lambda) d\lambda \end{aligned} \tag{9}$$

because $A\left(\hat{b}, \lambda\right)^2 = 1$. But the value of $A(\hat{a}, \lambda) A\left(\hat{b}, \lambda\right)$ can only be ± 1 so that (9) implies,

$$\left|E\left(\hat{a}, \hat{b}\right) - E(\hat{a}, \hat{c})\right| \leq \int \left(1 - A\left(\hat{b}, \lambda\right) A(\hat{c}, \lambda)\right) \rho(\lambda) d\lambda \tag{10}$$

Using (8) this gives, $\left|E\left(\hat{a}, \hat{b}\right) - E(\hat{a}, \hat{c})\right| \leq 1 + E\left(\hat{b}, \hat{c}\right)$ (11)

which is Bell's inequality, (6). QED.

Stronger inequalities than (6) have now been proved also, and forms of inequality which are better suited for certain experimental arrangements. However, (6) suffices to demonstrate the force of the strategy which permits experiments to refute local deterministic hidden variable theories rather than algebra alone.

The literature on Bell-type inequalities is large, as is the number of experiments to test such inequalities. The purist would claim, quite rightly, that the conclusion from these experiments is not strictly conclusive. Experimental loopholes exist through which local realistic hidden variable theories might still escape final discreditation. However experiments are constantly restricting the wriggle room more and more. The large majority of experiments violate a Bell-type inequality and are consistent with quantum mechanical expectations. Those few experiments which are exceptions have tended to come under suspicion of inaccuracy or have not proved reproducible. Although now rather out of date, Redhead (1987) presents a Table of experimental tests. Despite the loopholes, few people now expect local realistic hidden variable theories to survive.

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