Chapter 28

Sins of Intention

Can a Null Measurement Collapse the Wavepacket?

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The term "null measurement" refers to an experiment which includes an active measuring device but in which the device registers no measurement. Can a null measurement collapse the wavepacket? The answer is "yes", though it may be only a partial collapse. Note that wavepacket collapse by a null measurement is essentially identical to decoherence without an associated change in the state of the environment (which is illustrated in Chapter 22). The only difference is that the 'environment' is being regarded as a measuring device. Hence if its state does not change then it registers no measurement. The principle is so important that it bears repeating in this different guise. These phenomena are also closely related to the observability of counterfactuals, as illustrated in Chapter 8 by the Elitzur-Vaidman bomb test and the possibility of counterfactual computation. Indeed, a null measurement *is* a counterfactual.

The *gedanken* experiment to be considered has been attributed to Renninger, Ref.[1]. Suppose the decay of a radioactive nucleus gives rise to a particle (or radiation) in a spherically symmetric state. The particle may be emitted in any direction with equal probability. Imagine that we have arranged a large number of particle detectors around the surface of some sphere of radius R, with the radioactive source at its centre. For simplicity we shall assume the detectors cover all 4π steridians of solid angle and are 100% efficient, so that we can guarantee that an emitted particle will be detected in one of them. The spherically symmetric wavefunction representing the emitted particle can be considered as the sum of a large number, N, of isotropically disposed plane waves,

$$|\psi\rangle = \sum_{i=1}^{N} |u_i\rangle \qquad (1)$$

(see Appendix). Note that the phase relationship between the contributing plane waves in (1), namely no phase difference, is crucial to the sum being equivalent to a spherical S-wave. The individual states in (1) are normalised to $\langle u_i | u_i \rangle = 1/N$ and are orthogonal so that (1) represents just one particle.

The detectors can be considered as two-state devices, registering 0 when no particle has been detected or 1 when a particle has been detected. Since we shall consider only one particle to be present at a time, at most one detector can register a particle before the whole apparatus is reset (prior to a subsequent event). The state of the set of all detectors is therefore written $|ei\rangle$ where $i = 0$ means that no detectors have yet registered a particle, whereas $i = 1$ means that detector 1 (only) has registered a particle, $i = 2$ means that detector 2 (only) has registered a particle, etc. Moreover we have labelled the detectors so as to correspond to the labelling of the plane waves in (1), so that plane wave No.1 propagates into detector No.1, etc.

Suppose that the particle is emitted at time 0 and that it takes some time t_1 for the particle to travel from its source to the detectors. Then for times $0 < t < t_1$ the state of the combined particle-plus-apparatus is,

$$|\Psi\rangle = |\psi\rangle|e0\rangle \tag{2}$$

because the particle and the detectors have not yet had opportunity to interact. The reduced density matrix of the system is trivially just the system's density matrix at this time since there is only one 'environment state' to trace out, and it is given by,

$$\hat{\rho}_{sys}(0) = |\psi\rangle\langle\psi| = \left(\sum_i |u_i\rangle\right)\left(\sum_j \langle u_j|\right) = \sum_{i,j} |u_i\rangle\langle u_j| \tag{3}$$

In matrix notation the density matrix is thus,

$$(\rho_{sys}) = \frac{1}{N}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & 1 & 1 & \dots \\ & & & & & etc \end{pmatrix} \tag{4}$$

where N is the dimension of the matrix, i.e., the number of detectors and the number of terms in Equ.(1). The off-diagonal terms in (4) are witness to the coherence of the initial state.

At time t_1, when the interaction has occurred, the state evolves into,

$$|\Psi\rangle \to |\Psi'\rangle = \sum_i |u_i\rangle|ei\rangle \tag{5}$$

The density matrix representing the combined particle-plus-apparatus is thus,

$$\hat{\rho} = \left(\sum_i |u_i\rangle|ei\rangle\right)\left(\sum_j \langle u_j|\langle ej|\right) = \sum_{i,j} |u_i\rangle\langle u_j| \otimes |ei\rangle\langle ej| \tag{6}$$

The reduced density matrix representing the particle alone is obtained by tracing out the environment states, which here are represented by the detectors themselves. Hence,

$$\hat{\rho}^{red} = \sum_{ek} \langle ek|\hat{\rho}|ek\rangle = \sum_{i,j,k} |u_i\rangle\langle u_j| \otimes \langle ek|ei\rangle\langle ej|ek\rangle = \sum_{i,j,k} |u_i\rangle\langle u_j|\delta_{ki}\delta_{jk} = \sum_k |u_k\rangle\langle u_k| \tag{7}$$

See how different is (7) from (3). There are no longer any diagonal components. In matrix notation we have,

$$(\rho_{sys}) = \frac{1}{N}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ & & & & & etc \end{pmatrix} \tag{8}$$

Entanglement with the environment followed by tracing out the environment states results in decoherence. The particle is now in a classical mixed state. The transition from (4) to (8) is ***part of*** what is referred to as the collapse of the wavepacket. (The other part is the selection of one the particular states or detectors).

So far, so good: this is just standard decoherence theory. Things get more interesting, though, if the apparatus is modified by moving one hemisphere of detectors to a different radius. We now consider the left hemisphere to be at a radius R_1, with a time of flight t_1, but the right hemisphere is at a radius $R_2 > R_1$ with time of flight $t_2 > t_1$. The initial state, that is for $0 < t < t_1$, is still as given by (2-4). Similarly, the final state for $t > t_2$, is still as given by (5-8)[1]. The question is: what is the state of the particle for $t_1 < t < t_2$, after interaction with the left hemisphere has happened, but interaction with the right hemisphere has not yet happened?

The question is addressed by considering separately the sums over states/detectors in the left and right hemispheres. At the time in question, $t_1 < t < t_2$, the unitary evolution of the initial state has brought only left hemisphere particle states into alignment with left hemisphere detectors, whereas right hemisphere states are still associated with the null reference state of the detectors, i.e.,

$$\left|\Psi\right\rangle \to \left|\Psi'\right\rangle = \sum_{i\in L}\left|u_i\right\rangle\left|ei\right\rangle + \left(\sum_{i\in R}\left|u_i\right\rangle\right)\left|e0\right\rangle \tag{9}$$

where the first sum on the RHS of (9) is over the left hemisphere whereas the second sum is over the right hemisphere. The density matrix representing the combined particle-plus-apparatus is thus,

$$\begin{aligned}\hat{\rho} = &\sum_{i,j\in L}\left|u_i\right\rangle\left\langle u_j\right|\otimes\left|ei\right\rangle\left\langle ej\right| + \left(\sum_{i,j\in R}\left|u_i\right\rangle\left\langle u_j\right|\right)\left|e0\right\rangle\left\langle e0\right| + \\ &\sum_{i\in L}\sum_{j\in R}\left|u_i\right\rangle\left\langle u_j\right|\otimes\left|ei\right\rangle\left\langle e0\right| + \sum_{i\in R}\sum_{j\in L}\left|u_i\right\rangle\left\langle u_j\right|\otimes\left|e0\right\rangle\left\langle ej\right|\end{aligned} \tag{10}$$

Tracing out the environment states now means summing not just over $\{\left|ei\right\rangle, i = 1,2,3...\}$ but also including the state $\left|e0\right\rangle$ in the sum, since that remains a possible state of the detectors at this time. The second pair of terms (the cross terms) in (10) contribute nothing when traced out, whereas both the first two terms contribute, i.e.,

$$\hat{\rho}^{red} = \sum_{k=0}^{N}\left\langle ek\right|\hat{\rho}\left|ek\right\rangle = \sum_{k\in L}\left|u_k\right\rangle\left\langle u_k\right| + \left(\sum_{i,j\in R}\left|u_i\right\rangle\left\langle u_j\right|\right) \tag{11}$$

Hence decoherence has taken place over the states corresponding to the left hemisphere, whereas coherence remains for the right hemisphere states. In matrix notation,

[1] Actually this is only true in the "geometrical optics" approximation. It is not exactly true due to diffraction effects. This is discussed at the end of this Chapter.

$$\left(\rho^{red}\right)=\frac{1}{N}\begin{pmatrix} \begin{matrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & \ldots \\ & & & & & etc \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 1 & 1 & \ldots \\ & & & & & etc \end{matrix} \end{pmatrix} \tag{12}$$

where the left hemisphere corresponds to the first $N/2$ components and the right hemisphere to the second $N/2$ components. The density matrix (11,12) is the full description of the state of the particle at time $t_1 < t < t_2$ assuming that we have not yet looked at the left hemisphere detectors to see whether they have registered a particle. If they have, then the actual $\left|u_i, i \in L\right\rangle$ state has been found. If not, then the state of the particle is given by the density matrix for the remaining (right hemisphere) degrees of freedom alone, thus,

$$\hat{\rho}^{red} = 2\left(\sum_{i,j\in R}\left|u_i\right\rangle\left\langle u_j\right|\right) \tag{13}$$

The factor of 2 is to re-establish normalisation given that the sum in (13) now extends only over $N/2$ detectors. In matrix notation, if there is no particle registered on the left hemisphere the particle state becomes,

$$\left(\rho^{red}\right)=\frac{2}{N}\begin{pmatrix} \begin{matrix} 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots \\ & & & & & etc \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 1 & 1 & \ldots \\ & & & & & etc \end{matrix} \end{pmatrix} \tag{14}$$

What does this mean? It means that with the left hemisphere present but failing to detect a particle, the particle state is nevertheless changed from (4) to (14). The wavepacket is partially collapsed in that it is now concentrated on the right hemisphere states. However, full coherence is retained amongst the right hemisphere states, in contrast to the fully decohered density matrix (8). For example, if the right hemisphere of detectors was replaced by a screen with a pair of slits, a diffraction pattern could be developed.

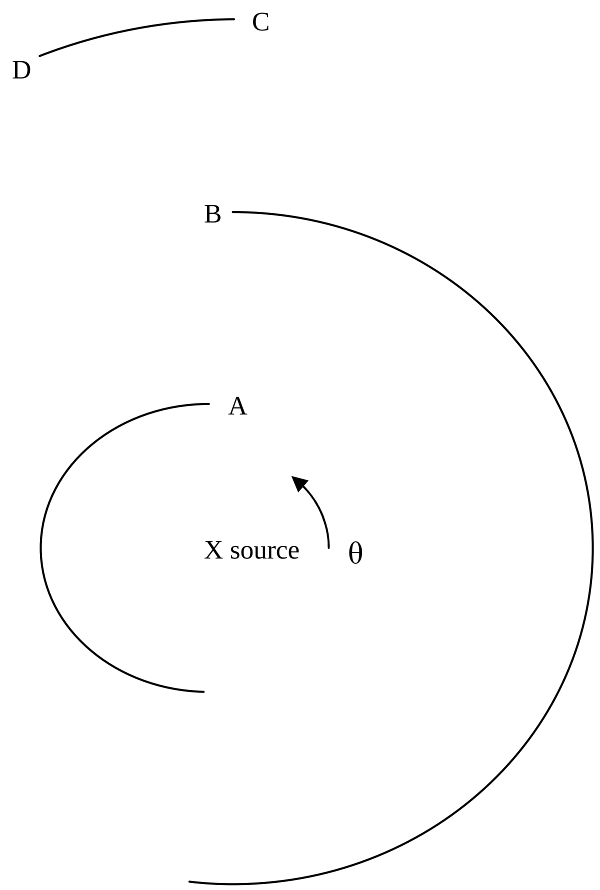
Diffraction Effects

To avoid misleading the reader we must now mention in passing that the above analysis is incomplete. The reason is that the presence of two hemispherical boundaries means that neither plane waves nor the "hemispherical" wave,

$$0 < \theta < \pi / 2: \qquad \psi(\bar{r}, t) \propto e^{-i\omega t} j_0(kr) \tag{15a}$$

$$\pi / 2 < \theta < \pi: \qquad \psi(\bar{r}, t) = 0 \tag{15b}$$

are actually solutions to the Schrodinger equation. For example, (15a) and (15b) are discontinuous over their mutual boundary at $\theta = \pi / 2$. So the true solution must smear (15a,b) so as to produce a continuous function (with continuous derivative). In the optical analogy, the physically effect which causes this modification to (15a,b) is diffraction from the edge of the inner hemisphere. Hence the detectors near the edge of the outer hemisphere would record a diffraction pattern. Moreover, diffraction would deflect some of the particles to angles $\theta > \pi / 2$ so that they would miss the outer hemisphere entirely. The total signal received by the two hemispheres would not sum to unity (there's a hole in our bucket!). The lost signal could be captured by putting another arc of detectors beyond the right hemisphere, thus…



Hence the diffraction around edge A causes some particles to pass to the left of point B, missing the right-hand hemisphere, but these particles are captured by the additional detectors placed on the arc CD. Of course, there will also be diffraction from edge B, which will cause some particles to miss the new detectors on CD by passing to the right of C. So you would really need another arc of detectors beyond CD (and so on, *ad infinitum*).

However these diffraction effects are mentioned only for completeness. They do not detract from the moral of the example: null measurements can partially decohere the density matrix (collapse the wavepacket).

Appendix

The spherically symmetric S-wave solution to the free Schrodinger equation, for a state of definite energy, is $\psi(\bar{r},t) \propto e^{-i\omega t} j_0(kr)$ where the energy is $\hbar\omega = (\hbar k)^2 / 2m$ and j_0 is the first spherical Bessel function, which is actually just the sinc function, i.e., $j_0(x) \equiv \sin x / x$. The plane wave solutions of the same energy are $\psi(\bar{r},t) \propto e^{-i\omega t} e^{i\bar{k}\cdot\bar{r}}$ so that an isotropic, in-phase sum of plane waves is,

$$\frac{1}{4\pi}\int e^{i\bar{k}\cdot\bar{r}}\,d\Omega = -\frac{1}{2}\int_0^{\pi} e^{ikr\cos\theta}\,d(\cos\theta) = \frac{1}{2}\cdot\frac{e^{ikr}-e^{-ikr}}{ikr} = \frac{2i\sin kr}{2ikr} = j_0(kr)$$

So the states $|u_i\rangle$ in (1) can be identified with plane waves of the same energy as the spherical wave (i.e., the same *k*), and noting that the plane waves in all directions contribute with the same amplitude and phase. There are therefore no expansion coefficients in (1). Obviously (1) becomes exact only in the limit of the subscript i becoming a continuous variable.

Reference

[1] Mauritius Renninger, *Messungen ohne Storung des Messobjekts (Measurement without disturbance of the measured objects)*, Zeitschrift für Physik, 1960; **158**(4): 417-421. An English translation by W. De Baere is available as arXiv:physics0504043.