Chapter 27

Noether's Theorem and the Origin of Spin

The derivation of conserved quantities from symmetry is one of the most fundamental features of modern physics. The quantities which are conserved as a result of the Poincare symmetries of Minkowski spacetime are derived here. Amongst other things this puts spin in its proper context.

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1. What is Noether's Theorem?

What is the greatest theorem in mathematical physics? Noether's Theorem would surely be one of the strongest candidates for the title. I recall the moment as an undergraduate when I first learnt that the conservation of energy and momentum could be *derived* from the translational symmetry of spacetime. It was one of those pivotal moments when you know that your understanding has just made a major leap forward. Such conceptual break-throughs are exactly what physicists constantly strive towards. Prior to appreciating the connection with symmetry, the conservation of energy was to be regarded as a physical Law. The connection with symmetry turns it instead into a mathematical theorem. It is one less thing which appears contingent upon how the physical world just happens to be, and one more thing whose logical necessity has been pinned down to a more fundamental level. This is what progress consists of in physics.

Noether's Theorem does not only elucidate the conservation laws in classical physics, it also provides the most cogent perspective on the formulation of quantum mechanics. When one first comes across the correspondence $E \to i\hbar\partial_t$ and $\overline{p} \to -i\hbar\overline{\nabla}$ in quantum mechanics one is likely to think, "For God's sake, why?" But these operators are the generators of translations in spacetime (strictly, generators of the Lie algebra of the Lie group of translational isometries). So a state sharing the full symmetry of the underlying spacetime should be unchanged by such operators, e.g., $i\hbar\partial_t\psi \propto \psi$. But Noether tells us that the conservation of energy and momentum also follows from this invariance. It is a short conceptual step to the eigenvalue equations such as $i\hbar\partial_t\psi = E\psi$ which underpin quantum mechanics. Noether had essentially set up the correspondence $E \to i\hbar\partial_t$ and $\overline{p} \to -i\hbar\overline{\nabla}$ before quantum mechanics (only just before, actually, by a few years).

The links symbolised by $E \to i\hbar\partial_t$ and $\overline{p} \to -i\hbar\overline{\nabla}$ are now so embedded within physical theory that mathematical developments are liable to treat them as mere definitions. This is not logically wrong. But it is in danger of loosing the connection with the physical world and robbing us of the achievements which have been made. It is salutary to recall that concepts of energy matured most in conjunction with the development of steam power in the nineteenth century. These were practical times. Engineers needed to stop mines flooding, and the mine owners wanted to burn as little coal as possible powering the steam engine driven pumps as in so doing. There was scant interest in philosophical speculations about the nature of energy amongst such men. But this should not dampen our enthusiasm for delving into the fundamental nature of things. The truly wonderful thing is this: the energy of interest to those nineteenth century engineers and that in $E \to i\hbar\partial_t$ are *the same thing*.

So what is Noether's Theorem? Noether's Theorem states that for every continuous symmetry of a Lagrangian dynamical system there corresponds a conserved quantity. A limitation of the theorem is that it applies only to theories expressed in Lagrangian form. This is hardly a serious limitation, however, since virtually the whole of physics can be so formulated. Examples are as follows,

- The absence of an explicit time dependence in the Lagrangian implies that the dynamical behaviour of the system will be the same tomorrow as it is today and was yesterday. The Lagrangian is invariant under the action of translation in time. Noether's Theorem tells us that there must be a conserved quantity. This conserved quantity is energy.
- The physical laws controlling the behaviour of a system are expected to be the same here as on Mars, or our space programmes will be in trouble. So the Lagrangian is expected to be invariant under spatial translations. Since there are three spatial dimensions, there are three conserved quantities – the three components of momentum.
- Space is also isotropic: there is no preferred direction. The Lagrangian should therefore be invariant under rotations of the coordinate system. The corresponding conserved quantity is angular moment. There are three components of angular momentum corresponding to the three perpendicular axes of rotation.

The rotational symmetry, leading to the conservation of angular momentum, is of particular interest. It is here that we discover spin. Spin is that part of the angular momentum which is intrinsic to the object itself and not due to the object's motion in space. For a long time I had the incorrect idea that spin was something essentially quantum mechanical. But spin is no more (or less) quantum mechanical than other forms of momentum. My misunderstanding was the result, of course, of the importance of spin in atomic, nuclear and particle physics. This false perspective regarding the essential quantum nature of spin was reinforced by Pauli (1924) who referred to it as "a non-classical two-valued-ness". Of course the spin of an electron is indeed non-classical, and intrinsically quantum mechanical. But then, so is its linear momentum and energy state. But spin is not intrinsically quantum mechanical in the general case, any more than is linear momentum or energy. I was brought up with the vague notion that the term "spin" was not to be taken too literally. But spin can be relevant in classical physics, in which context the word "spin" is precisely accurate. Quantum applications introduce their usual weirdness, of course, but spin is affected by this only in the same manner as orbital angular momentum. So Pauli's "non-classical two-valued-ness" is a fair description of the spin of electrons, but the spin of a tennis ball is also spin.

All the above examples of conserved quantities arise from symmetries under the Poincare group of transformations. But we missed out the Lorentz boosts. The equivalence of inertial observers is also a symmetry. What quantity is conserved as a result? There is such a conserved quantity, as there must be by Noether's Theorem. However the status of this quantity is rather different. This is explained below.

The Poincare group does not provide the only possible symmetries of a Lagrangian. There are broadly two other classes of symmetry: dynamical and internal. *Should I also mention gauge symmetries?* A dynamical symmetry is an algebraic invariance of the Lagrangian which depends upon the particular form of the interaction. For example, in the Kepler problem or the hydrogen atom (see Chapters 15 and 35) the

Runge-Lenz vector is conserved as a consequence of the $1/r$ form of the gravitational and Coulomb potentials. This is a dynamical symmetry. Internal symmetries arise when several different fields are involved and the Lagrangian is invariant under some transformation which involves algebraic combinations of these fields alone. For example, a Lagrangian for a complex field may depend only upon absolute magnitudes and hence be invariant if the field is multiplied by a phase factor (which is just a linear transformation of the real and imaginary parts). This leads to a conserved quantity which can be interpreted as electric charge. In a similar manner the various internal symmetry groups of particle physics [SU(2), SU(3)] lead to conserved quantum numbers.

In this Chapter we shall concentrate on the Poincare symmetries as they apply to fields of arbitrary spin in flat (Minkowski) spacetime. However we shall also indicate how the conserved quantities corresponding to internal symmetries can be derived in the simplest case, the conservation of charge.

2. The Poincare Symmetries and Their Conserved Quantities

Throughout this Chapter we work in units with $c=1$. The fields in question will be written $\phi_r(x)$. The subscript r serves two different purposes. Firstly it labels the spin degrees of freedom of a given field. Secondly it can also address the presence of more than one field in the system. The Lagrangian is assumed to be a function of the fields and their first derivatives, $L(\{\phi_r\},\{\phi_{r,\mu}\})$. The principle of minimum action, $\delta\int Ld^4x=0$, leads to the Euler-Lagrange equations (see Chapter 13),

$$\frac{\partial L}{\partial \phi_r}=\partial_\mu\left(\frac{\partial L}{\partial \phi_{r,\mu}}\right) \tag{1}$$

This applies for all *r*, covering all spin degrees of freedom of all contributing fields.

It is important to distinguish clearly between changes due to coordinate changes and changes due to the transformations amongst the spin degrees of freedom at a point. We shall use x and x' to refer to the <u>same</u> *point in spacetime* as seen by two observers, S and S'. Similarly, the <u>same</u> *spinorial or tensorial field* seen at this point has components ϕ_r as seen by S, but components ϕ'_r as seen by S'. For a scalar field there is therefore no difference, $\phi'=\phi$. However, care is needed because a spacetime transformation can affect both the coordinates of the point in question and the components of a spinorial/tensorial field. The effect of a general, but infinitesimal, Poincare transformation of the coordinates is,

$$x'_\alpha=x_\alpha+\varepsilon_{\alpha\beta}x^\beta+\Delta_\alpha \tag{2}$$

The $\varepsilon_{\alpha\beta}$ and Δ_α are a set of infinitesimal, real parameters, arbitrary apart from being anti-symmetric, $\varepsilon_{\beta\alpha}=-\varepsilon_{\alpha\beta}$. [It is easy to see that $x'_\alpha=x_\alpha+\varepsilon_{\alpha\beta}x^\beta$ is the most general homogeneous Lorentz transformation to first order in the small quantities $\varepsilon_{\alpha\beta}$. This would follow by appeal to the specific matrix representations for these transformations, e.g., as given for the rotations in Chapter 32. But this is not necessary. The Lorentz transforms preserve the Minkowski metric, so we require $x'_\alpha x'^\alpha=x_\alpha x^\alpha$. But,

$$x'_\alpha x'^\alpha = \left(x_\alpha + \varepsilon_{\alpha\beta} x^\beta\right)\left(x^\alpha + g^{\alpha\mu}\varepsilon_{\mu\beta} x^\beta\right) = x^\alpha x_\alpha + \varepsilon_{\alpha\beta} x^\beta x^\alpha + g^{\alpha\mu}\varepsilon_{\mu\beta} x^\beta x_\alpha + O\left(\varepsilon^2\right)$$
$$= x^\alpha x_\alpha + \varepsilon_{\alpha\beta} x^\beta x^\alpha + \varepsilon_{\mu\beta} x^\beta x^\mu + O\left(\varepsilon^2\right) = x^\alpha x_\alpha + O\left(\varepsilon^2\right)$$

where the last step requires only the antisymmetry of the transformation coefficient matrix, $\varepsilon_{\beta\alpha} = -\varepsilon_{\alpha\beta}$. Hence, to first order in the small quantities $\varepsilon_{\alpha\beta}$, the Minkowski metric is preserved by $x'_\alpha = x_\alpha + \varepsilon_{\alpha\beta} x^\beta$. Moreover, there are six independent $\varepsilon_{\alpha\beta}$ parameters which clearly produce six linearly independent transformations of x_α. So (2) must be the most general Poincare transformation].

Recall that x and x' refer to the same point in spacetime, and the transformation (2) describes how to find the coordinates seen by observer S' given those seen by S. However they are both observing the same field at the same point. Observer S' sees the fields $\phi'_r(x')$ whereas S sees the fields $\phi_r(x)$. Though these are the same field at the same point, the differing orientation and/or states of motion of the two observers means that they do not see the same numerical field components. Instead they are related by whatever spin/tensor representation of the Lorentz group is appropriate for that field type. Hence we can write, for an arbitrary but infinitesimal transformation,

Spin > 0: $$\phi'_r(x') = \phi_r(x) + \frac{1}{2}\varepsilon_{\alpha\beta} S_{rs}^{\alpha\beta}\phi_s(x) \tag{3a}$$

Spin = 0: $$\phi'_r(x') = \phi_r(x) \tag{3b}$$

where repeated indices are summed, in the case of Greek indices over the 4 spacetime dimensions, and for Latin indices over the spin degrees of freedom for the field in question. If there is more than one field, Equ.(3) applies separately for each one (i.e., for different ranges of the subscript *r*). In (3) the rotation/boost parameters $\varepsilon_{\alpha\beta}$ are, of course, necessarily identical to those appearing in (2), since they are defined by the relationship between S and S'. Hence, since the $\varepsilon_{\alpha\beta}$ are anti-symmetric, the representation matrices of the Lorentz group appearing in (3a) can be assumed anti-symmetric in these indices also, i.e., $S_{rs}^{\alpha\beta} = -S_{rs}^{\beta\alpha}$.

I used to get tied in knots when deriving Noether's conserved quantities as a result of the different field differentials that occur. It is important to distinguish between three things,

The local differential: $$\delta_L\phi_r(x) \equiv \phi'_r(x') - \phi_r(x) = \frac{1}{2}\varepsilon_{\alpha\beta} S_{rs}^{\alpha\beta}\phi_s(x) \tag{4a}$$

Or, for spin zero (scalar): $$\delta_L\phi_r(x) \equiv \phi'_r(x') - \phi_r(x) = 0 \tag{4b}$$

The functional differential: $$\delta\phi_r(x) = \phi'_r(x) - \phi_r(x) \tag{5}$$

The gradient: $$\partial\phi_r(x) \equiv \phi_r(x') - \phi_r(x) = \frac{\partial\phi_r(x)}{\partial x_\alpha}\delta x_\alpha \tag{6}$$

The three are related since $\delta\phi_r(x') + \partial\phi_r(x) = \phi'_r(x') - \phi_r(x') + \phi_r(x') - \phi_r(x) = \delta_L\phi_r(x)$. To first order in small quantities this gives,

$$\delta_L\phi_r(x) = \delta\phi_r(x) + \partial\phi_r(x) \tag{7}$$

since $\delta\phi_r(x)$ only differs from $\delta\phi_r(x')$ by a second order term. Re-arranging and substituting (4) and (6) gives,

$$\delta\phi_r(x) = \delta_L \phi_r(x) - \partial\phi_r(x) = \frac{1}{2}\varepsilon_{\alpha\beta} S_{rs}^{\alpha\beta} \phi_s(x) - \frac{\partial\phi_r(x)}{\partial x_\alpha}\delta x_\alpha \tag{8}$$

In (8) it is understood that the first term on the RHS is dropped for a scalar field.

Now the two observers must see the same Lagrangian, i.e.,

$$L(\{\phi'_r(x')\}, \{\phi'_{r,\mu}(x')\}) = L(\{\phi_r(x)\}, \{\phi_{r,\mu}(x)\}) \tag{9}$$

(The alert reader may be worried that, because *L* is really a Lagrange density, the change in volume element between observers might mean that *L* should not be invariant. This would indeed be the case if *L* were a density in the sense of "per unit spatial volume". But, in fact, *L* is a density per volume element of spacetime, i.e., per d^4x not per d^3x, and the 4-volume element is a scalar).

In general, the total change of a function-of-a-function when the independent functions are changed and the point at which they are evaluated is also changed, is given by, for example,

$$df(g(x), h(x), \ldots) = \frac{\partial f}{\partial g}\delta g + \frac{\partial f}{\partial h}\delta h + \ldots + \frac{\partial f}{\partial x}\delta x \tag{10}$$

Applying this to the difference between the LHS and RHS of (9) gives, (11)

$$dL = L(\{\phi'_r(x')\}, \{\phi'_{r,\mu}(x')\}) - L(\{\phi_r(x)\}, \{\phi_{r,\mu}(x)\}) = \frac{\partial L}{\partial\phi_r}\delta\phi_r + \frac{\partial L}{\partial\phi_{r,\alpha}}\delta\phi_{r,\alpha} + \frac{\partial L}{\partial x_\alpha}\delta x_\alpha = 0$$

Note that the fact that the RHS of (11) is zero is an expression of the symmetry, i.e., the invariance, (9). On the RHS of (11), evaluation of all quantities at *x* is understood. Here $\delta\phi_r$ corresponds to the differential defined by (5) and given by (8). The Euler-Lagrange equations, (1), allows us to re-write (11) as,

$$\partial_\mu\left(\frac{\partial L}{\partial\phi_{r,\mu}}\right)\delta\phi_r + \frac{\partial L}{\partial\phi_{r,\alpha}}\delta\phi_{r,\alpha} + \frac{\partial L}{\partial x_\alpha}\delta x_\alpha = \partial_\mu\left(\frac{\partial L}{\partial\phi_{r,\mu}}\delta\phi_r\right) + \frac{\partial L}{\partial x_\alpha}\delta x_\alpha = 0 \tag{12}$$

Now substituting for $\delta\phi_r$ from (8) gives,

$$\partial_\mu\left(\frac{\partial L}{\partial\phi_{r,\mu}}\left[\frac{1}{2}\varepsilon_{\alpha\beta} S_{rs}^{\alpha\beta}\phi_s(x) - \frac{\partial\phi_r(x)}{\partial x_\alpha}\delta x_\alpha\right]\right) + \frac{\partial L}{\partial x_\alpha}\delta x_\alpha = 0 \tag{13}$$

Again it is understood in (13) that the term in $S_{rs}^{\alpha\beta}$ is dropped for a scalar field. This is the master equation from which all the conserved quantities resulting from the symmetry under Poincare transformations, (2), can be derived.

2.1 The Energy-Momentum Tensor

It is convenient to consider spacetime translations separately from rotations and boosts. In (2), the former have non-zero Δ_α but $\varepsilon_{\alpha\beta} = 0$, whereas for the latter it is the other way around. For spacetime translations, $\delta x_\alpha = x'_\alpha - x_\alpha = \Delta_\alpha$, and (13) becomes,

$$\left\{\partial_\mu\left(\frac{\partial L}{\partial\phi_{r,\mu}}\left[\frac{\partial\phi_r(x)}{\partial x_\alpha}\right]\right) - \frac{\partial L}{\partial x_\alpha}\right\}\Delta_\alpha = 0 \tag{14}$$

But note that $\frac{\partial L}{\partial x_\alpha} \equiv \partial^\alpha L = \eta^{\alpha\mu}\partial_\mu L$, where $\eta^{\alpha\mu}$ is the Minkowski metric tensor. So (14) becomes,

$$\left\{\partial_\mu\left(\frac{\partial L}{\partial\phi_{r,\mu}}\left[\frac{\partial\phi_r(x)}{\partial x_\alpha}\right] - \eta^{\mu\alpha}L\right)\right\}\Delta_\alpha = 0 \tag{15}$$

This is true for arbitrary Δ_α and hence the rank 2 tensor,

$$T^{\mu\alpha} \equiv \frac{\partial L}{\partial\phi_{r,\mu}}\left[\frac{\partial\phi_r(x)}{\partial x_\alpha}\right] - \eta^{\mu\alpha}L \tag{16}$$

must have zero divergence,

$$\partial_\mu T^{\mu\alpha} = 0 \tag{17}$$

Hence the four quantities defined by,

$$P^\alpha = \int T^{0\alpha}\cdot d^3x \tag{18}$$

are conserved, i.e., (17) implies that,

$$\frac{dP^\alpha}{dt} = 0 \tag{19}$$

In (18) it is understood that the spatial integral extends over the whole of space. Note, however, that being divergence free, (17), also expresses the more general property of continuity. The rate of change of the quantity in question within any region of space is balanced by the flux of the quantity out of the boundary of the region,

$$\frac{\partial}{\partial t}\int_V T^{0\alpha}\cdot d^3x = \int_V \partial_i T^{i\alpha} d^3x = \oiint_{\delta V} T^{i\alpha} dS_i \tag{20}$$

This leads to the interpretation of $T^{i\alpha}$ as being the flux of P^α, through unit area with normal in direction i per unit time. Global conservation, (19), follows from local continuity, (20), in the limit that the region V is the whole of space and assuming that the fields vanish at infinity such that $\oiint_{\delta V} T^{i\alpha} dS_i \to 0$ as the boundary $\delta V \to \infty$.

From these purely mathematical observations we conclude that there are 4 types of 'stuff', labelled by the index α, whose density is $T^{0\alpha}$ and whose vectorial flux is $T^{i\alpha}$. Specific examples whose interpretation is already understood show that the P^α can be interpreted as the energy-momentum 4-vector. In unfamiliar cases, (18) is taken to define the energy-momentum 4-vector. As noted in §1 this is consistent with quantum mechanics, in which we have the correspondence $\hat{P}^\alpha \leftrightarrow i\hbar\partial^\alpha$, and these operators generate the representation (2) of the translation sub-group of the Poincare group. Consequently $T^{\mu\alpha}$ is called the energy-momentum tensor.

Assuming this interpretation is sound, it follows that T^{00} is energy density; T^{i0} is the vectorial flux of energy through unit area per unit time; T^{0i} is the density of momentum in the direction i; and T^{ji} is the flux (per unit area per unit time) in the direction j of the ith component of momentum.

2.2 The Symmetry of the Energy-Momentum Tensor

The physical interpretations described above suggest that the energy-momentum tensor should be symmetrical. This can be seen as follows. The flux of 'stuff' is the density of stuff times its velocity. So the energy flux is expected to be energy density times velocity, $T^{i0} = \frac{dE}{dV}\bar{v}$, where dE is the energy in volume dV. On the other hand, momentum density is expected to be $T^{0i} = \frac{d\bar{p}}{dV} = \frac{dm}{dV}\bar{v}$, where dm is the mass in volume dV (suitably allowing for relativistic effects so that $dm = \gamma dm_0$). But provided that Einstein was right and we can equate mass and energy, $E = m$ (recalling that we are using $c = 1$), then these two things are the same, i.e., $T^{0i} = T^{i0}$. Similarly, the flux in direction j of the i$^{\text{th}}$ component of momentum is $T^{ji} = \frac{dp^i}{dV}v^j = \frac{dm}{dV}v^i v^j$ and hence is also expected to be symmetric.

We note that the symmetry of the stress energy tensor is actually deeply embedded within relativity theory in the form of the Einstein field equations of general relativity, $G_{\mu\nu} = 8\pi G \cdot T_{\mu\nu}$. The symmetry of $T_{\mu\nu}$ is required by these equations because $G_{\mu\nu}$ is identically symmetric. Indeed this leads to an alternative definition $T^{\mu\nu} = 2\frac{\partial L}{\partial g_{\mu\nu}}$.

However, the definition (16) of the energy-momentum tensor does not itself ensure that the energy-momentum tensor is symmetric. Instead, the physical requirement for $T_{\mu\nu}$ to be symmetric is a constraint upon suitable Lagrangians. The Lagrangian which is often employed in classical electromagnetism, $L = -F_{\mu\nu}F^{\mu\nu}/4$, where $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$, does *not* lead to (16) being symmetrical. Texts generally adjust the canonical energy-momentum tensor given by (16) to be symmetrical by adding an extra term whose divergence is identically zero and whose contribution to the 4-momentum is zero. I have always regards the procedure as a blemish on the theory. One might have hoped for a Lagrangian which led directly to a symmetrical $T_{\mu\nu}$. The Lagrangian $L \propto F_{\mu\nu}F^{\mu\nu}$ is problematical in quantum field theory also, because the canonically conjugate fields have an identically zero time component which undermines the canonical quantisation procedure. So QED uses an alternative such as $L = -(\partial_\mu A_\nu)(\partial^\mu A^\nu)/2$ which does produce a symmetric energy-momentum tensor. However this Lagrangian only reproduces Maxwell's equations if a Lorentz gauge such that $\partial_\mu A^\mu = 0$ is assumed.

2.3 The Conservation of Angular Momentum and the Origin of Spin

We now consider a general Lorentz transformation of coordinates but with no translation, i.e., $\delta x_\alpha = x'_\alpha - x_\alpha = \varepsilon_{\alpha\beta}x^\beta$. Substitution into (13) gives,

$$\partial_\mu\left(\frac{\partial L}{\partial \phi_{r,\mu}}\left[\frac{1}{2}\varepsilon_{\alpha\beta}S_{rs}^{\alpha\beta}\phi_s(x) - \frac{\partial \phi_r(x)}{\partial x_\alpha}\varepsilon_{\alpha\beta}x^\beta\right]\right) + \frac{\partial L}{\partial x_\alpha}\varepsilon_{\alpha\beta}x^\beta = 0 \qquad (21)$$

Putting $\frac{\partial L}{\partial x_\alpha} \equiv \partial^\alpha L = \eta^{\alpha\mu}\partial_\mu L$ this becomes,

$$\partial_\mu \left(\frac{\partial L}{\partial \phi_{r,\mu}} \left[\frac{1}{2} S_{rs}^{\alpha\beta} \phi_s(x) - \frac{\partial \phi_r(x)}{\partial x_\alpha} x^\beta \right] + \eta^{\alpha\mu} L x^\beta \right) \varepsilon_{\alpha\beta} = 0 \qquad (22)$$

Note that the term x^β can be taken inside the derivative ∂_μ because $\partial_\mu \left(\eta^{\alpha\mu} x^\beta \varepsilon_{\alpha\beta} \right) = \eta^{\alpha\mu} \varepsilon_{\alpha\beta} \delta_\mu{}^\beta = \eta^{\alpha\beta} \varepsilon_{\alpha\beta} = 0$ by virtue of the anti-symmetry of $\varepsilon_{\alpha\beta}$. The second and third terms in (22) can be re-written in terms of the energy-momentum tensor using (16) as,

$$\partial_\mu \left(\frac{1}{2} \cdot \frac{\partial L}{\partial \phi_{r,\mu}} S_{rs}^{\alpha\beta} \phi_s(x) - T^{\mu\alpha} x^\beta \right) \varepsilon_{\alpha\beta} = 0 \qquad (23)$$

However we cannot conclude that the coefficient of $\varepsilon_{\alpha\beta}$ in (23) is zero because the anti-symmetry of $\varepsilon_{\alpha\beta}$ means that it is only the part of this which is antisymmetric in $\alpha\beta$ which must be zero. Hence we write,

$$\Im^{\mu\alpha\beta} = \frac{\partial L}{\partial \phi_{r,\mu}} S_{rs}^{\alpha\beta} \phi_s(x) + \left(x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} \right) \qquad (24)$$

Note that $S_{rs}^{\alpha\beta} = -S_{rs}^{\beta\alpha}$, so that $\Im^{\mu\alpha\beta} = -\Im^{\mu\beta\alpha}$, and $\partial_\mu \Im^{\mu\alpha\beta} \varepsilon_{\alpha\beta} = 0$ then implies,

$$\partial_\mu \Im^{\mu\alpha\beta} = 0 \qquad (25)$$

[*We note in passing that for a scalar field, (24) and (25) imply that* $T^{\alpha\beta} = T^{\beta\alpha}$*, so that the difficulty noted in §2.2 will not arise for spinless particles].*

Since $\Im^{\mu\alpha\beta} = -\Im^{\mu\beta\alpha}$, (25) gives us six further conserved quantities defined by,

$$M^{\alpha\beta} = \int \Im^{0\alpha\beta} \cdot d^3x \qquad (26)$$

As before, $\Im^{i\alpha\beta}$, for spatial indices ^{i}, can be interpreted as the flux of these quantities. But what are they? In this section we will consider the spatial components, M^{ij} (the components M^{0i} are considered in §2.4). If we look firstly at the second term in (24) – which would be the only term for a scalar field – we find,

$$M_{orbit}^{ij} = \int d^3x \left(x^i T^{0j} - x^j T^{0i} \right) \qquad (27)$$

But T^{0j} is the jth component of the density of momentum, so $d^3x \cdot x^i T^{0j}$ is the *ij* component of angular momentum due to the field in a small volume element, i.e., due to the momentum in the *j*-direction multiplied by the distance in the *i*-direction. So if *ijk* is a permutation of 1,2,3, then $d^3x \cdot \left(x^i T^{0j} - x^j T^{0i} \right)$ is the angular momentum in the *k*-direction due to the fields in this small region. So we can interpret M_{orbit}^{ij} as the angular momentum of a scalar field, or that part of the angular momentum of a field with spin degrees of freedom which does not depend upon the spin. Specifically, because M_{orbit}^{ij} arises from products of linear momentum and perpendicular distance, it is the ***orbital*** angular momentum of the field.

The first term in (24) must have an interpretation compatible with the second term, otherwise it would make no sense to add them together. This allows us finally to interpret the first term in (24) as the angular momentum which arises from the spin degrees of freedom. This justifies the use of the word "spin" for the degrees of freedom associated with the rotational transformations under the representation $S_{rs}^{\alpha\beta}$ of the Lorentz group. The term "spin" appropriately encapsulates that form of angular momentum intrinsic to the field, rather than resulting from orbital angular momentum.

$$M_{spin}^{ij} = \int d^3x \cdot \frac{\partial L}{\partial \phi_{r,0}} S_{rs}^{ij} \phi_s(x) \tag{28}$$

The definition of the field conjugate to ϕ_r is,

$$\pi_r = \frac{\partial L}{\partial \phi_{r,0}} \tag{29}$$

So (28) can also be written,

$$M_{spin}^{ij} = \int d^3x \cdot \pi_r S_{rs}^{ij} \phi_s(x) \tag{30}$$

The total angular momentum is,

$$M^{ij} = M_{orbit}^{ij} + M_{spin}^{ij} \tag{31}$$

It is only this *total* angular momentum which is conserved as a consequence of (25), i.e.,

$$\frac{dM^{ij}}{dt} = 0 \tag{32}$$

This is the origin of spin. It is required for any theory formulated in Lagrangian terms and for which the fields transform under rotations according to a representation of the Lorentz group other than the trivial one-dimensional representation (i.e., other than scalar fields). This is the reason why fields which are vectorial or tensorial of rank 2 in classical physics correspond to quantum fields with quanta which possess spins of 1 or 2 $\hbar$-units respectively. The tensorial/spinorial degrees of freedom of a field defined over the spacetime continuum correspond, and give rise to, the spin of its quanta.

Note, however, that there is nothing essentially quantum-mechanical about spin in general. The fields in question could be purely classical. So why does spin generally appear to be an overtly quantum mechanical concept? The reason, I suspect, is due to the rotation group being the only compact subgroup of the Poincare group. This means that it is the only subgroup which has finite dimensional unitary representations. Consequently it is only the rotations which correspond to eigenstates (angular momentum states) which have discrete eigenvalues (j, m). Because the angular momentum states are discrete, the quantum properties are more apparent. For example, a spin half particle can be spin up or spin down with respect to a given measurement direction – but no intermediate case will be measured. This is obviously and overtly quantum.

Interpreted classically there is no reason why the spin angular momentum given by (28) should not take a continuum of values, rather than being confined to the eigenvalues of a discrete representation, S. This is true even if S is the ½-spinor representation or the vector (spin 1) representation. Thus, the spin angular momentum of the classical electromagnetic field can be evaluated using (28), but it is not (classically) confined to discrete values. Thus it is the measurement theory of quantum mechanics which distinguishes the quantum from the classical, not the mere existence of spin *per se*. This holds that measurements can only produce results which are eigenvalues of the operator representing the quantity measured. Where the representation is finite, as for the rotations, these eigenvalues are necessarily a discrete set.

Whilst the same measurement theory also applies to, say, the measurement of energy or linear momentum, these correspond to a non-compact subgroup of the Poincare group. Now in the quantum mechanics of single particles (as opposed to field theory) symmetries must be enacted on quantum states as unitary operators (though anti-unitary operators will also do - Wigner's theorem). But non-compact Lie groups have no finite dimensional unitary representations. Hence translations in Minkowski spacetime must be implemented in quantum mechanics as infinite dimensional unitary representations, i.e., as differential operators acting on a space of continuous functions. Their eigenvalues then take a continuum of values. Hence, whilst the measurement theory of QM still applies, it is not so obvious since there is no discreteness in the set of possible energy or momentum values.

The translation subgroup can be made compact by confining the particle to a box with suitable boundary conditions. This causes the possible energy and momentum states to become quantised into a discreet set, labelled by quantum numbers *l,m,n* in the manner familiar from solving the Schrodinger equation for this problem. But we see now that the origin of this discreteness is the compactness of the relevant Lie group arising from the box boundary conditions. ***Am I happy with all this?***

2.4 Is There a Conserved Quantity Corresponding to Covariance under Boosts?

Well, we know there is – we have already derived it in the form of Equ.(25). The boosts correspond to one of $\alpha\beta$ being zero. The conserved quantities are,

$$-M^{j0} = M^{0j} = \int \Im^{00j} \cdot d^3x \qquad (33)$$

But, before we examine (33) more closely, two little problems arise. The first is that, whilst we could easily have guessed the previous conservation laws, i.e., the conservation of energy, momentum and angular momentum, there now appears to be no obvious conserved quantity left. What can it be?

The second problem seems to confirm the first. It comes from considering the commutation properties of the generators of the Poincare group (or more correctly, the Lie algebra). These are,

(34)

$$[L_j, L_k] = i\in_{jkn} L_n \qquad [K_j, K_k] = i\in_{jkn} L_n \qquad [L_j, K_k] = i\in_{jkn} K_n$$

$$[P_j, L_k] = i\in_{jkn} P_n \qquad [P_j, K_k] = i\delta_{jk} P_0 \qquad [P_\mu, P_\nu] = 0$$

$$[P_0, L_k] = 0 \qquad [P_0, K_k] = -iP_k$$

where $\overline{L}$, $\overline{K}$, P_μ are the generators of rotations, boosts and spacetime translations respectively. The commutators of the generators of the homogeneous Lorentz group, L_j and K_j, are discussed in the context of finite dimensional representations in Chapter 35, the group elements being given by $\exp\{i\overline{\theta} \cdot \overline{L} + \overline{\beta} \cdot \overline{K}\}$ for the six arbitrary real parameters $\bar{\theta}$ and $\bar{\beta}$. However the easiest way to check the commutators (34) is through the infinite dimensional (continuum) representation as differential operators, $P_\mu = i\partial_\mu$; $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$; $K_j = M_{0j} = i(t\partial_j - x_j \partial_t)$; $L_j = \in_{jnm} M_{nm}/2$. Need to check this correctly reproduces (34). Are the contravariant/covariant indices ok? Make notation for Poincare operators consistent between all Chapters. Relevant Chapters include 3, 15, 27, 32, 35 and 42.

Now in quantum mechanics the energy P_0 will be equated with the Hamiltonian, and we will interpret the rate of change of a quantity with the commutator: $i\hbar \frac{dQ}{dt} = [Q, P_0]$. Consequently conserved quantities should be associated with operators which commute with P^0. This works for linear momentum, since $[P_j, P_0] = 0$. It also works for angular momentum since $[P_0, L_k] = 0$. And it trivially works for energy, since obviously $[P_0, P_0] = 0$. However, it would appear that the value taken by the boost operator $\bar{K}$ is not conserved because $[P_0, K_k]$ is *not* zero. So just what ***is*** the meaning of (33)?

It is quite common in texts for authors to shy away from this question altogether. And even when they do not, they usually only address the meaning of *part* of (33). We shall also offer only a partial explanation here, putting off the rest until §?? We have,

$$M^{0j} = \int d^3x \cdot \left\{ \pi_r S_{rs}^{0j} \phi_s(x) + \left(x^0 T^{0j} - x^j T^{00} \right) \right\} \tag{35}$$

We shall initially consider the meaning only of the second term, i.e., the sole term which would occur for a scalar field,

$$M_{scalar}^{0j} = \int d^3x \cdot \left\{ \left(x^0 T^{0j} - x^j T^{00} \right) \right\} \tag{36}$$

But $\int d^3x \cdot x_0 T^{0j} = tP^j$ using (18). Similarly, if we normalise $\int d^3x \cdot x^j T^{00}$ by the total energy, $E = \int d^3x \cdot T^{00}$ we see that,

$$\xi^j = \frac{\int d^3x \cdot x^j T^{00}}{\int d^3x \cdot T^{00}} \tag{37}$$

is the position of the centre of mass of the field system. In this context a better name might be "centre of energy", since that is what (37) literally produces. But, of course, there is no difference in relativity theory (apart from the units, i.e., that factor of c^2). Putting these together we see that the quantity which Noether tells us is conserved is,

$$M_{scalar}^{0j} = tP^j - E\xi^j \tag{38}$$

Now since we already know that energy and momentum are conserved, by (19), the conservation of the quantity (38) gives,

$$\frac{d}{dt} M_{scalar}^{0j} = \frac{d}{dt}\left(tP^j - E\xi^j \right) = P^j - E\frac{d\xi^j}{dt} = 0 \tag{39}$$

This tells us that,

$$P^j = E\frac{d\xi^j}{dt} \tag{40}$$

So, the total momentum equals the total energy times the velocity of the centre of mass-energy (recall we are working in units with $c = 1$). This is the expected result, of course, and corresponds exactly to the relativistic momentum of a single particle ($p = Ev = \gamma mv$) and reduces to the non-relativistic value of momentum for $v << c$ ($p = mv$). However, for an arbitrary system of fields, quite possibly many different fields undergoing complicated interactions, we had no right to assume that this relation would hold. The energy and momentum might be widely distributed over a large region. So Noether's theorem has a non-trivial – indeed a crucial – implication when applied to boosts.

Nevertheless we can ask what is the meaning of the quantity in (38) which is conserved? We note that its value depends upon the coordinate system, both the spatial system and the origin chosen for time. It is this latter feature which renders the quantity rather different from the other conserved quantities. By a suitable choice of the zero datum for time, the quantity in (38) can always be chosen to be zero. And since it is conserved, it is always zero.

So now we see the resolution of the apparent conflict between the implications of the Poincare commutators, (34), under a Hamiltonian interpretation, and Noether's theorem, under a Lagrangian interpretation. The commutators are quite right in indicating that there is no ***physical quantity*** of interest arising from invariance under boosts. The entity resulting from the application of Noether's theorem, (38), can be taken to be identically zero, forever.

But, on the other hand, the application of Noether's theorem to the boosts *does* produce an extremely important ***physical equation***, namely (40) which tells us what the total momentum is in terms of the total energy and the centre-of-mass velocity. This is important and non-trivial and results from the time derivative of (38) being zero, despite the fact that the quantity in (38) is itself of no interest.

So how does the Hamiltonian / commutator perspective reflect this? The answer is that, although $[P_0, K_k] = -iP_k$ implies that there is no interesting conserved quantity associated directly with $\overline{K}$, nevertheless the fact that $[P_0, P_k] = 0$ means $[P_0, [P_0, K_k]] = 0$. Since $[P_0, K_k] = -i\hbar \frac{dK_k}{dt}$ this means that $\left[P_0, \frac{dK_k}{dt}\right] = 0$ and so the object whose conservation is interesting will be $\frac{dK_k}{dt}$, rather than $\overline{K}$ itself. This aligns exactly with what has been found from Noether's theorem. Simpler still, we can consider the special case that $P_i = 0$ in which case $[P_0, K_k] = 0$ and the conserved quantity, from (40), is just ξ_i: the centre of mass position.

3. Examples of Noether Conservation for Other Symmetries – The Conservation of Charge

Consider now the case of purely internal symmetries. That is, transformations which mix the fields $\{\phi_r\}$ but have no affect upon spacetime. We can write,

$$\phi'_r(x') = \phi'_r(x) = \phi_r(x) + \varepsilon_\alpha S^\alpha_{rs} \phi_s(x) \tag{41}$$

Since $\delta x_\alpha = 0$, (13) becomes,

$$\partial_\mu\left(\frac{\partial L}{\partial \phi_{r,\mu}}\varepsilon_\alpha S^\alpha_{rs}\phi_s\right)=0 \quad (42)$$

And hence the conserved quantities are,

$$\aleph^{\mu\alpha}=\frac{\partial L}{\partial \phi_{r,\mu}}S^\alpha_{rs}\phi_s \quad (43)$$

with
$$\partial_\mu \aleph^{\mu\alpha}=0 \quad (44)$$

Hence there is one conserved quantity for each value of α, corresponding to each continuous parameter of the internal symmetry, (41). The simplest example is provided by a complex field whose Lagrangian is invariant under an arbitrary change of phase of the field, i.e.,

$$\phi\to\phi'=e^{i\varepsilon}\phi\approx\phi+i\varepsilon\phi \quad (45a)$$

The Hermetian conjugate field transforms as,

$$\phi^+\to\phi'^+=e^{-i\varepsilon}\phi^+\approx\phi^+-i\varepsilon\phi^+ \quad (45b)$$

Note that the field and its conjugate count as separate fields, i.e., the subscript *r* in (43) takes two values, say 1 and 2, for which $S_{11}=i$ and $S_{22}=-i$, and $S_{12}=0$. Hence (43) gives,

$$\aleph^\mu=\frac{\partial L}{\partial\phi,_\mu}i\phi+\frac{\partial L}{\partial\phi,^+_\mu}(-i)\phi^+ \quad (46)$$

Suppose we have a Lagrangian whose only terms in the derivatives of these fields is $\frac{1}{2}(\partial_\mu\phi)^+\partial^\mu\phi$. Then (46) gives,

$$\aleph^\mu=i\left[(\partial^\mu\phi^+)\phi-(\partial^\mu\phi)\phi^+\right]\equiv -i\phi^+\overleftrightarrow{\partial}^\mu\phi \quad (47)$$

where $\overleftrightarrow{\partial}^\mu$ is the antisymmetric derivative defined by (47). The Noether conserved quantity is given by,

$$Q=\int\aleph^0 d^3x=\int i\left[\pi\phi-\pi^+\phi^+\right]d^3x \quad (48)$$

where we have used the conjugate fields $\pi=\frac{\partial L}{\partial\phi_{,0}}=\phi^+_{,0}$. In quantum field theory this conserved quantity can be interpreted as the electric charge (or, at least, if we multiply by the quantum of charge it can, $\aleph^\mu\to ie\phi^+\overleftrightarrow{\partial}^\mu\phi$). This is because it can be written in terms of the number operators for particles and antiparticles, $\hat{N}$ and $\hat{N}^a$, as,

$$Q=e\sum_{\bar{k}}\left[a^+_{\bar{k}}a_{\bar{k}}-b^+_{\bar{k}}b_{\bar{k}}\right]=e\sum_{\bar{k}}\left(\hat{N}_{\bar{k}}-\hat{N}^a_{\bar{k}}\right)=e\left(\hat{N}-\hat{N}^a\right) \quad (49)$$

as shown in Chapter 43. This is a consistent interpretation because particles and antiparticles are oppositely charged, so the total charge is just the difference in their numbers, as given by (49). Consequently $\aleph^\mu$ can be interpreted as the 4-current density.

Note that invariance under a phase change is effectively a symmetry under group U(1). Consequently we see that the conservation of charge is due to U(1) symmetry.

Loose Ends

In discussing the interpretation of the Noether quantity conserved under boosts in §2.3, we confined attention to the case of a scalar field. This leaves unexplained what might be the meaning or significance of the spin dependent term in (35).