Chapter 23

Cosmic Geometry: FLRW Spacetimes

Space and time in general relativistic cosmology; Friedmann-Lemaître-Robertson-Walker spacetimes; positive and negative curvature and whether the universe is finite or infinite; the critical density revisited; the cosmological constant and its implications for universal expansion.

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1. General Relativity and Cosmology

At the risk of heresy, whether general relativity is essential in cosmology would make an interesting debate. The defining features of general relativity are the absence of a preferred coordinate system and the curvature of spacetime. Yet the standard model of cosmology is based upon a preferred cosmic time coordinate, and observational evidence is consistent with spacetime being flat. Be that as it may, general relativity provides a broader forum for discourse than a Newtonian outlook and looses nothing by admitting the possibility of spacetime curvature, to be decided by observation. Personally I have always found the notion of a positively curved space comforting, providing as it does an elegant answer to every child's question, "what is beyond the edge of the universe?". Only general relativity provides the neat answer that space can be finite but unbounded. Many people, however, would prefer the universe to be infinite. Perhaps it is.

The expansion of the universe is probably the best argument in favour of the general relativistic perspective. This could possibly be understood dynamically rather than geometrically, for example by the Newtonian cosmology model (see Chapter 57). But this is obviously fraught with problems. If comoving observers are both equivalent and experience non-uniform relative motion, then some form of relativity of non-uniform motions appears unavoidable – enter general relativity. The apparent acceleration in the cosmic expansion is a further motivation for a general relativistic formulation. An explanation in terms of a cosmological constant is mathematically natural in the Einstein field equations (if physically mysterious).

Consequently, this Chapter explains the geometrical aspects of the standard general relativistic cosmological models and how they expand.

2. The Einstein Field Equations

The field equations in general form are stated here only for completeness. The far simpler (Friedmann) equations suffice to discuss the standard homogeneous cosmological models. However, the point is that the Friedmann equations are the result of applying certain simplifications to the Einstein field equations. It is important to appreciate this, though it will not be proved here.

Newtonian gravity is based on a single scalar potential which obeys a Poisson equation, $\nabla^2\phi = 4\pi G\rho$. In this equation, the mass density, ρ, is the source of the gravitational potential, ϕ. Thus, a point source $\rho = M\delta^3(r)$ gives the familiar solution $\phi = -GM/r$. In general relativity, the single scalar potential is replaced by ten potential functions which form the components of a symmetrical rank two tensor in the four dimensions of space-time, $g_{\alpha\beta}$. This tensor is the metric tensor of space-time, defined so that the line element ds between any two neighbouring points (events) in space-time is a co-ordinate system invariant quadratic form in the co-ordinate differences between the points, i.e.,

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \quad (1)$$

This line element with respect to any curvilinear coordinate system in spacetime is the generalisation of the Minkowski line element in special relativity, $ds^2 = c^2 dt^2 - |d\bar{r}|^2$.

That the metric tensor is also the potential field of gravitation is the major insight of general relativity and leads to elucidation of the equality of inertial and gravitational mass. In place of the linear Poisson equation we have the non-linear Einstein field equations. There are ten of these, which, with suitable boundary conditions, may suffice to determine the ten unknown potentials, $g_{\alpha\beta}$. They can be written,

$$E_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \quad (2)$$

where, $$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}; \quad R \equiv R^a{}_\alpha \equiv g^{\alpha\beta} R_{\alpha\beta} \quad (3)$$

and, $$R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu} \equiv g^{\alpha\beta} R_{\beta\mu\alpha\nu} \quad (4)$$

and, $$R^\alpha{}_{\mu\beta\nu} \equiv \Gamma^\alpha{}_{\mu\nu,\beta} - \Gamma^\alpha{}_{\beta\mu,\nu} - \Gamma^\alpha{}_{\tau\beta}\Gamma^\tau{}_{\nu\mu} + \Gamma^\alpha{}_{\tau\nu}\Gamma^\tau{}_{\beta\mu} \quad (5)$$

and, $$\Gamma^\alpha{}_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\tau} \left[g_{\beta\gamma,\tau} - g_{\beta\tau,\gamma} - g_{\gamma\tau,\beta} \right] \quad (6)$$

where Greek indices stand for the four spacetime coordinates and we have used the convention that repeated indices are summed. A comma before an index implies ordinary partial differentiation, e.g., $X_{\alpha,\beta} \equiv \frac{\partial X_\alpha}{\partial x^\beta}$. The metric tensor can be used to lower or raise indices to convert between covariant and contravariant forms. Note that we have written (2) using a time-like convention for the metric, with $g_{00} > 0$, which requires the minus sign on the RHS of (2).

The Einstein tensor has been denoted $E_{\mu\nu}$, and is defined in terms of the metric tensor, $g_{\alpha\beta}$, and its first and second derivatives through (3-6). The field equations, (2), are linear in the second derivatives of the metric tensor, but quadratic in the first derivatives. However there are also various factors of g around. In short, the field equations are nastily non-linear. A more detailed explanation of these matters is beyond the scope of this Chapter, but see for example Adler, Bazin and Schiffer (1975) or Misner, Thorne and Wheeler (1973).

The LHS of the field equation, (2), is the most general expression which is linear in the second derivatives of the metric and also generally covariant (i.e. a second rank tensor with respect to arbitrary curvilinear coordinate transformations). It was these two requirements which led Einstein to these equations. An additional requirement is that permissible spacetimes must be locally Lorentzian. This means that within any small region the spacetime can be approximated by a Minkowski coordinate system.

Whatever appears on the RHS of the field equations, (2), acts as the source of gravity. In the Einstein equations, (2), the energy-momentum tensor occurs in place of what was just the mass density in the Newtonian Poisson equation. We have written (2) so that $T_{\mu\nu}$ has units of energy density. The $_{00}$ component of the energy-momentum tensor is the nearest equivalent to the Newtonian density, just as the $_{00}$ component of the metric tensor is the nearest equivalent to the Newtonian potential (modulo unity). Also on the RHS of (2) is *G*, the universal gravitation constant, identical to Newton's universal constant of gravity (6.67 x 10^{-11} $m^3kg^{-1}s^{-2}$). General relativity introduces no new universal constants (in contrast to the standard model of particle physics).

Finally, the scalar Λ on the LHS of (2) is the infamous Cosmological Constant. Mathematically this can appear in the equation because $E_{\mu\nu} + \Lambda g_{\mu\nu}$ is the most general rank two tensor which is linear in the second derivatives of the metric. Both $E_{\mu\nu}$ and Λ have dimensions L^{-2} (MKS unit m^{-2}), the dimensions of curvature. This matches the units of the RHS of (2). Note that $g_{\alpha\beta}$ is dimensionless. Thus, the cosmological constant has units consistent with an interpretation as a form of curvature or, alternatively, $\xi_\Lambda \equiv \frac{\Lambda c^4}{8\pi G}$ can be considered as an energy density. This latter perspective yields the interpretation of the cosmological constant as some form of "dark energy". Moving the cosmological constant term to the RHS of (2), it appears as a source term. The RHS is then $-\frac{8\pi G}{c^4}\left(T_{\mu\nu} + \xi_\Lambda g_{\mu\nu}\right)$. You might be forgiven for concluding from this (erroneously) that a positive cosmological constant, $\xi_\Lambda > 0$, would act like a positive mass-energy density and hence produce an attractive gravitational effect. This would be the case if only the g_{00} component were relevant in this expression. But this is not the case. If the metric is approximately Minkowskian then the total effect of the cosmological constant is given by a tensorial source,

$$\xi_\Lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (7)$$

The negative pressure terms turn out to be dominant and result in a form of repulsive gravity.

3. The Friedmann Equations and the FLRW Metric

The general formulation of the field equations, (2-6), is rather forbidding. Fortunately the assumptions that underlie the standard cosmological model lead to a dramatic simplification. These assumptions are,

- There is a universal (cosmic) time co-ordinate, *t*;
- The spatial 3-spaces defined by *t* = constant are isotropic and homogeneous.

The latter condition is essentially a generalised Copernican principle: there is no preferred place or orientation in the universe. The idea is that the universe is filled with a substratum of material, energy, etc., which is uniform on a suitably large spatial scale. A "co-moving observer" is one who moves along with this material substratum. The centres of gravity of sufficiently large clusters of galaxies are expected to be co-moving: they are embedded in the "Hubble flow". In contrast, individual galaxies, and matter on all smaller scales, will be affected by peculiar motions – local motions which depart from the Hubble flow.

The most general solution of the Einstein field equations under the above simplifying assumptions is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric,

$$ds^2 = c^2 dt^2 - R(t)^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta . d\phi^2 \right\} \tag{8}$$

In (8), r is a dimensionless coordinate marker. The range of values it is permitted to take is not yet clear and will be clarified below. It is *not* in general valid to assume that r can take any positive value. The variables θ, ϕ take the same range of values as polar coordinates in flat space, i.e., $0 \le \phi < 2\pi$, $0 \le \theta \le \pi$. In (8), k is a dimensionless constant which will be seen to correspond to the spatial curvature: $k > 0$ being positive curvature, $k < 0$ being negative curvature, and $k = 0$ being spatially flat. The magnitude of k will not alter the underlying geometry since it can be removed from the metric (8) by the replacements $\tilde{r}^2 = |k| r^2$, $\tilde{R}(t)^2 / |k| = R(t)^2 / |k|$. However the *sign* of k is crucial.

The spatial dimensions of the metric are carried by the term $R(t)$. The solution, (8), is not a unique metric but a class of metrics depending upon the functional dependence of $R(t)$ on the time coordinate, t. This in turn depends upon the contents of the universe, i.e., the source terms $T_{\mu\nu}$ and Λ in (2). The sources must be consistent with the assumed homogeneity and isotropy of the spacetime. A simple assumption for the stress-energy tensor consistent with homogeneity and isotropy is that it corresponds to a fluid of density ρ and pressure p, and hence,

$$\{T_{\mu\nu}\} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{9}$$

where homogeneity requires that ρ, p do not depend upon the spatial coordinates but will depend upon time in general. The condition that (8) and (9) be a solution to the Einstein field equations, (2), reduces to just two equations which must be obeyed by $R(t)$,

$$\ddot{R} = -4\pi G \left(\rho + \frac{3p}{c^2} \right) \frac{R}{3} + \frac{\Lambda c^2}{3} R \tag{10a}$$

$$\dot{R}^2 = \frac{8\pi}{3} G \rho R^2 - kc^2 + \frac{\Lambda c^2}{3} R^2 \tag{10b}$$

These are the Friedmann equations. In the case that the contents of the universe are matter dominated and with negligible pressure, conservation of mass means that ρR^3 will be constant. In this case, differentiating (10b) allows (10a) to be derived for $p = 0$, so the two equations are not independent. More generally, when pressure is not negligible, differentiating (10b) and subtracting from (10a) gives the so-called *fluid equation*,

$$c^2 \frac{d}{dt}\left(\rho R^3\right) + p\frac{d}{dt}\left(R^3\right) = 0 \qquad (11)$$

For a given universal expansion, $R(t)$, (11) relates the pressure and density of the fluid: it is the equation of state for the fluid. Modulo some numerical factor involving π, the first term in (11) is just the rate of change of the amount of mass-energy in a sphere of radius *R*. Including the same numerical factor, the second term in (11) is pdV per time interval *dt*, i.e., the work done by the radiation and matter in expanding during that period. Hence we recognise (11) as the thermodynamic statement that the change in internal energy balances against the work done by the fluid in expanding. Note that (11) makes rigorous our previous claim that ρR^3 is necessarily constant in FLRW spacetimes if $p = 0$, whatever the underlying geometry. The pair of Friedmann equations, (10a) and (10b), can equivalently be replaced by (10b) and (11).

We can now see why a positive cosmological constant causes a form of repulsive gravity. Firstly we note that if $\Lambda > 0$ then the cosmological term on the RHS of (10a) has the opposite sign from the matter term involving the fluid density and pressure. Since the LHS of (10a) is acceleration, this implies a type of repulsive gravity causing positive acceleration, $\ddot{R}$. But why ***is*** the sign of the Λ term on the RHS of (10a) positive? This can be understood as follows. The cosmological term, (7), can be mimicked by a fluid energy-momentum tensor, (9), if we choose the density to be $\rho = \xi_\Lambda / c^2$ and the pressure to be negative, $p = -\xi_\Lambda$. The effect of the cosmological term can then be deduce from the *first* term on the RHS of (10a), which becomes,

$$-\frac{4\pi}{3}GR\left(\rho + \frac{3p}{c^2}\right) = -\frac{4\pi}{3}\frac{GR}{c^2}\left(\xi_\Lambda - 3\xi_\Lambda\right) = \frac{8\pi}{3}\frac{GR}{c^2}\xi_\Lambda = \frac{\Lambda c^2}{3}R \qquad (12)$$

in agreement with the second term on the RHS of (10a). The point of this is that it shows that the coefficient of the Λ term becomes positive as a result of the tensorial source, (7), having negative pressure. The 'density' part of dark energy produces normal attractive gravity. It is its negative pressure which causes the repulsive effect, and this dominates.

In the case of zero cosmological constant and zero pressure, the first Friedmann equation, (10a), is identical to the equation of motion for a Newtonian cosmology - see Equ.(4.1) of Chapter 57 – because the RHS is then just the Newtonian gravitational force on a unit test mass, $-4\pi RG\rho/3 = -GM/R^2$. Hence the second Friedmann equation with $\Lambda = 0$ also applies in the Newtonian model, see Equ.(4.5) of Chapter 57, but the constant k now arises as an integration constant, to be determined by some initial conditions, rather than as spatial curvature.

Actually we have already come across the second Friedmann equation with $\Lambda = 0$ in another guise, namely in Chapter 11, Equ.(3). This gave the total non-relativistic energy, E, of a small test mass, m, in terms of the kinetic and gravitational potential energies. Re-arranging it gives,

$$\dot{R}^2 = \frac{8\pi}{3}G\rho R^2 + \frac{2E}{m} \qquad (13)$$

Comparison with (10b) leads to another interpretation of the parameter k as proportional to the total energy per unit mass, but of opposite sign, $kc^2 = -2E/m$. The elementary arguments of Chapter 11 have already allowed us to conclude that the universe will expand indefinitely if $E > 0$ but will reach a maximum size and re-contract if $E < 0$. Consequently, in the case of zero cosmological constant we can expect that $k < 0$ will correspond to an indefinitely expanding universe, whereas $k > 0$ will produce a universe which reaches a maximum size and re-contracts. We shall see shortly that this is indeed the case. But we will also see that a non-zero cosmological constant changes this simple picture dramatically.

4. FLRW Space with *k* > 0 as a Hypersphere Embedded in 4D Euclidean Space

So far we have not explained why the FLRW metric, (8), with $k > 0$ should be regarded as having "positive curvature". Note that we are concerned here with the curvature of the 3D sub-spaces defined by a constant time coordinate, not the curvature of spacetime. In general the spatial part of a solution to the Einstein field equations cannot necessarily be embedded in a higher dimensional Euclidean space. However it so happens that this can be done for the FLRW metric when $k > 0$. In fact, the spatial part can be interpreted as the surface of a 3-sphere embedded in a 4D Euclidean space. This is the most convincing way to demonstrate that it is positively curved. But note that this does not imply that there is anything physically real about the 4D embedding space.

Call the Cartesian co-ordinates of the hypothetical 4D Euclidean embedding space x_1, x_2, x_3, x_4. A 3-sphere of radius R in this space is defined by,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2 \tag{14}$$

Note that neither x_4, nor any of the other co-ordinates in (14), have anything whatsoever to do with the time co-ordinate, t, in spacetime. We are considering only the spatial part of spacetime, at some fixed time.

The line element in the 4D embedding space will be denoted ds_4, whereas a line element within the 3D sub-space spanned by x_1, x_2, x_3 will be denoted ds_3. In terms of spherical polar co-ordinates in this latter 3D space we can write,

$$ds_3^2 = d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right) \tag{15}$$

Note the usual convention that $d\rho^2$ means $(d\rho)^2$, not $d(\rho^2)$. Where the latter is meant it will be written explicitly. We have,

$$x_1 = \rho\sin\theta\cos\phi\,, \qquad x_2 = \rho\sin\theta\sin\phi\,, \qquad x_3 = \rho\cos\theta \tag{16}$$

i.e.,

$$\rho^2 = x_1^2 + x_2^2 + x_3^2 \tag{17}$$

and the 4D line element is given by $ds_4^2 = ds_3^2 + dx_4^2$ (18)

Now by taking the derivative of (14) we can find dx_4 as follows,

$$x_1dx_1 + x_2dx_2 + x_3dx_3 + x_4dx_4 = 0 \Rightarrow dx_4 = -\frac{\sum_{i=1}^{3} x_i dx_i}{x_4} \tag{19}$$

But,

$$\sum_{i=1}^{3} x_i dx_i = \frac{1}{2}\cdot d\left(x_1^2 + x_2^2 + x_3^2\right) = \frac{1}{2}d(\rho^2) \tag{20}$$

and from (14) and (17)

$$x_4^2 = R^2 - \rho^2 \tag{21}$$

Hence, (19) becomes, $$(dx_4)^2 = \frac{\left[d(\rho^2)\right]^2}{4\left(R^2-\rho^2\right)} = \frac{\rho^2 d\rho^2}{\left(R^2-\rho^2\right)} \tag{22}$$

Hence, (18) with (15) and (22) give,

$$\begin{aligned} ds_4^2 &= d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right) + \frac{\rho^2 d\rho^2}{\left(R^2-\rho^2\right)} \\ &= \frac{R^2 d\rho^2}{\left(R^2-\rho^2\right)} + \rho^2\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right) \end{aligned} \tag{23}$$

Both ρ and R have dimensions of length. Switching to the dimensionless coordinate,

$$r = \rho / R \tag{24}$$

(23) becomes,

$$ds_4^2 = R^2\left[\frac{dr^2}{\left(1-r^2\right)} + r^2\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right)\right] \tag{25}$$

So we can identify (25) with the spatial part of the FLRW metric, (8), for $k > 0$. Since a hypersphere clearly has positive curvature, this establishes that the spatial part of the $k > 0$ FLRW metric has positive curvature.

This embedding directly demonstrates that the positively curved FLRW space is finite. This conclusion will be reinforced shortly by calculating its volume.

Moreover, the correspondence between (8) and (25) means that we can interpret the parameter $R(t)$ appearing in the FLRW metric as the radius of the hypersphere. This 'radius' lives in the embedding space, not in our spacetime, and hence need have no physical reality. Nevertheless, this embedding provides a homely interpretation of the positively curved FLRW metric which is unfortunately not possible for the $k < 0$ case.

5. Can the k <0 FLRW Space be Embedded in 4D Euclidean Space?

The reason why the embedding of the positively curved FLRW space in 4D is helpful is because it is sufficiently similar to a sphere in 3D, which we can visualise. Unfortunately, the equivalent negatively curved 2-surface, the hyperbolic plane, cannot be embedded in 3D Euclidean space (though that has not deterred Daina Taimina and others from creating attempts at its likeness in wool, see Henderson and Taimina (2001) and Figure 1). We are therefore unable to visualise even a 2D negatively curved surface, let alone the 3D negatively curved FLRW space. Moreover, the negatively curved FLRW space is not embeddable in 4D Euclidean space anyway, so even if we could visualise four dimensions it would not help.

Figure 1: A Rendering of the Hyperbolic Plane in Wool (by Daina Taimina, 2005) need permission

We will not attempt a general proof of the latter claim. However it is instructive to make a few obvious attempts. For example one might guess that the FLRW space with $k < 0$ might relate to a hyperboloid in 4D such as,

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = R^2 \tag{26a}$$

Or,

$$-\left(x_1^2 + x_2^2 + x_3^2\right) + x_4^2 = R^2 \tag{26b}$$

However, if the 4D space in question is Euclidean so that its line element is $ds_4^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$, as in §4, then neither of (26a,b) reproduce the FLRW metric, as is readily shown by following the same approach as in §4. Let us demonstrate this using (26b). The spherical polars for the subspace x_1, x_2, x_3 are employed unchanged from before. By taking the derivative of (26b) we find dx_4 as follows,

$$-\left(x_1 dx_1 + x_2 dx_2 + x_3 dx_3\right) + x_4 dx_4 = 0 \Rightarrow dx_4 = + \frac{\sum_{i=1}^{3} x_i dx_i}{x_4} \tag{27}$$

But, $$\sum_{i=1}^{3} x_i dx_i = \frac{1}{2} \cdot d\left(x_1^2 + x_2^2 + x_3^2\right) = \frac{1}{2} d(\rho^2) \tag{28}$$

and from (26b) $$x_4^2 = R^2 + \rho^2 \tag{29}$$

Hence, (27) becomes, $$\left(dx_4\right)^2 = \frac{\left[d(\rho^2)\right]^2}{4\left(R^2 + \rho^2\right)} = \frac{\rho^2 d\rho^2}{\left(R^2 + \rho^2\right)} \tag{30}$$

If the 4D embedding space is assumed to be Euclidean then (15-18) hold and so $ds_4^2 = ds_3^2 + dx_4^2$ with (30) leads to,

$$\begin{aligned} ds_4^2 &= d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right) + \frac{\rho^2 d\rho^2}{\left(\rho^2 + R^2\right)} \\ &= \frac{\left(R^2 + 2\rho^2\right) d\rho^2}{\left(R^2 + \rho^2\right)} + \rho^2\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right) \end{aligned} \tag{31}$$

which is *not* the same as the FLRW metric, (8), as we anticipated. However, an embedding is possible if we assume the 4D embedding space is itself Minkowskian, that is assuming the metric,

$$ds_4^2 = ds_3^2 - dx_4^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 = d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right) - dx_4^2 \tag{32}$$

in place of (18). In this case (31) becomes,

$$\begin{aligned} ds_4^2 &= d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right) - \frac{\rho^2 d\rho^2}{\left(\rho^2 + R^2\right)} \\ &= \frac{R^2 d\rho^2}{\left(R^2 + \rho^2\right)} + \rho^2\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right) \\ &= R^2\left[\frac{dr^2}{\left(1 + r^2\right)} + r^2\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right)\right] \end{aligned} \tag{33}$$

where $r = \rho / R$, and (33) ***is*** the same as the spatial part of the FLRW metric, (8), when $k < 0$. Note that this requires an embedding of a 3D space-like region into a 4D Minkowski space and so does not aid true visualisation since we cannot really visualise Minkowski space. The 3-surface given by (26b) is a hyperboloid when plotted on (x_1, x_2, x_3, x_4) coordinates. This plot can be realised by suppressing one of the first three dimensions and plotting in, say, (x_2, x_3, x_4) coordinates. The FLRW space appears as a hyperboloid of revolution. This is misleading, though, since the (x_2, x_3, x_4) plot encourages one to imagine that (x_2, x_3, x_4) is Euclidean, but really it is Minkowskian. However it is indicative in one respect: the hyperboloid is infinite in extent, which faithfully indicates the fact that space in the negatively curved FLRW spacetime is indeed infinite. This will be demonstrated directly shortly.

Note that this embedding of the spatial part of the negatively curved FLRW metric induces an embedding of the whole 4D spacetime into a flat 5D manifold with signature $(-,-,+,+,+)$. How would we distinguish between the two candidate time directions?

Rather surprisingly, any solution of the Einstein field equations can be embedded in a Ricci-flat 5D manifold, a result known as the Campbell-Magaard theorem, Campbell (1926), Magaard (1963).

6. But Why Does $k<0$ Mean Negative Curvature?

Strictly §5 has not conclusively shown that the $k<0$ solution is negatively curved. Had we shown it equivalent to the hyperboloid (26b) in Euclidean space then that would have clinched the matter. But it is not obvious (to me, anyway) that negative curvature is implied by the embedding in Minkowski space. The issue can be decided by evaluating the Ricci scalar of curvature for the 3D part of the metric. Misner, Thorne and Wheeler (1973), §27.6, gives an expression for the Riemann tensor in any 3D space of uniform curvature. From this it follows that the Ricci scalar is $6k/R^2$, check hence the sign of k is the sign of the curvature. However there is another way of demonstrating this which I find more satisfactory.

Consider the ratio of the circumference of a circle to its radius. In flat space this is 2π. In a positively curved space the ratio will be $<2\pi$. To see this just imagine a circle on a sphere. The radial arc lying on the surface of the sphere is longer than the true radius through the inside of the sphere. But as far as the intrinsic geometry is concerned, the radial arc *is* the radius – so the ratio is $<2\pi$. Conversely the signature of negative curvature is that this ratio is $>2\pi$.

Take the circle to be $\theta=\pi/2, r=1$. An element of its circumference, for either sign of k, has $d\theta=dr=0$ and hence is $ds=Rd\phi$, so the circumference is $2\pi R$ in both cases. However, for $k>0$ the radius of the circle is evaluated by integrating the line elements along $d\theta=0, d\phi=0$, that is, radius $=\int_0^1 R\frac{dr}{\sqrt{1-r^2}}=R\sin^{-1}(1)=\frac{\pi}{2}R$. Hence the ratio of circumference to radius for $k>0$ is $\frac{2\pi}{\pi/2}=4<2\pi$ and so is positively curved. For $k<0$ the radius is $\int_0^1 R\frac{dr}{\sqrt{1+r^2}}=R\sinh^{-1}(1)=0.8814R$ and hence the ratio of circumference to radius is $\frac{2\pi}{0.8814}>2\pi$ and so is negatively curved.

7. FLRW Space: Finite or Infinite?

7.1 Positive Curvature, *k* > 0

For $k>0$, we have tacitly assumed that the term in $\{\ldots\}$ in (8) is spacelike. However, taking $k=1$, this is only true if $|r|<1$ so we must restrict r to lie in this interval. This can also be seen from the embedding as a hypersphere in 4D Euclidean space, §4, since this requires, from (21) and (24),

$$x_4^2=R^2-r^2R^2 \tag{34}$$

and on the hypersphere we require $-R\le x_4\le R$ so that the possible range of r is $[0,1]$. (Strictly there is a larger, analytically continued space which includes the manifold at negative r, but we will not discuss that here). This suggests an alternative co-ordinate system in which, using (16),

$$r = \frac{\rho}{R} = \sin\psi\,; \quad x_4 = R\cos\psi \tag{35a}$$

$$x_1 = R\sin\psi\sin\theta\cos\phi\,; \quad x_2 = R\sin\psi\sin\theta\sin\phi\,; \quad x_3 = R\sin\psi\cos\theta \tag{35b}$$

where ψ is a hyperspherical polar angle with respect to the x_4 axis, whilst R is the radius of the hypersphere, and ρ is the projection onto the subspace x_1, x_2, x_3 of the hypothetical 4D Euclidean space x_1, x_2, x_3, x_4. Thus the whole of space is covered by the finite angular ranges,

$$0 \le \psi \le \pi; \quad 0 \le \theta \le \pi; \quad 0 \le \phi < 2\pi \tag{36}$$

In terms of ψ the space-time metric becomes,

$$ds^2 = c^2 dt^2 - R^2(t)\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\cdot d\phi^2\right)\right] \tag{37}$$

It is tempting to conclude that because the co-ordinate ranges in (36) are all finite, this implies that the space in question is finite. However, that would not be a correct argument (though, as it happens, the answer is). The reason is that a finite coordinate range can be made infinite by a suitable co-ordinate transformation, for example $\tan(\psi/2)$ will range from 0 to infinity for $\psi \in [0,\pi]$. To determine if space is finite we need to calculate its volume. Because volume is a scalar the result is independent of what coordinate system we use. Of course, we already know that the volume is finite since it equals the 'surface area' of the hypersphere derived in §4 (and hence is $2\pi^2 R^3$). This can be evaluated explicitly as follows: an element of the hypersphere's 'area' is given by,

$$dV = Rd\psi.dS_\rho \tag{38}$$

where dS_ρ is the element of area of an ordinary sphere of radius ρ, i.e.,

$$dS_\rho = \rho^2 d(\cos\theta)d\phi = R^2\sin^2\psi d(\cos\theta)d\phi \tag{39}$$

Hence, $$V = R^3 \int_0^\pi\int_0^\pi\int_0^{2\pi} d\psi d\theta d\phi \sin^2\psi\sin\theta = 4\pi R^3\int_0^\pi d\psi\sin^2\psi = 2\pi^2 R^3 \tag{40}$$

Thus, we have established that the FLRW spacetime with $k > 0$ consists of 3D spacelike subspaces which are finite in the sense of having finite volume. However, this volume will vary with time since R is time dependent. However, at any time, R is a true measure of the size of the universe, since $2\pi^2 R^3$ is the genuine, finite volume of the universe.

This also means that a $k > 0$ FLRW spacetime would start from a single point at the Big Bang, since R reduces to zero at $t = 0$ (see later).

7.2 Negative Curvature, $k < 0$

Unlike $k > 0$, for $k < 0$ the {…} in (8) remains a spacelike interval (i.e., positive) for any real r. We may immediately suspect that this implies that space is infinite in this case, and this suspicion has been reinforced by the infinite hyperboloid embedding demonstrated in §5. However neither of these arguments is quite rigorous. We need to investigate whether the volume is infinite.

We note that the substitution (35a) is not valid in this case since it would restrict the possible range of r and we must permit any real r. The whole of the real line is available to r if we make the substitution,

$$r = \sinh\psi \tag{41}$$

The equivalent of (37) is easily derived to be,

$$ds^2 = c^2dt^2 - R^2(t)\left[d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right)\right] \tag{42}$$

identical to (37) except for the replacement of $\sin\psi$ by $\sinh\psi$. However, the permitted co-ordinate ranges are also modified, thus,

$$0 \le \psi \le \infty; \quad 0 \le \theta \le \pi; \quad 0 \le \phi < 2\pi \tag{43}$$

The volume is found as follows: the metric (42) tells us the length associated with the change in each co-ordinate. For a change $d\psi$ with other co-ordinates constant the length element is $Rd\psi$; for a change $d\theta$ with other co-ordinates constant the length element is $R\sinh\psi d\theta$; for a change $d\phi$ with other co-ordinates constant the length element is $R\sinh\psi\sin\theta d\phi$. The product of these three lengths gives the element of volume (because the coordinates are orthogonal, or, equivalently, because the metric is diagonal),

$$dV = R^3\sinh^2\psi\sin\theta d\psi d\theta d\phi \tag{44}$$

This is the same as (38,39) except for the replacement of $\sin\psi$ by $\sinh\psi$. Attempting to integrate (44) over the full range of the co-ordinates as given by (4.29) reveals that this space has infinite volume. Whilst the integral over the angular co-ordinates θ, ϕ evaluates to 4π, the usual solid angle, the integral of $\sinh^2\psi$ over $\psi \in [0, \infty]$ is clearly divergent. Thus, we conclude that negatively curved FLRW space is infinite.

Note that space is infinite at all times $t > 0$ for which R is non-zero. Thus, at the first instant after the Big Bang, when t is just one quantum of time, space is already infinite. Thus, FLRW spacetime with negative curvature must have been created infinite. So it is ***not*** correct in the $k < 0$ case to think of the universe as springing from a single point at the Big Bang. Rather, all points in an already infinite space were created at the first instant, and all points explode simultaneously. Nevertheless, if we consider any finite region at a finite time t, this region will shrink to zero size as we go backwards in time towards $t = 0$. This is true no matter how large a finite region we consider. Thus, the density must be infinite at $t = 0$. Such solutions therefore have an initially infinite density at all points in an infinitely large volume.

8. Cosmological Parameters Defined

It is useful to define certain key parameters which are most significant in the large scale structure of the universe. The first cosmological parameter is already familiar. It is the Hubble parameter, given by,

$$H(t) = \frac{\dot{R}}{R} \tag{45}$$

The second cosmological parameter we have also already met, in Chapter 11. It is the critical density, but now we define it more precisely as,

$$\rho_{crit}(t) = \frac{3H^2}{8\pi G} \tag{46}$$

It so happens that in our universe at the present epoch we have $H \approx 1/t$, to a good approximation, so that we can also write $\rho_{crit} = 3/8\pi Gt^2$. However this is not true in general, i.e., for different values of k or Λ, or even for our universe in earlier epochs. The next cosmological parameter is perhaps the most important of all, and we also met it previously in Chapter 11. It is the ratio of the total density of all forms of matter, ρ_m, to the critical density,

$$\Omega_m(t) = \frac{\rho_m}{\rho_{critical}} = \frac{8\pi G\rho_m}{3H^2} \tag{47}$$

All these parameters are time dependent, as we have emphasised by the notation in (45-47). However in general we will not display the time dependence from now on.

The total matter density is often divided into two parts: the so-called baryonic part (this is, ordinary matter), ρ_b, and that due to dark matter, denoted ρ_c. Hence we have two dimensionless density parameters,

$$\Omega_b = \frac{\rho_b}{\rho_{critical}} \qquad \Omega_c = \frac{\rho_c}{\rho_{critical}} \qquad \Omega_m = \Omega_b + \Omega_c \tag{48}$$

We saw in §2 that the cosmological constant, Λ, was equivalent to an energy density, $\xi_\Lambda \equiv \frac{\Lambda c^4}{8\pi G}$, so a further density ratio is,

$$\Omega_\Lambda = \frac{\xi_\Lambda}{c^2\rho_{crit}} = \frac{\Lambda c^2}{3H^2} \tag{49}$$

which may be regarded as the density parameter for 'dark energy'. So we can define the sum of all density parameters as,

$$\Omega = \Omega_m + \Omega_\Lambda \tag{50}$$

It is also convenient to define the reciprocal of the Hubble parameter,

$$\tau = \frac{1}{H} \tag{51}$$

Evaluated at the present epoch it is denoted τ_0. The age of the universe equals τ_0 to within a factor close to unity (see Chapter spare). Finally, the dimensionless deceleration parameter is defined as,

$$q(t) = -\frac{R\ddot{R}}{\dot{R}^2} \tag{52}$$

The deceleration parameter was defined in an era when cosmologists still thought that the acceleration of the universe would be slowing down, so that $\ddot{R}$ was expected to be negative and hence *q* positive. It is currently believed that $\ddot{R}$ is positive and hence *q* negative - perhaps due to the effects of a positive cosmological constant or dark energy.

Note that $H, \rho_b, \rho_c, \Omega_b, \Omega_c, \Omega_\Lambda, q$ and, of course, *R*, all vary with time. Only Λ and *k* (and *G* and *c*) are constants.

Relationships between these parameters are derived as follows. Firstly we express the curvature parameter, k, in terms of the other parameters. The second Friedmann equation, (10b), gives, in the case of matter dominance so that $\rho = \rho_m$,

$$\begin{aligned} kc^2 &= \frac{8\pi}{3} G\rho R^2 + \frac{\Lambda c^2}{3} R^2 - \dot{R}^2 = \Omega_m H^2 R^2 + \Omega_\Lambda H^2 R^2 - H^2 R^2 \\ &= H^2 R^2 (\Omega_m + \Omega_\Lambda - 1) = H^2 R^2 (\Omega - 1) \end{aligned} \tag{53}$$

For convenience we define an alternative dimensionless curvature parameter, Ω_k, as,

$$\Omega_k = -\frac{kc^2}{H^2 R^2} = -\frac{kc^2}{\dot{R}^2} \tag{54}$$

So (53) becomes simply,

$$\Omega_k + \Omega = 1 \tag{55}$$

(55) is actually just a re-writing of the Friedmann equation, (10b). Once again (55) tells us that if the total density, from all sources, is less than the critical density, $\Omega < 1$, then $\Omega_k > 0$ and $k < 0$ and the universe is infinite and negatively curved. Conversely, if the total density, from all sources, is greater than the critical density, $\Omega > 1$, then $\Omega_k < 0$ and $k > 0$ and the universe is finite and positively curved. The new element to this observation is that dark energy (or the cosmological constant) is included in the total density inventory via (49,50).

Using the Friedmann equations, (10a,b), the deceleration parameter can be written,

$$q = \frac{4\pi G\left(\rho + \frac{3p}{c^2}\right)\frac{R^2}{3} - \frac{\Lambda c^2}{3} R^2}{\frac{8\pi}{3} G\rho R^2 - kc^2 + \frac{\Lambda c^2}{3} R^2} \tag{56}$$

In the case $p = 0$, dividing (56) by H^2 gives,

$$q = \frac{\frac{1}{2}\Omega_m R^2 - \Omega_\Lambda R^2}{\Omega_m R^2 + \Omega_\Lambda R^2 + \Omega_R R^2} = \frac{1}{2}\Omega_m - \Omega_\Lambda \tag{57}$$

Or, $$\Omega_m = 2(q + \Omega_\Lambda) \qquad \text{for } p = 0 \tag{58}$$

Thus, only two of these three parameters are independent in a matter dominated universe with negligible pressure. (58) shows that we will have a universe with accelerating expansion ($q < 0$) if the dimensionless cosmological constant (Ω_Λ) is positive and greater than half the dimensionless matter density, Ω_m. This condition appears to be easily met in our universe at present, see §9.

Finally, the following are also ways of re-writing the second Friedmann equation, (10b), in the case of matter dominance,

$$\frac{\Omega_m}{\Omega - 1} R = \frac{A^2}{kc^2} \qquad \text{and} \qquad \frac{\Omega_m}{|\Omega - 1|^{3/2}} \tau = \frac{A^2}{|k|^{3/2} c^3} \tag{59}$$

where, $$A^2 = \frac{8\pi}{3} G\rho R^3 \tag{60}$$

The point here is that A^2 is a constant when $p = 0$ (matter dominance). So (59) relates the size scale and the Hubble time to the two density parameters Ω and Ω_m.

9. Observational Values of the Cosmological Parameters

The current best estimates of the cosmological parameters are,

$$\Omega_b = 0.045 \pm 0.003$$

$$\Omega_c = 0.222 \pm 0.026$$

$$\Omega_m = 0.267 \pm 0.026$$

$$\Omega_\Lambda = 0.734 \pm 0.029$$

$$\Omega = 1.00 \pm 0.01$$

$$\Omega_k = 0 \pm 0.01$$

$$q = -0.60 \pm 0.03$$

$$H_0 = 71 \pm 2.5 \text{ km/s/Mpc} = (2.30 \pm 0.08) \times 10^{-18} \text{ s}^{-1}$$

$$\tau_0 = 13.75 \pm 0.15 \,\text{Gyr}$$

It appears that only ~4.5% of the universe consists of ordinary baryonic matter.

Precision measurements of the CMB from the WMAP satellite play a large part in refining the above estimates, Jarosik *et al* (2011). The first intimation that q is negative, i.e., that the universal expansion rate is accelerating, came from observations of Type Ia supernovae in 1997/8, work which won Perlmutter, Schmidt and Riess the 2011 Nobel prize in physics, S.Perlmutter *et al.* (1997) and Riess *et al* (1998).

The universe is within 1% of being flat, in the sense that $\Omega = 1.00 \pm 0.01$. However it seems unlikely that exact flatness could ever be observationally confirmed. On the other hand, it is feasible that a small curvature, whether positive or negative, could be underwritten by observational evidence in due course. Consequently it *may* be possible to determine whether the universe is spatially finite or infinite in the future, though only if it departs from exact flatness. As present, however, the matter is undecided.

It is important to realise that the above parameter values assume that the standard Λ – plus-cold-dark-matter model is correct. There is more than enough room for doubt given that we have no idea what dark matter is nor what dark energy (Λ) is. In terms of understanding the nature of the contents of the universe we score an abysmal 4.5%.

10. Solutions to the Friedmann Equations for p = 0 and Λ = 0

Having explored the qualitative characteristics of the FLRW spacetimes for both signs of curvature, we now consider how the universe expands, i.e., the solutions of the Friedmann equations for $R(t)$. Only the case of negligible pressure, $p \approx 0$ is considered. This can only be applicable in the matter dominated era. Exact closed form solutions can be given for the case of zero cosmological constant. Until 1997 these solutions were generally regarded as the ones most likely to apply to our universe. This is no longer the case now that it appears that $\Lambda > 0$, solutions for which are discussed in §11. For p = 0 and Λ = 0 it is simply checked that the following are the exact closed form solutions of the Friedmann equations, (10a,b),

p = 0, Λ = 0, k = 0

$$R = \left(\frac{3}{2}At\right)^{2/3} \tag{61}$$

where A is given by (59) or (60) and is a constant when the universe is matter dominated.

p = 0, Λ = 0, k > 0

$$R = \frac{A^2}{2kc^2}(1-\cos 2\psi) \quad \text{and} \quad t = \frac{A^2}{2c^3k^{3/2}}(2\psi - \sin 2\psi) \tag{62}$$

Also
$$\Omega_m = \frac{2}{1+\cos 2\psi} \tag{63}$$

A is again given by (59) or (60). At $\psi = 0, t = 0$ we get $R = 0$ and $\Omega_m = 1$ (in agreement with Chapter 11, Figure 2). The universe reaches its maximum size of $R_{\max} = \frac{A^2}{kc^2}$ at $\psi = \frac{\pi}{2}$ at time $t = \frac{\pi R_{\max}^{3/2}}{2A}$ when $\Omega_m \to \infty$ (again in agreement with Chapter 11). For $k > 0$ the universe is finite and $A^2 = \frac{4}{3\pi}GM_u$ where M_u is the mass of the universe.

p = 0, Λ = 0, k < 0

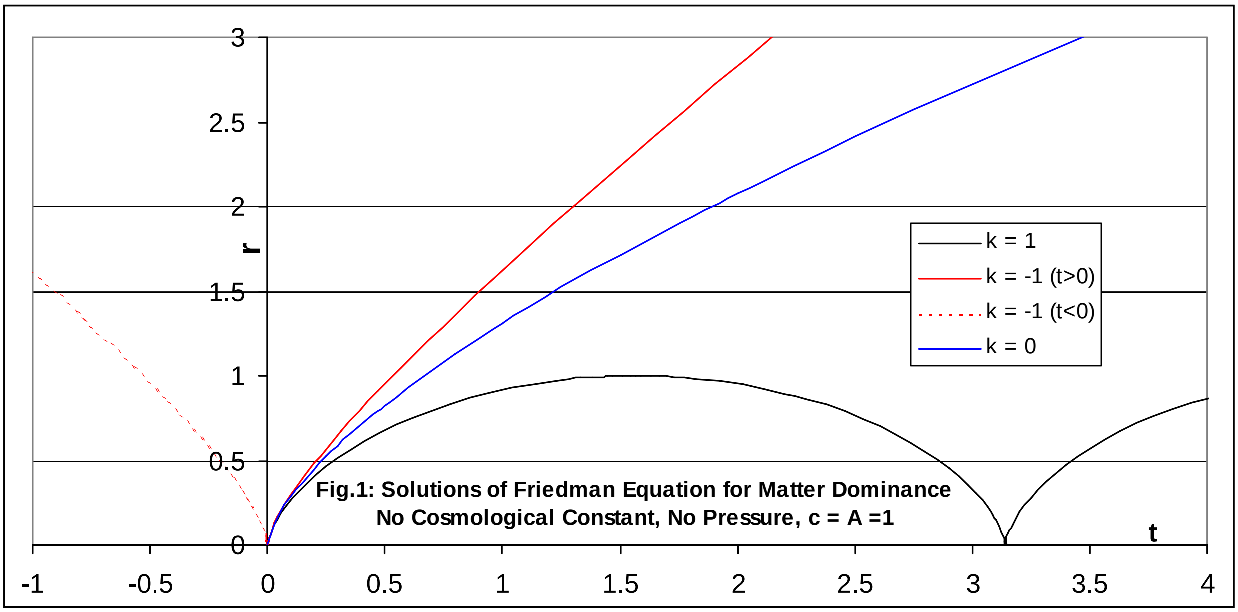
$$R = \frac{A^2}{2c^2|k|}(\cosh 2\psi - 1) \quad \text{and} \quad t = \frac{A^2}{2c^3|k|^{3/2}}(\sinh 2\psi - 2\psi) \tag{64}$$

Also
$$\Omega_m = \frac{2}{1+\cosh 2\psi} \tag{65}$$

A^2 is again given by (59) or (60). At $\psi = 0, t = 0$ we get $R = 0$ and $\Omega_m = 1$ (in agreement with Chapter 11, Figure 2). But as $\psi \to \infty, t \to \infty$ and the universe expands forever, $R \to \infty$ and we get $\Omega_m \to 0$ (again in agreement with Chapter 11, Figure 2).

The above solutions for $p = \Lambda = 0$ are illustrated in Figure 2.

Figure 2: Exact Solutions of Friedmann Equations, (10a,b), for a Matter Dominated Universe with Zero Pressure and No Cosmological Constant (setting c = A = 1)



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11. Solutions to the Friedmann Equations for $p = 0$ but $\Lambda \neq 0$

Finally we come to the geometries which are believed most likely to describe our universe in the present epoch. For completeness all the solutions with non-zero cosmological constant and zero pressure are discussed, not just those which are credible candidates for describing our universe. It is not necessary to derive closed form solutions since the qualitative behaviour of $R(t)$ can be discerned directly from the Friedmann equation, (10b). Only the matter dominated, zero pressure cases are discussed here, so even those solutions identified as most likely to describe our universe will not be applicable before matter-radiation equality at about 75,000 years.

The equation whose solution is being considered is the second Friedmann equation, (10b), which can be written,

$$\dot{R}^2 = F(R) = \frac{A^2}{R} - kc^2 + \frac{\Lambda c^2}{3} R^2 \qquad (66)$$

where A is a constant, (60). Recall that for $p = 0$ this is sufficient to ensure that the first Friedmann equation, (10a), is also obeyed.

The qualitative behaviour of $R(t)$ is simple to deduce, as explained below. We will see that a non-zero cosmological constant causes major qualitative differences to the nature of spacetime. In particular, although when $\Lambda = 0$ positive curvature is necessarily associated with a universe which reaches a maximum size, this is no longer the case when $\Lambda > 0$. Similarly, when $\Lambda = 0$, negative curvature is necessarily associated with a universe which expands forever, but this is no longer the case when $\Lambda < 0$.

11.1 $\Lambda < 0$

This is not currently believed to represent our universe.

The first term in *F*, (66), is positive and reduces monotonically from infinity, at $R = 0$, towards zero at large *R*. In the case of negative cosmological constant, the third term also reduces monotonically, starting at zero at $R = 0$ and diverging to minus infinity at large *R*. It follows that *F* itself reduces monotonically from plus infinity to minus infinity. Hence there is a unique 'radius', R_0, at which *F* is zero. For $R > R_0$, *F* is negative, and, since $\dot{R}^2$ cannot be negative, it follows that $R > R_0$ is not possible. Consequently the solution is confined to $R < R_0$. The universe cannot expand forever when $\Lambda < 0$ whatever the value of the curvature, k.

For sufficiently small *R* the last term in *F* will be small compared with the first two terms. Hence, the small *R* solution approximates to the solutions of (61), (62) or (64). For $k > 0$ and sufficiently small $|\Lambda|$, the solution may be little different from (62). For larger $|\Lambda|$ the maximum size of the universe will be smaller.

On the other hand, for $k < 0$ and $\Lambda < 0$, the solution is always qualitatively different from (64) for large *R*, even when $|\Lambda|$ is numerically small. This is because, for large *R*, approaching R_0, the third term will become important and causes the rate of expansion to reduce to zero. Thus R_0 is always a finite maximum size scale of the universe, irrespective of the sign of '*k*'. The solution with $k < 0$ is a spatially infinite universe of negative curvature whose size scale reaches a maximum then reduces, an impossible situation for $\Lambda = 0$.

Physically, a negative cosmological constant produces an additional attractive gravitational force. This force increases as R increases, rather than decreasing as normal gravity does, as can be seen from (66). Consequently, the Λ-induced force is bound to dominate eventually and causes the universe to re-contract. This happens irrespective of the curvature.

11.2 $\Lambda > 0$ and $k \le 0$

This case is qualitatively particularly simple. All three terms in (67) are positive and hence it is clear that *R* is monotonically increasing. Moreover, at large *R* the third term is dominant and hence the solution will tend asymptotically to an exponential 'explosion', i.e.,

For large t:

$$R \propto \exp\left\{\sqrt{\frac{\Lambda c^2}{3}} \cdot t\right\} \qquad (67)$$

Physically, a positive cosmological constant represents a repulsive gravitational force. Since this repulsive force grows proportionally as *R*, see (66), it inevitably dominates eventually and exponential growth results.

This is one of the solutions currently regarded as consistent with the observational evidence in our universe. However, at sufficiently early times, the first term in $F(R)$ will dominate and the universal expansion will initially decelerate. Only some time later does the third term become sufficient to cause the expansion rate to accelerate. Our universe is not yet in the asymptotic regime given by (67), but its expansion rate has already been accelerating for a substantial fraction of the age of the universe. If a cosmological constant term is responsible for the acceleration then the universe is destined to follow the exponential growth given by (67). The curve of $R(t)$ as it appears to be for our universe is plotted in Chapter spare Figure ??, where the change from deceleration to acceleration is apparent.

A special case is when both $k = 0$ and $\rho = 0$, for which (67) is then the complete solution for all times. This is the de Sitter solution and has some interesting properties. However it is hardly physical since it represents a universe which is empty apart from dark energy.

11.3 $\Lambda > 0$ and $k > 0$

This case has a variety of qualitatively different possible solutions. The first and third terms of F are both positive, but the second term is negative. The qualitative nature of the solution depends upon whether there is a radius at which F becomes zero. There is definitely a unique minimum of F at radius,

$$R_{\min}^3 = \frac{3A^2}{2\Lambda c^2} \tag{68}$$

where A is given by (59) or (60). The minimum of F is thus,

$$F_{\min} = \frac{1}{R_{\min}}\left\{\frac{3}{2}A^2 - kc^2 R_{\min}\right\} \tag{69}$$

Hence, the critical value of Λ (Λ_c) such that the minimum of F is zero is given by,

$$R_{\min} = R_c = \frac{3A^2}{2kc^2} \tag{70}$$

Combining this with (68) gives,

$$\Lambda_c = k\left(\frac{2kc^2}{3A^2}\right)^2 = \frac{k}{R_c^2} \tag{71}$$

Hence, for $\Lambda > \Lambda_c$ $R_{\min} < R_c$ and hence $F_{\min} > 0$ and hence $F > 0$ for any R. Conversely for $\Lambda < \Lambda_c$ $R_{\min} > R_c$ and hence $F_{\min} < 0$ so that F could be positive or negative and there are two distinct values of R where $F = 0$. (71) can also be written, using (49,50,51,59),

$$\Lambda_c c^2 = \frac{4}{9}\cdot\frac{H^2(\Omega-1)^3}{\Omega_m^2} \qquad \text{or} \qquad \Omega_\Lambda = \frac{4}{27}\cdot\frac{(\Omega-1)^3}{\Omega_m^2} \tag{72}$$

Three possible cases can be identified, as follows,

$\Lambda > \Lambda_c$

In this case F cannot be negative for any R, so R is monotonically increasing. For sufficiently large R the expansion becomes exponential and the asymptotic behaviour is again given by (67). This is another instance of the cosmological term winning over the curvature effect (since $k > 0$ would tend to oppose the expansion). If our universe has $k > 0$ then this is would be the correct model for our universe. Nevertheless, for completeness we consider the two final cases.

$0 < \Lambda < \Lambda_c$

In this case there are two real radii at which $F = 0$. Call them R_1 and R_2 with $R_1 < R_2$.

The allowed regimes, for which $\dot{R}^2 > 0$, are [0, R_1] and [R_2, ∞]. These two disconnected regimes correspond to two distinct solutions. They are illustrated in Figure 3. In the first, [0, R_1], there is a Big Bang at R = 0 and the expansion reverses at a maximum radius of R_1. We can think of this solution as being one in which R is restricted to sufficiently small values that the cosmological term does not get a chance to dominate and cause exponential growth.

Conversely, the second solution, with $R \in [R_2, \infty]$, can be thought of as one in which the size scale is confined to a range where the cosmological constant is always dominant. The universe can be imagined to have been given an initial inward velocity at some large radius, thus making it move initially against the repulsive Λ-effect. Its speed of contraction slows, until the universe 'bounces' back at radius R_2, at which point the velocity is zero. This is the first instance of a universe with no Big Bang. A Big Bang has R = 0, and hence infinite density and infinite initial speed, at some finite time designated as t = 0. At the 'bounce' of this universe, R is non-zero, the speed is zero and the density finite, so the 'bounce' is not a Big Bang by any of these criteria.

Figure 3: Solution for k > 0 and $0 < \Lambda < \Lambda_c$

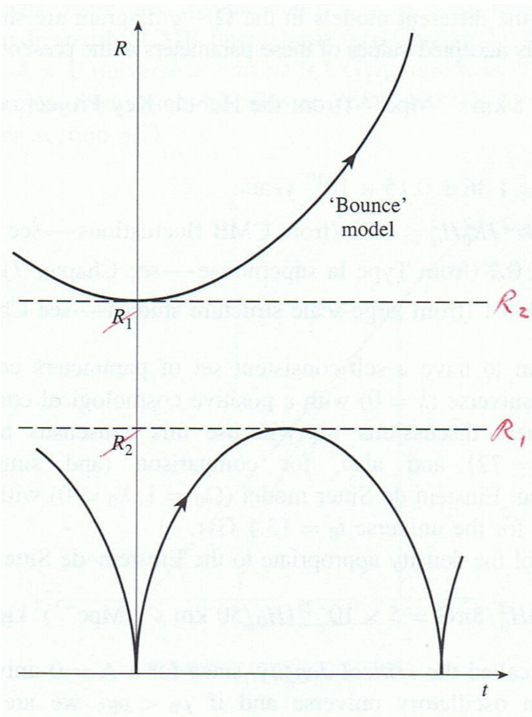

Fig. 4.14 $k = +1, 0 < \Lambda < \Lambda_c$ models.

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<u>$\Lambda = \Lambda_c$</u>

This case is more subtle. One solution is the static Einstein universe. Putting $R = R_c$ we have $\dot{R} = 0$, by definition. Moreover, substitution of R_c and Λ_c into Equ.(10a) shows that $\ddot{R}$ is also zero. Consequently, the universe can be balanced at constant R provided that its size and the cosmological constant are chosen appropriately. 'Balanced' is the correct word since this comes about because the attractive force of gravity is matched against the repulsive cosmological force. This was the cosmology put forward by Einstein (1917). It is an unstable equilibrium, as was first pointed out by Lemaitre.

There are two stable solutions for which the radius is confined respectively to either $[0, R_c]$ or $[R_c, \infty]$. They are illustrated in Figure 4. They differ from the solutions for $0 < \Lambda < \Lambda_c$ because, as R approaches R_c, not only does the speed approach zero, but the acceleration or deceleration approaches zero also. This means the solution 'gets stuck' at $R = R_c$. Thus, for the Einstein-Lemaitre model EL1, the universe only tends asymptotically to its maximum size of R_c, and never contracts. Conversely, the Einstein-Lemaitre model EL2 might start off infinitesimally larger than R_c and initially grow very slowly. Eventually, however, the cosmological repulsive force catches hold and the universe will eventually expand exponentially.

The Einstein-Lemaitre model EL1 is unique. It is the only instance of a universe which has a finite maximum size, but which approaches it asymptotically and never contracts. EL1 does have a Big Bang, however. EL2 is unusual in having no Big Bang, but rather a non-zero initial size. It is similar to the 'bounce' model in this respect.

An interesting special case is worth mentioning, known as the Lemaitre model. This is obtained for $\Lambda = (1 + \varepsilon)\Lambda_c$ where ε is very small, but positive. Of course this is actually a special case of $\Lambda > \Lambda_c$ and hence is a solution which starts at R = 0 and expands monotonically to infinity. However, its proximity to the EL1 and EL2 solutions gives it some characteristics of these solutions. Specifically, it appears to be like an EL1 followed by an EL2. In other words this universe expands rapidly to near R_c. It then stays near R_c for a long time. Eventually, however, the cosmological repulsion wins and an exponential expansion takes over, as illustrated in Figure 5.

Figure 4: Solutions for k > 0 and $\Lambda = \Lambda_c$

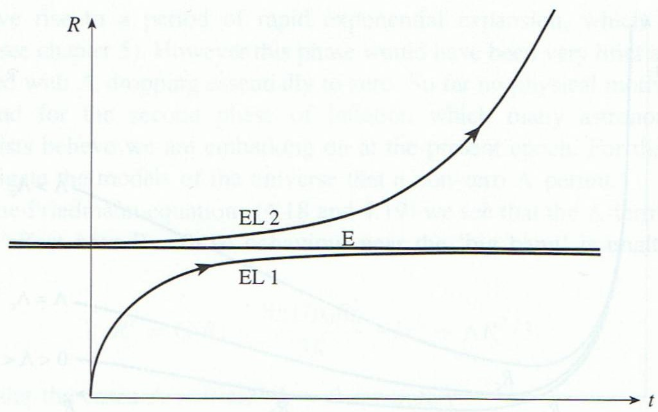

Fig. 4.12 $k = +1$, $\Lambda = \Lambda_c$ models. E = Einstein static model; EL1, EL2 = Eddington–Lemaître models.

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Figure 5: Solution for k > 0 and $\Lambda = (1 + \varepsilon)\Lambda_c$ ($\varepsilon \rightarrow 0$)

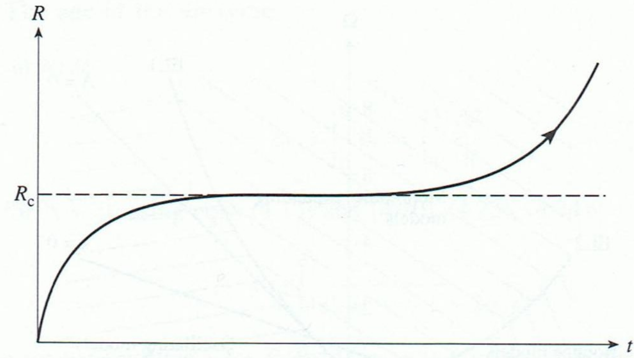

Fig. 4.13 Lemaître models, with long 'quasi-stationary' period.

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The qualitative features of all the solutions we have discussed are summarised in Table 1, the models most representative of this universe are shown in bold blue. Our current best estimate of the shape of the $R(t)$ curve for our universe is illustrated in Chapter spare, Figure ?

Table 1 Summary of Qualitative Features of Cosmological Models

Λ	k	Starting Radius[1]	Finite Maximum Radius?[2]	Re-Contraction?
0	> 0	0	Yes	Yes
0	≤ 0	0	No	No
< 0	any	0	Yes	Yes
> 0	**≤ 0**	**0**	**No**	**No**
> Λ_c	**> 0**	**0**	**No**	**No**
$0 < \Lambda < \Lambda_c$	> 0	0	Yes	Yes
		≠ 0 ('bounce')	No	No
$\Lambda = \Lambda_c$	> 0	≠ 0 ('E')	Yes	No (Static)
		0 ('EL1')	Yes	No
		≠ 0 ('EL2')	No	No

[1]'Zero' also implies infinite initial speed, i.e. a Big Bang. Note that the initial volume of the universe may be infinite even though the size scale, R, is zero. This is always determined simply by *k*.
[2]'No' implies the universe expands to R → ∞

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