Chapter 21

Big Bang Nucleosynthesis

The first few minutes: the shifting balance of protons and neutrons and the escape into the helium sanctuary; the light element abundance of the universe.

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1. The First Few Minutes in Outline

The universe is particularly simple a few milliseconds after the Big Bang. This is because the temperature is low enough by then to ensure that the only particles that can be created out of the available thermal energy are electrons, positrons, neutrinos and, of course, photons. We shall say nothing of dark matter purely out of ignorance. At slightly earlier times the next lightest particles, the muons, would have been abundant (mass ~105 MeV). At still earlier times the hadronic degrees of freedom multiply rapidly and give way to a QCD phase transition. The physics is more complicated from that time forward. We will avoid it. Let us start the story after the first few milliseconds when $k_B T \sim 20\,\text{MeV}$.

At this time the typical thermal energy is much larger than the rest mass of the electron. Hence, the electrons, positrons and the electron and muon neutrinos are all present in numbers comparable with the photons (as determined by the relativistic fermion black-body spectrum, and hence the same as the photon density apart from factors of order unity). The same may apply to the tau neutrinos, though their mass limit presently makes this uncertain. However, there are also neutrons and protons. These are the survivors from the earlier period of complex hadronic physics. Exactly what mechanism underlies the failure of all hadrons to annihilate with their antiparticles need not concern us. But the vast majority of the hadrons present when $k_B T$ was comparable with their masses do annihilate. In lieu of a theory of baryogenesis we accept instead an extra universal constant, 2 x 10^9, the photon:baryon ratio.

The nucleon mass is far too high for nucleon-antinucleon pairs to be formed from the thermal energy after a few milliseconds. However the neutron-proton mass *difference* is only 1.29 MeV, small compared with the thermal energy prevailing at that time. Consequently, thermal equilibrium ensures that there are roughly equal numbers of protons and neutrons. This thermal equilibrium is mediated by the leptonic reactions $n + \nu_e \Leftrightarrow p + e$ and $n + \bar{e} \Leftrightarrow p + \bar{\nu}_e$. At ~0.2 seconds $k_B T$ becomes comparable with $M_n - M_p$. It was then significantly more difficult to make neutrons out of protons than vice-versa, and the neutron:proton ratio would have fallen significantly below 50:50. As the universe cools further the number of neutrons remaining might be expected to reduce indefinitely. But this does not happen because at around 1 second the leptonic reactions, above, which convert neutrons into protons cease. This is a result of cosmic expansion (a phenomenon generally called "freeze-out", which will be explained shortly).

However, this is not enough to save the neutrons because neutrons are unstable as isolated particles. They decay through beta decay, $n \rightarrow p + e + \bar{\nu}_e$ with a mean life of 882 seconds (half-life 611 seconds). By rights the neutrons should disappear from the early universe. Indeed as free particles they do. But most of the neutrons remaining after the freeze out of the leptonic reactions are saved by the occurrence of nucleosynthesis: the formation of stable nuclei - starting with deuterium. Once bound in a stable nucleus, the neutrons are saved from decay. During the first minute or two the high temperatures cause deuterium to be unstable due to photodisintegration. But deuterium becomes stable after a few minutes, sufficiently soon compared with the free neutron half-life that most neutrons are saved, firstly as deuterium and ultimately mostly as helium-4.

The rest of this chapter justifies quantitatively the scenario outlined above.

2. Phase 1: The Equilibrium Ratio of Neutrons to Protons

In this first phase, the nucleon-lepton reactions $n + \nu_e \Leftrightarrow p + e$, $n + \bar{e} \Leftrightarrow p + \bar{\nu}_e$ are active. They provide the mechanism by which thermal equilibrium between the number densities of neutrons and protons arises. These reactions can proceed in the forward or backward direction. However, the forward reactions can occur even at vanishingly small neutrino or positron kinetic energies. In contrast, the reverse reactions can occur only if the combined kinetic energies of the particles on the RHS exceeds the mass difference, $M_n - (M_p + M_e)$ for the first reaction or, $M_n - M_p + M_e$ for the second reaction. Consequently, as the temperature falls, the neutron density will reduce and the proton density will increase. At thermal equilibrium the ratio of the neutron and proton particle densities is given by the Boltzmann distribution,

$$\frac{N_n}{N_p} = e^{-\Delta M / k_B T} \quad \text{where,} \quad \Delta M = M_n - M_p = 1.293\,\text{MeV} \qquad (1)$$

We have already seen in Chapter 11 that the time-temperature relationship prior to ~14 seconds, when annihilation of the electron/positron pairs takes place, is,

$$T = \frac{0.95 \times 10^{10}}{\sqrt{t(\sec s)}} \qquad (2)$$

Hence, using (1) and (2) we can deduce the neutron:proton ratio at any time as long as equilibrium prevails. The result is given in Table 1.

Table 1: Neutron:Proton Ratio at Different Times and Temperatures

t (s)	T (K)	N:P	N (%)	P (%)
0.001	3.00E+11	0.951	48.8	51.2
0.01	9.50E+10	0.854	46.1	53.9
0.03	5.48E+10	0.761	43.2	56.8
0.1	3.00E+10	0.607	37.8	62.2
0.3	1.73E+10	0.421	29.6	70.4
0.5	1.34E+10	0.328	24.7	75.3
0.6	1.23E+10	0.295	22.8	77.2
0.7	1.14E+10	0.267	21.1	78.9
0.8	1.06E+10	0.244	19.6	80.4
0.9	1.00E+10	0.224	18.3	81.7
1	9.50E+09	0.206	17.1	82.9
1.05	9.27E+09	0.198	16.6	83.4
1.1	9.06E+09	0.191	16.0	84.0

3. Phase 2: The Freeze-Out of the Leptonic Reactions

Why do the reactions $n + \nu_e \Leftrightarrow p + e$, $n + \bar{e} \Leftrightarrow p + \bar{\nu}_e$ cease – "freeze out"? The reason is that these reactions are occurring in an expanding universe. Providing that the universe does not expand very much in the time between nucleon/lepton interactions the expansion makes little difference. However, suppose that, for a given particle, the typical time between interactions is T_I and that a particle typically travels a distance L_I in this time. What if the universe expands sufficiently in this time interval that the distance L_I stretches to more than $2L_I$? This would mean that, at the end of the period T_I, the particle had a greater distance still to travel than it had at the start! In these circumstances the particles will never react and the reaction freezes out. The criterion for freeze out is therefore that the fractional change in size due to cosmic expansion in the reaction time should exceed unity.

How much does L_I stretch in time T_I? The fractional change in size due to cosmic expansion in this time is, from Chapter 11, $\frac{1}{R}\frac{dR}{dt}T_I = HT_I$, so the criterion for freeze-out is $HT_I > 1$. But the reaction rate per particle is $1/T_I$ so the criterion is simply,

Freeze-Out if: $H >$ reaction rate per particle (3)

So to evaluate when the leptonic reactions freeze-out we need to know the reaction rate. The strength of the weak nuclear force will enter the calculation at this point. An accurate calculation of the leptonic reactions with the nucleons must take account of the internal hadronic structure of the nucleons. This could be accomplished by the use of appropriate forms factors, for example. However, we will not attempt an accurate calculation. Instead we shall opt for a simple heuristic approach based upon analogy with pseudo-elastic muon-neutrino scattering $\mu + \nu_e \to e + \nu_\mu$. In the centre of mass frame the total cross section of this purely leptonic reaction is,

$$\sigma(\mu + \nu_e \to e + \nu_\mu) = \frac{4}{\pi}\frac{G_F^2 E_\nu^2}{(\hbar c)^4} \tag{4}$$

where E_ν is the energy of the incoming electron-neutrino, M_p is the proton mass, and G_F is Fermi's constant for the weak nuclear force, $G_F = 1.44 \times 10^{-62}$ Jm^3. Note that the use of a cross-section in the centre of mass frame is adequate for our purposes because the neutrino thermal energies are tiny compared with the nucleon mass at these times.

In applying (4) to a nucleon target, we shall assume that the three quarks of which the nucleon is composed each behave like the muon in $\mu + \nu_e \to e + \nu_\mu$, and moreover that each quark behaves independently. Now a neutron is composed of the quarks *udd*, and a proton of *uud*. Thus, the reactions $n + \nu_e \Leftrightarrow p + e$, $n + \bar{e} \Leftrightarrow p + \bar{\nu}_e$ involve the conversion of a *d* quark into an *u* quark. For example $n + \nu_e \Leftrightarrow p + e$ may be re-written as $udd + \nu_e \Leftrightarrow uud + e$. Hence, just two of the three quarks in the neutron are relevant, that is the *d* quarks. Thus, we shall assume as a rough estimate that the nucleon cross-section for the reaction $n + \nu_e \Leftrightarrow p + e$ is just twice (4),

$$\sigma(n + \nu_e \to p + e) = \frac{8}{\pi}\frac{G_F^2 E_\nu^2}{(\hbar c)^4} \tag{5}$$

To use (5) we shall take the neutrino energy to be the average of their (fermion) spectrum, i.e.,

$$\langle E_\nu \rangle = 3.15 k_B T \tag{6}$$

The speed of the neutrinos is close to c, hence, for a cross-section σ, a given nucleon sweeps out a volume σc in unit time. If the number density of neutrinos is ρ_ν the number of neutrinos with which our nucleon will interact per unit time is therefore $\sigma c \rho_\nu$. This is the reaction rate per nucleon,

$$\text{reaction rate per nucleon} = \frac{1}{T_I} = \sigma c \rho_\nu \tag{7}$$

Now the number density of electron neutrinos is given by the fermion black body distribution,

$$\rho_\nu = 0.0913 \left(\frac{k_B T}{\hbar c} \right)^3 \tag{8}$$

Thus, at a given temperature, (5-8) will give the reaction rate for $n + \nu_e \rightarrow p + e$ per neutron. But there is also the reaction $n + \bar{e} \rightarrow p + \bar{\nu}_e$ to consider. The argument is exactly the same. The cross-section per positron spin state is assumed to be the same as (5), but accounting for the two spin states gives an extra factor of two. The number density of positrons is also taken to be the same as that of the neutrinos, to a sufficient approximation (noting that the positrons are relativistic at these times). Overall, therefore, the two reactions $n + \nu_e \Leftrightarrow p + e$ and $n + \bar{e} \Leftrightarrow p + \bar{\nu}_e$ produce a total reaction rate which is three times that given by (5-8).

Note that this argument implicitly assumes that the leptonic reactions freeze out well before electron/positron annihilation take place at ~14 seconds. This is important since otherwise the density of positrons would not be given by an expression like (8). Rather, there would be virtually no positrons left and the second reaction, $n + \bar{e} \rightarrow p + \bar{\nu}_e$, would contribute negligibly to the reaction rate per neutron.

Putting (5-8) together, including the factor of three for the positron reaction, gives,

$$\frac{1}{T_I} = 3 \frac{8}{\pi} \frac{G_F^2 3.15^2 (kT)^2}{(\hbar c)^4} 0.0913 \left(\frac{kT}{\hbar c} \right)^3 = 6.921 \frac{G_F^2 (kT)^5 c}{(\hbar c)^7} \tag{9}$$

Having found the reaction rate per particle, (9), we now need the Hubble parameter. This is given in terms of the density of the universe as discussed in Chapter 11, and the density of the universe is simply that of blackbody radiation, taking account not just of photons but also electrons, positrons and all the neutrinos. Hence,

$$H^2 = \frac{8\pi G}{3} \rho \quad \text{and} \quad \rho = \chi . \frac{4\sigma_S T^4}{c^3} \text{ and Stefan's constant, } \sigma_S = \frac{\pi^2 k_B^4}{60 c^2 \hbar^3} \tag{10}$$

The factor χ accounts for the electrons, positrons and neutrinos as well as the photons. With the help of the numerology explained in the Appendix to Chapter 11 this factor is seen to be,

$$\chi = 1 + \frac{\frac{7}{8}(2 \times 2 + 3 \times 1 \times 2)}{2} = \frac{43}{8} \tag{11}$$

where the terms in the numerator of the fraction represent the electrons/positrons and the three neutrino species respectively. (The denominator of 2 is required because two photon modes are included in the definition of Stefan's constant). Hence, using (10,11) and equating the resulting H to $1/T_I$, as given by (9), we get,

$$H = \sqrt{\frac{8\pi G}{3}\frac{43}{8}\frac{\pi^2 k_B^4}{60c^2\hbar^3}\frac{4T^4}{c^3}} = \frac{1}{T_I} = 6.921\frac{G_F^2 (kT)^5 c}{(\hbar c)^7} \quad (12)$$

which reduces to,

$$(k_B T)^3 = 0.786\frac{\sqrt{G(\hbar c)^{11}}}{(G_F c)^2} = 1.93\times 10^{-39} \quad (13)$$

hence $$T = 0.90\times 10^{10}\ K \quad (14)$$

Consulting Table 1 we see that this temperature occurs at 1.1 seconds when the neutron:proton ratio is **16% : 84%**. Note that this confirms that the leptonic reactions freeze out well before electron/positron annihilation take place at ~14 seconds, as we assumed in the calculation.

4. Phase 3 – Beta Decay of the Neutrons

Once the nucleon/lepton reactions $n + \nu_e \Leftrightarrow p + e$ and $n + \bar{e} \Leftrightarrow p + \bar{\nu}_e$ have ceased, the only means by which the number of neutrons can reduce is through beta decay, $n \to p + e + \bar{\nu}_e$. Thus, after 1.1 seconds, the neutron number density reduces exponentially according to,

$$N_n(t) = N_n(1.1)e^{-(t-1.1)/882} \quad (15)$$

where t is the time in seconds, and 882 s is the mean life of a free neutron. Because the total number of nucleons is fixed, N_n in (15) may be interpreted either as the absolute number density of neutrons, or as the percentage or fraction of the total nucleons which are neutrons. In the latter case we have $N_n(1.1) = 16\%$.

Equ.15 holds from the time of freeze out of the leptonic reactions (1.1 seconds) until the time when nuclei become stable, t_D. Once the neutrons are safely within a nucleus they are stable. So the final fraction of nucleons which are neutrons, now all within nuclei, is,

$$N_n = 16\% \times e^{-(t_D - 1.1)/882} \quad (16)$$

We must now calculate t_D.

5. Phase 4 – The Formation of the First Nuclei: Deuterons

The lightest compound nucleus is the deuteron, np. The deuteron is quite lightly bound, the binding energy being only 2.22MeV. Equating the average photon energy ($2.7k_B T$) to this binding energy, we might expect that the deuterons would be stable against thermal fission at temperatures below ~10^{10} K. Since the temperature is less than this after ~1 sec, it might be thought that the deuterons would be stable from this time onwards. However, this pays insufficient respect to the huge abundance of photons compared to neutrons.

Even at temperatures, say, only one-tenth of 10^{10} K, a small percentage of the photons will have energies in excess of the 2.22MeV required to fission a deuteron into its proton and neutron constituents. Since there are so many photons, only a very small proportion need have energies above 2.22MeV in order to fission all the deuterons. In fact, we need only as many photons with energies >2.22MeV as there are deuterons.

We have seen that at 1.1 seconds about 16% of the nucleons are neutrons. We must now assume a value for the photon:baryon ratio, η, namely $\eta \approx 2 \times 10^9$. This can be deduced from observations, either by estimating directly the total quantity of baryonic material in the cosmos, or, more accurately, by appeal to precision measurements of the microwave background by satellites such as WMAP, see Jarosik *et al* (2011). The number density of neutrons is therefore,

$$N_n = 0.16 \; \frac{N_\gamma}{2 \times 10^9} = 0.8 \times 10^{-10} N_\gamma \tag{17}$$

and the number density of photons is given by the black body formula,

$$N_\gamma = 0.2436 \left(\frac{k_B T}{\hbar c} \right)^3 \tag{18}$$

The blackbody spectrum giving the number of photons per unit volume with energy between E and $E + dE$ is,

$$dN_\gamma = \frac{1}{\pi^2 (\hbar c)^3} \cdot \frac{E^2 dE}{e^{E/kT} - 1} \tag{19}$$

Hence, from (18) and (19), the fraction of all photons which have energy greater than E_1 is,

$$\frac{1}{N_\gamma} \int dN_\gamma = \frac{1}{0.2436 \pi^2 (kT)^3} \cdot \int_{E_1}^{\infty} \frac{E^2 dE}{e^{E/kT} - 1} = 0.416 \int_{x_1}^{\infty} \frac{x^2 dx}{e^x - 1} \tag{20}$$

where $x_1 = E_1 / k_B T$. Comparing with (17), we see that the fraction given by the RHS of (20) must be less than 0.8 x 10^{-10} if the deuterons are to be stable against photodisintegration. Hence, the condition for stable deuteron formation is,

$$\int_{x_1}^{\infty} \frac{x^2 dx}{e^x - 1} < 1.92 \times 10^{-10} \tag{21}$$

with $x_1 = 2.22 \, \text{MeV} / k_B T$. Since this clearly corresponds to $x_1 >> 1$, a very good approximation for the integral is,

$$\cdot \int_{x_1}^{\infty} \frac{x^2 dx}{e^x - 1} \approx \int_{x_1}^{\infty} e^{-x} x^2 dx = \left(2 + 2x_1 + x_1^2\right) e^{-x_1} \tag{22}$$

from which we find that x_1 = 29.19, and hence T = 8.8 x 10^8 K. Recalling from Chapter 11 that, after ~14 secs, the time-temperature relation is $T\sqrt{t} \approx 1.33 \times 10^{10}$, this temperature is found to occur at time ~228 secs. Finally, substituting this time into Equ.(16) gives the neutron fraction when nuclei are first stable as,

$$N_n = 16\% \times e^{-227/882} = 16\% \times 0.77 = 12.3\% \tag{23}$$

This will be the final fraction of nucleons which survive as neutrons – albeit now locked up inside nuclei.

6. Phase 5 – The Cosmic Helium Abundance

The prediction of the cosmic helium abundance requires no extra work. We need only note that once deuterons become stable, all the neutrons then present quickly end up in the form of helium-4 ($^{4}_{2}He$). Essentially this is because the nuclei heavier than the deuteron have larger binding energies, with helium-4 being the largest (and hence the tightest bound) of those that can be formed during this early stage of the universe. The binding energies are given in Table 2.

Table 2: Binding Energies of the Lightest Nuclei

name	nucleus	total binding energy, MeV	binding energy per nucleon, MeV
deuterium*	$^{2}_{1}D$	2.22	1.11
tritium	$^{3}_{1}H$	8.48	2.83
helium-3	$^{3}_{2}He$	7.72	2.57
helium-4	$^{4}_{2}He$	28.30	7.07
lithium-7	$^{7}_{3}Li$	39.24	5.61

**Strictly deuterium is the element (neutral atom) whereas the np nucleus is a deuteron. Similarly, hydrogen is the element or atom whereas its nucleus is, of course, a proton. Helium-4 also has a named nucleus, the alpha particle. However other nuclei do not have names which distinguish them from their element. Consequently it is common in astrophysics to speak of hydrogen, deuterium and helium when we actually mean protons, deuterons and alpha particles. And for the other nuclei we have no choice. Remember that astrophysicists are people who think that the elements beyond helium in the periodic table are all metals!*

Hence, assuming for the moment that negligible quantities of the elements beyond lithium are formed, it is to be expected that most of the neutrons will end up in helium-4 since this has the maximum binding energy per nucleon.

Helium-4 involves 2 protons as well as 2 neutrons so it follows that the cosmic helium abundance *by mass* should be double that predicted for the neutrons when the formation of nuclei starts, i.e., 2 x 12.3% = 24.6%. (Note that this is only ~7.5% of the total nuclei, the remainder of which are virtually all protons).

The measured cosmic helium abundance is indeed within a fraction of a percent of this theoretical estimate, Peimbert (2008). Admittedly we have been rather lucky getting quite so close with our crude method of calculation, but the calculation can be done more accurately, of course, and with essentially the same result. Considering that there is no *a priori* reason why the helium abundance should not be 99.9% or 0.0001%, this agreement is one of the substantial triumphs of the Big Bang theory.

7. Phase 6 – BBN: The Abundance of the Trace Nuclei

In §6 it was possible to deduce the final quantity of helium-4 simply because virtually all the surviving neutrons end up in helium-4. However there will also be residual traces of deuterium and helium-3. Tritium is also produced but is unstable and hence will decay later (and its concentrations are always very low). The abundance of lithium-7 is also used as a validation case for BBN, but we shall not follow the nuclear reactions that far. In this section we show how the residual quantities of deuterium and helium-3 may be calculated. To be done properly, this requires numerical integration of the reaction network. However we also make a rough estimate “by hand” in §8. This has the advantage of clarifying physically why the deuterium and helium-3 abundances are four orders of magnitude smaller than hydrogen and helium-4.

The period of Big Bang nucleosynthesis lasts only a few minutes. There are two reasons for this. The first is that temperature and density are reducing rapidly in the expanding universe. Since nuclear fusion reactions are extremely sensitive to temperature, and also proportional to the product of the reactant densities, this leads to rapidly reducing reaction rates. In addition, reaction freeze-out by universal expansion turns out to be important in establishing the termination of some reactions, and hence the final abundances. Almost all of the fusion activity occurs in the period 100 to 300 seconds.

Once the universe has cooled sufficiently that the deuteron becomes stable to photodisintegration, a sequence of possible fusion reactions becomes possible. All the possible reactions leading to helium are,

		Change in Binding Energy (MeV)
[1]	${}_1^1p + {}_0^1n \to {}_1^2D + \gamma$	2.2245 (0 → 2.2245)
[-1]	${}_1^2D + \gamma \to {}_1^1p + {}_0^1n$	-2.2245 (2.2245 → 0)
[2]	${}_1^1p + {}_1^2D \to {}_2^3He + \gamma$	5.4935 (2.224 → 7.7181)
[3]	${}_2^3He + {}_2^3He \to {}_2^4He + 2{}_1^1p$	12.8595 (2 x 7.7181 →28.2957)
[a]	${}_1^2D + {}_1^2D \to {}_2^4He + \gamma$	23.8467 (2 x 2.2245 → 28.2957)
[b]	${}_1^2D + {}_1^2D \to {}_1^3T + {}_1^1p$	4.0328 (2 x 2.2245 → 8.4818)
[c]	${}_1^2D + {}_1^2D \to {}_2^3He + {}_0^1n$	3.2691 (2 x 2.2245 → 7.7181)
[d]	${}_1^3T + {}_1^2D \to {}_2^4He + {}_0^1n$	17.5894 (8.4818 + 2.2245 → 28.2957)
[e]	${}_2^3He + {}_0^1n \to {}_1^3T + {}_1^1p$	0.7637 (7.7181 → 8.4818)
[f]	${}_0^1n + {}_1^2D \to {}_1^3T + \gamma$	6.2573 (2.2245 → 8.4818)
[g]	${}_1^3T + {}_1^1p \to {}_2^4He + \gamma$	19.8139 (8.4818 → 28.2957)
[h]	${}_2^3He + {}_0^1n \to {}_2^4He + \gamma$	20.5776 (7.7181 → 28.2957)
[i]	${}_2^3He + {}_1^2D \to {}_2^4He + {}_1^1p$	18.3531 (7.7181 + 2.2245 → 28.2957)
[j]	${}_2^3He + {}_1^3T \to {}_2^4He + {}_1^2D$	14.3203 (7.7181 + 8.4818 → 28.2957 + 2.2245)
[k]	${}_2^3He + {}_1^3T \to {}_2^4He + {}_0^1n + {}_1^1p$	12.0958 (7.7181 + 8.4818 → 28.2957)

With the exception of [-1], these reactions have all been written with the forward direction being exothermic. The figures on the right give the increase in binding energy in MeV, and, in brackets, the individual binding energies on the two sides of the reaction. The increase in the binding energy is the gain in kinetic energy of the products (carried off predominantly by the lighter particle). In principle, all these reactions are reversible. However, the endothermic energy requirements for the reverse reactions are larger than that for photodisintegration of a deuteron, [-1], in all cases except [e]. Since we are only considering times (temperatures) at which the deuteron has become stable, it follows that all the reverse reactions [2] through [k] will be slow and can be ignored to a first approximation. The exceptional reaction, [e], has a negligible reaction rate in any case.

Reactions rates are formulated in the following manner. It is conventional to express the particle number densities, e.g., ρ_p, ρ_n, etc., in mole/cm^3 (the absolute particle number density per cm^3 being obtained by multiplying by Avogadro's number, 6.022 x 10^{23}). The rate of deuterium production by the reaction ${}^1_1\text{p} + {}^1_0\text{n} \rightarrow {}^2_1\text{D} + \gamma$, for example, is proportional to the product of the reactant densities times some reaction rate. Writing [1] for the rate of reaction 1, this can be written,

Due to reaction 1 only: $$\frac{d\rho_D}{dt} = \rho_p \rho_n [1]$$

The rates of two-particle reactions are conventionally expressed in s^{-1} per mole/cm^3, so that reactant densities in mole/cm^3 give a rate of product density increase in mole/cm^{3}s. An alternative way of looking at this is in terms of the reaction rate per proton. This is simply $\rho_n[1]$ with units s^{-1}, the reciprocal of which is the mean time a proton will wait before reaction [1] consumes it.

Fits to the rates of the reactions [1] to [k], in units s^{-1} per mole/cm^3, which are applicable for the temperature range of interest in BBN are,

Rate [1] = 4.37 x 10^4 (24.1)

Rate [2] = 1.529 x 10^{-10} T$^{1.3785}$ (24.2)

Rate [3] = 1.3185 x 10^{-32} T$^{4.1119}$ (24.3)

Rate [a] = 1.99 x 10^{-10} T$^{1.0284}$ (24.a)

Rate [b] = 1.593 x 10^{-8} T$^{1.6321}$ (24.b)

Rate [c] = 5.35 x 10^{-9} T$^{1.6956}$ (24.c)

Rate [d] = 4.57 x 10^8 + 0.06T (24.d)

Rate [e] ≈ 0 (24.e)

Rate [f] = 69 (24.f)

Rate [g] = 3.128 x 10^{-11} T$^{1.5466}$ (24.g)

Rate [h] = 4 (24.h)

Rate [i] = 1.408 x 10^{-6} T$^{1.5271}$ (24.i)

Rate [j] = 7.301 x 10^{-11} T$^{1.8150}$ (24.j)

Rate [k] = 8.712 x 10^{-11} T$^{1.8250}$ (24.k)

These expressions will not be accurate outside the temperature range 5 x10^{8}K to 10^{9}K.

The one reaction whose rate we have not given above is that of photodisintegration of deuterium, reaction [-1], the one reverse reaction to be taken into account. A closed-form expression for the cross-section of the reaction ${}^2_1\text{D} + \gamma \rightarrow {}^1_1\text{p} + {}^1_0\text{n}$ can be found via simple Schrodinger wave-functions based on magnetic dipole and electric dipole interactions. From these a closed-form expression for the rate of this reaction at any temperature T can derived by integration over the photon energy spectrum. The result is,

$$R[T] \approx \frac{3.40x10^{-23}}{\sqrt{\varepsilon_{th}}} \left\{ \frac{15}{8} + \frac{3}{2}\varepsilon_{th} + \frac{1}{2}\varepsilon_{th}^2 \right\} \left(\frac{kT}{\hbar c} \right)^3 \exp\{-\varepsilon_{th}\} \tag{25}$$

where $\varepsilon_{th} = B/kT$. Unlike the other reactions, the rate given by (25) is in units s^{-1}, the reciprocal of which is the mean life of the deuteron to photodisintegration, the reason being that the black body photon density is built into the reaction rate.

It was claimed in §5, based on an argument independent of this reaction rate, that deuterons become stable against photodisintegration at about 228 seconds (T = 8.8 x 10^8 K). The reaction rate (25) is consistent with this in the sense that a little earlier (say, $T = 10^9$ K) the reaction time, $1/R[T]$, would have been of the order of milliseconds. Since BBN requires the order of minutes, deuterons would have been too unstable at that time. In contrast, a little later (say, T = 7 x 10^8 K) the reaction time has increased dramatically to the order of minutes.

From the rate of each contributing reaction we can now write down the total rate of production and rate of consumption of each species. Using a notation in which $[r]$ stands for the rate of reaction 'r', and ρ_X is the number density of nucleus X in mole/cm^3, and denoting by $_\uparrow$ and $_\downarrow$ the production and consumption rates respectively, we have,

$$\dot{\rho}_{p\uparrow} = \rho_D^2[b] + 2\rho_{He3}^2[3] + \rho_{He3}\rho_D[i] + \rho_{He3}\rho_T[k] + \rho_D[-1] \tag{26}$$

$$\dot{\rho}_{p\downarrow} = \rho_p\rho_n[1] + \rho_p\rho_D[2] + \rho_T\rho_p[g] \tag{27}$$

$$\dot{\rho}_{n\uparrow} = \rho_D^2[c] + \rho_T\rho_D[d] + \rho_{He3}\rho_T[k] + \rho_D[-1] \tag{28}$$

$$\dot{\rho}_{n\downarrow} = \rho_p\rho_n[1] + \rho_n\rho_D[f] + \rho_n\rho_{He3}[h] \tag{29}$$

$$\dot{\rho}_{D\uparrow} = \rho_p\rho_n[1] + \rho_{He3}\rho_T[j] \tag{30}$$

$$\begin{aligned}\dot{\rho}_{D\downarrow} = {} & \rho_p\rho_D[2] + 2\rho_D^2([a]+[b]+[c]) + \rho_T\rho_D[d] + \rho_n\rho_D[f] + \\ & \rho_{He3}\rho_D[i] + \rho_D[-1]\end{aligned} \tag{31}$$

$$\dot{\rho}_{T\uparrow} = \rho_D^2[b] + \rho_n\rho_D[f] \tag{32}$$

$$\dot{\rho}_{T\downarrow} = \rho_T\rho_D[d] + \rho_T\rho_p[g] + \rho_{He3}\rho_T([j]+[k]) \tag{33}$$

$$\dot{\rho}_{He3\uparrow} = \rho_p\rho_D[2] + \rho_D^2[c] \tag{34}$$

$$\dot{\rho}_{He3\downarrow} = \rho_{He3}\rho_n[h] + 2\rho_{He3}^2[3] + \rho_{He3}\rho_D[i] + \rho_{He3}\rho_T([j]+[k]) \tag{35}$$

$$\begin{aligned}\dot{\rho}_{He4\uparrow} = {} & \rho_D^2[a] + \rho_{He3}^2[3] + \rho_T\rho_D[d] + \rho_T\rho_p[g] + \rho_{He3}\rho_n[h] + \\ & \rho_{He3}\rho_D[i] + \rho_{He3}\rho_T([j]+[k])\end{aligned} \tag{36}$$

The net increase in the density of any species is simply $\dot{\rho}_\uparrow - \dot{\rho}_\downarrow$ for that species. Hence, solving for the densities of all the nuclei at any time is simply a matter of numerically integrating the reaction network, Equs.(26-36), by time stepping. The initial conditions are the known starting densities for protons and neutrons, and the fact that all other species have zero initial densities. However the effects of universal expansion must be taken into account.

Most obviously, universal expansion causes both the temperature and the absolute particle densities to change over time. This is easily accommodated by explicitly including $T = 1.33 \times 10^{10} / \sqrt{t}$ in the time integration, noting that, unlike (2), this applies *after* electron/positron annihilation at 14 seconds. In addition it is necessary to scale all particle densities by a factor and $(t / t + \delta t)^{3/2}$ on each time increment, δt. (This is because the linear scale is expanding $\propto \sqrt{t}$, see Chapter 11). The second effect of cosmic expansion is the potential to freeze-out the nuclear reactions. A crude means of accounting for this is simply to switch a reaction off when its rate falls below $1/2t$ (see Chapter 11). BBN calculations can, of course, be done a great deal better than the crude methodology outlined here. Our purpose is only to indicate the principles involved, and make a quick estimate.

The results of the above time integration are shown in Figure 1. The abundance of protons, neutrons, deuterons, helium-3 and helium-4 nuclei are plotted against time as a fraction of the total number of baryons. Note that the D and He3 abundances are on a different scale (on the right) to that of p, n and He4 (on the left). Figure 1 plots the ratio of *particle number* densities. Relative abundances of deuterium and helium-3 by mass are obtained from these by multiplying by 2 and 3 respectively.

Figure 1: Light Element Abundances Predicted by Integrating Equs.(26-36)

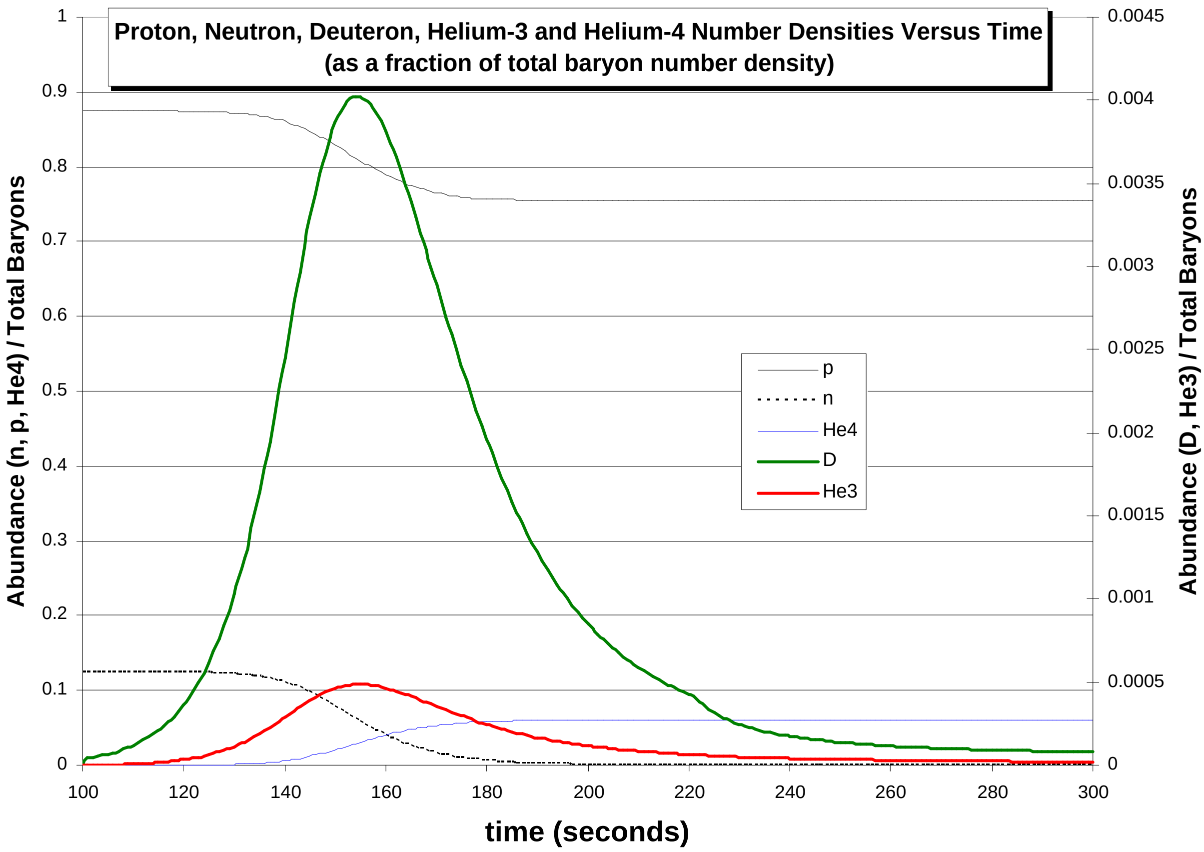

Table 3 compares the calculated abundances of deuterium and helium-3 with estimates of the actual abundances. The latter are subject to considerable uncertainty, but even our simple estimates are consistent with them. This is another significant triumph of the Big Bang hypothesis, though debate over the finer details continues.

Table 3: Abundances (Number Density) of D and He3 as a Fraction of Total

Nucleus	Observed Cosmic Abundance	Our Calculation
D	$10^{-5} – 10^{-4}$	5×10^{-5}
He3	$10^{-6} – 3 \times 10^{-5}$	8×10^{-6}

The unsatisfactory feature of the preceding calculation is that it requires a 'black box' numerical integration by computer. It is always desirable to make a purely pen-and-paper estimate, partly to check the computer code but also to provide some degree of physical understanding of the key features which determine the result. This attempted next.

8. Simple Approximate Estimate of D and He3 Abundances

The final abundances of D and He3 occur when the reactions producing and consuming them have ceased to operate. To make our crude estimate we concentrate on the freeze-out by universal expansion of the deuteron consumption reactions. By this time the photodisintegration of deuterium, reaction [-1], will be inactive so the remaining reactions which consume deuterium are reactions 2, a, b, c, d, f and i. Approximations are made as follows,

- Reaction a has a negligible rate and can be ignored;
- Similarly, reaction f has a rate which is far slower than reactions b, c, d and i and can also be ignored;
- Reaction 2 will also be slower than reactions b plus c provided that the deuteron abundance is not less than $\sim 2 \times 10^{-5}$ (which turns out to be just valid);
- Reaction d requires tritium, and the full integration shows that tritium densities are always extremely small, hence reaction d can also be ignored.

We are left with reactions b, c and i.

Figure 1 shows that BBN is essentially complete by ~300 seconds. At this time the temperature is 7.7×10^8 K and the sum of the reaction rates of b and c is 1.1×10^7 $s^{-1}(\text{mole/cm}^3)^{-1}$. Both reactions b and c involve two deuterons. Hence, reactions b and c will be frozen out if $\rho_D \times 7.5 \times 10^6 < 1/2t$. This suggests freeze out of deuteron consumption by reactions b and c for $\rho_D < 1.5 \times 10^{-10}$ mole/cm^3. The total baryon density at this time is 7.6×10^{-6} mole/cm^3 so freeze out occurs for $\rho_D/\rho_b < 2 \times 10^{-5}$. This is in reasonable order of magnitude agreement with the result of the numerical integration, 5×10^{-5} (Table 3) given the crude nature of the approximation, particularly the degree of arbitrariness about the time chosen for freeze-out.

We must also ensure that deuterons are not still being consumed by reaction i either. The rate of reaction i at 300 seconds is 5.2×10^7 $s^{-1}(mole/cm^3)^{-1}$. Hence, reaction i will be frozen out as regards consumption of deuterons if $\rho_{He3} \times 5.2 \times 10^7 < 1/2t$. This suggests total freeze-out of deuteron consumption if $\rho_{He3} < 3.2 \times 10^{-11}$ mole/cm^3, i.e., $\rho_{He3}/\rho_b < 4 \times 10^{-6}$. This is also in order-of-magnitude agreement with the results of the numerical integration, 8×10^{-6} (Table 3).

These arguments are not rigorous, of course. They are intended only to illustrate why the final abundances of D and He3 are of the order they are, namely about four orders of magnitude less than the dominant components, p and He4.

References

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