Chapter 17

Going Round in Circles

How to calculate motion with respect to a rotating coordinate system; derivation of the centrifugal and Coriolis forces; are they fictitious? Is gravity? The Foucault pendulum; Why is the behaviour of gyroscopes counter-intuitive? How to demonstrate anti-gravity to the unwary.

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1. Rotating Coordinate Systems

You will have learnt your lessons well at school and you will know that centrifugal force is fictitious. You may even be reluctant to use the term "centrifugal force" for fear of being reprimanded. But are you sure that you have not been brainwashed?

Let us first consider non-relativistic dynamics and how this can be formulated in a frame of reference which is rotating. To avoid unnecessary complications the origin of the rotating coordinate system will be assumed to be stationary at the origin of our non-rotating (inertial) coordinate system. The non-rotating coordinate system, $S_{inertial}$, is defined by Cartesian axes with unit vectors $\hat{x}_i, i \in [1,3]$ whereas the rotating coordinate system, S_{rot}, has Cartesian unit vectors $\hat{n}_i, i \in [1,3]$, which are time-dependent. The rotation of S_{rot} with respect to $S_{inertial}$ is defined by a vector $\overline{\varpi}$ whose direction is the axis of rotation and whose magnitude is the angular rotation rate. The rotation rate of S_{rot} need not be constant, both the rotation rate and the axis of rotation may vary.

An arbitrary vector, $\overline{a}$, may be described in either coordinate system, thus,

$$\overline{a} = a_i \hat{x}_i = \tilde{a}_i \hat{n}_i \tag{1}$$

(summation assumed). The coordinates $\tilde{a}_i$ of the vector with respect to the rotating coordinate system will necessarily differ from those in the stationary system, a_i, if they are to represent the same vector. Equ.(1) applies for any vector, and in particular for the position vector of (say) a point mass. The components of the velocity of this point mass as seen by the inertial and rotating observers will differ, being given by,

$$\overline{v} = \dot{a}_i \hat{x}_i \qquad \text{and} \qquad \overline{\tilde{v}} = \dot{\tilde{a}}_i \hat{n}_i \tag{2}$$

where the dot denotes the time derivative as usual. The time coordinate is the same in both expressions because we are dealing with non-relativistic motion only. The second of the expressions (2) follows from the fact that the rotating observer regards his co-moving coordinate system, $\hat{n}_i$, to be stationary. But of course the inertial observer sees the $\hat{n}_i$ as time-dependent. By taking the time derivative of (1) we get,

$$\overline{v} = \dot{a}_i \hat{x}_i = \dot{\tilde{a}}_i \hat{n}_i + \tilde{a}_i \dot{\hat{n}}_i = \overline{\tilde{v}} + \tilde{a}_i \dot{\hat{n}}_i \tag{3}$$

The second term on the RHS of (3) must be the velocity which would be seen by $S_{inertial}$ if the particle were stationary with respect to S_{rot}. Considering an arbitrary vector $\overline{B}$ which is stationary wrt S_{rot}, Figure 1 shows that the change in this vector due to the rotation $\overline{\varpi}\delta t$ over a small interval of time δt must be $\delta\overline{B} = \overline{\varpi} \times \overline{B}\delta t$, i.e.,

For $\overline{B}$ co-moving with S_{rot}:
$$\frac{d\overline{B}}{dt} = \overline{\omega} \times \overline{B} \tag{4}$$

Figure 1 Demonstration of Equ.(4)

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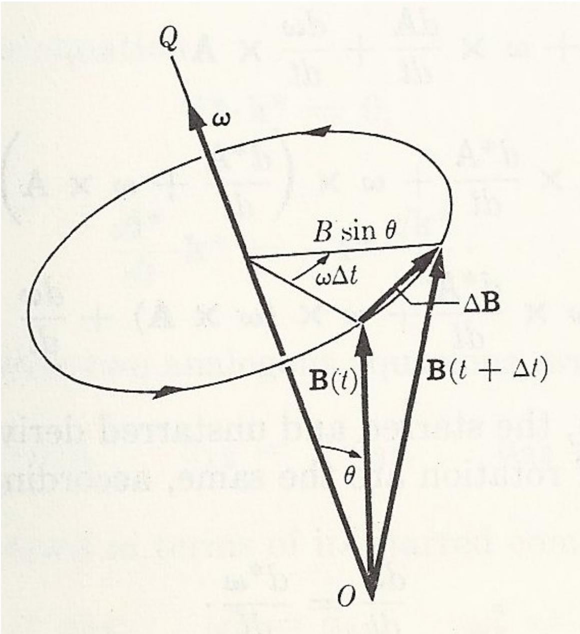


In particular Equ.(4) applies for the position vector $\overline{a}$ of a particle which is stationary wrt S_{rot}. Consequently (3) becomes,

$$\overline{v} = \tilde{\overline{v}} + \overline{\omega} \times \overline{a} \tag{5}$$

Recalling that $\overline{v} = d\overline{a} / dt$ we can write (5) in operator form as,

$$\frac{d}{dt} \equiv \frac{\tilde{d}}{dt} + \overline{\omega} \times \tag{6}$$

where it is understood that this operator acts on vectors, and the time derivative operator with the tilda acts only upon the coordinates not upon the unit vectors, i.e.,

$$\frac{\tilde{d}}{dt}\overline{a} \equiv \dot{\tilde{a}}_i \hat{n}_i \equiv \tilde{\overline{v}} \tag{7}$$

The relationship between the accelerations of the particle seen by the two observers now follows simply from,

$$\frac{d\overline{v}}{dt} = \frac{d^2\overline{a}}{dt^2} \equiv \left(\frac{d}{dt}\right)^2 \overline{a} = \left(\frac{\tilde{d}}{dt} + \overline{\varpi}\times\right)^2 \overline{a} = \left(\frac{\tilde{d}}{dt} + \overline{\varpi}\times\right)\left(\tilde{\overline{v}} + \overline{\varpi}\times\overline{a}\right)$$
$$= \frac{\tilde{d}\tilde{\overline{v}}}{dt} + \frac{\tilde{d}}{dt}\left(\overline{\varpi}\times\overline{a}\right) + \overline{\varpi}\times\left(\tilde{\overline{v}} + \overline{\varpi}\times\overline{a}\right)$$
$$= \frac{\tilde{d}\tilde{\overline{v}}}{dt} + \overline{\varpi}\times\tilde{\overline{v}} + \overline{\varpi}\times\left(\tilde{\overline{v}} + \overline{\varpi}\times\overline{a}\right) + \left(\frac{\tilde{d}\overline{\varpi}}{dt}\right)\times\overline{a} \tag{8}$$
$$= \frac{\tilde{d}\tilde{\overline{v}}}{dt} + 2\overline{\varpi}\times\tilde{\overline{v}} + \overline{\varpi}\times\left(\overline{\varpi}\times\overline{a}\right) + \left(\frac{d\overline{\varpi}}{dt}\right)\times\overline{a}$$

noting that $\frac{\tilde{d}\overline{\varpi}}{dt} = \frac{d\overline{\varpi}}{dt}$ by (6) because $\overline{\varpi}\times\overline{\varpi}\equiv 0$. Equ.(8) gives the acceleration seen by $S_{inertial}$ in terms of the acceleration seen by S_{rot}, the first term on the RHS of (8), and three other terms. This expression applies for arbitrary motion of S_{rot}, assuming a fixed origin, so that $\overline{\varpi}(t)$ may be any function of time. In the case that S_{rot} rotates with constant angular velocity the last term in (8) is zero and we get,

$$\frac{d\overline{v}}{dt} = \frac{\tilde{d}\tilde{\overline{v}}}{dt} + 2\overline{\varpi}\times\tilde{\overline{v}} + \overline{\varpi}\times\left(\overline{\varpi}\times\overline{a}\right) \tag{9}$$

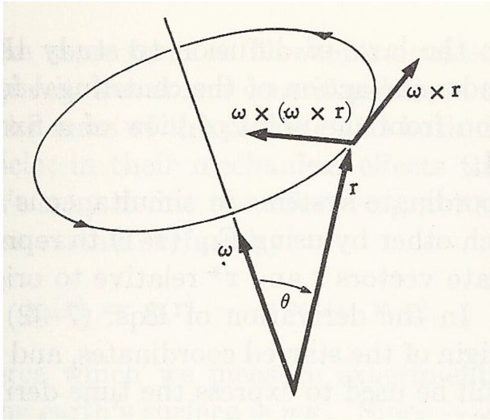
Suppose that our point mass, m, is subject to a 'real' force $\overline{F}$ as seen by $S_{inertial}$, then $\overline{F} = m\frac{d\overline{v}}{dt}$ and hence re-arranging (9) gives,

$$\tilde{\overline{F}} \equiv m\frac{\tilde{d}\tilde{\overline{v}}}{dt} = \overline{F} - 2m\overline{\varpi}\times\tilde{\overline{v}} - m\overline{\varpi}\times\left(\overline{\varpi}\times\overline{a}\right) \tag{10}$$

Here we have defined $\tilde{\overline{F}}$ as the *apparent* force seen by S_{rot}, defined simply as the mass times the acceleration seen by S_{rot}. Because the accelerations seen by the two observers differ, as shown by (9), the apparent force seen by S_{rot} is also necessarily different from $\overline{F}$. The two extra terms in (10) are respectively called the Coriolis force and the centrifugal force. Because they arise only due to the use of a rotating coordinate system they are often referred to as fictitious forces. There is no Coriolis force and no centrifugal force with respect to the inertial coordinate system.

Note that the vector $\overline{\varpi}\times\left(\overline{\varpi}\times\overline{a}\right)$ points towards the centre of rotation, as illustrated by Figure 2. Consequently it is called the *centripetal acceleration* (since *centripetal* means "towards the centre"). Consequently the fictitious force $-m\overline{\varpi}\times\left(\overline{\varpi}\times\overline{a}\right)$ acts in the opposite direction and hence the term *centrifugal force* is appropriate (since centrifugal means "away from the centre"). Suppose the particle is at rest wrt S_{rot}. The inertial observer regards the particle as having a centripetal acceleration $\overline{\varpi}\times\left(\overline{\varpi}\times\overline{a}\right)$ which must be due to being acted upon by a real centripetal force given by $\overline{F} = m\overline{\varpi}\times\left(\overline{\varpi}\times\overline{a}\right)$. The rotating observer S_{rot} must see no net force, in order for the particle to appear at rest. He thus regards the real centripetal force as exactly balanced by the equal and opposite centrifugal force.

Figure 2 Centripetal Acceleration **Need to draw my own version**



2. Example: A Falling Body on Earth

Release a body from rest with respect to the Earth's surface at a small distance above the Earth's surface: in which direction does it fall? You will now know better than to say "towards the centre of the Earth". This would be the case if the Earth were not rotating. Because it *is* rotating, the equation of motion of the body with respect to the coordinate system, $\tilde{a}_i$, comoving with the Earth is,

$$m\frac{\tilde{d}^2\bar{a}}{dt^2} = m\bar{g} - m\bar{\varpi}\times(\bar{\varpi}\times\bar{a}) - 2m\bar{\varpi}\times\frac{\tilde{d}\bar{a}}{dt} \tag{11}$$

where $\bar{\varpi}$ is the Earth's angular rotation rate vector, whose direction is along the Earth's axis. The vectorial acceleration due to gravity, $\bar{g}$, does indeed point towards the Earth's centre of mass. For this purpose we are idealising the Earth as perfectly spherically symmetric. In reality $\bar{g}$ varies with latitude for reasons discussed below, as well as having interesting local variations due to surface topography and local density variations. However, even for a perfectly spherically symmetric planet, the Coriolis and centrifugal forces will cause the acceleration of the body to deviate from the non-rotating value of $\bar{g}$. Even supposing that the velocities are small enough that the Coriolis force, $-2m\bar{\varpi}\times\frac{\tilde{d}\bar{a}}{dt}$, can be ignored, there is still the centrifugal force, $-m\bar{\varpi}\times(\bar{\varpi}\times\bar{a})$. It is convenient to absorb the centrifugal term into the definition of an effective acceleration due to gravity, $\bar{g}_e$, thus,

$$\bar{g}_e = \bar{g} - 2\bar{\varpi}\times(\bar{\varpi}\times\bar{r}) \tag{12}$$

where $\bar{r}$ is the position on the Earth's surface of interest. In the northern hemisphere this effective acceleration due to gravity acts towards a point a little south of the centre of the Earth, as illustrated in Figure 3. g_e is the apparent acceleration due to gravity which would be measured by an experiment in a terrestrial laboratory and is rather less than the value of g for a non-rotating planet, except at the poles. At latitude θ the magnitude of the apparent acceleration due to gravity, as a fraction of the non-rotating value, is,

$$\frac{g_e}{g} = \left[(1-\lambda)^2\cos^2\theta + \sin^2\theta\right]^{1/2} \qquad \text{where,} \qquad \lambda = \frac{\omega^2 r}{g} \tag{13}$$

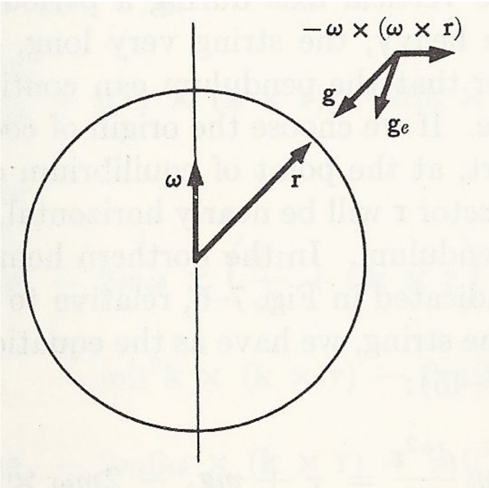
Using $\omega = 7.29 \times 10^{-5}$ s^{-1} and the average radius of the Earth at the equator, $r = 6378$ km, the acceleration due to gravity at the equator is less than at the poles by about 0.35% (or 0.034 ms^{-2}). The angle α between the measured and non-rotating accelerations, $\bar{g}_e$ and $\bar{g}$, is given by,

$$\cos\alpha = \frac{1 - \lambda\cos^2\theta}{\sqrt{1 - \left(2\lambda - \lambda^2\right)\cos^2\theta}} \qquad (14)$$

This angular deviation of the apparent gravitational centre of the Earth has its maximum value of ~0.1^o at a latitude of $\pm 45^o$.

Actually (13) and (14) under-estimate the effects of the Earth's rotation on the apparent $\bar{g}_e$ and its variation with latitude. They would be correct if the Earth were spherically symmetric. But the rotation caused the formative Earth to adopt an oblate ellipsoid shape, the semi-major axis at the equator exceeding the semi-minor axis at the poles by about 21 km on average. The greater distance between the Earth's surface and its centre at the equator further decreases the gravity at the equator compared with the poles. The combined effects of the centrifugal force and the oblate shape of the Earth means that bodies actually weigh about 0.5% less at the equator than at the poles, quite a substantial difference. The angular deviation of the direction of $\bar{g}_e$ compared to $\bar{g}$ is also increased due to the equatorial bulge, since this places more mass to the south of a point in the northern hemisphere. The total deviation is perhaps double that implied by (14).

Figure 3 Effective Acceleration Due To Gravity



3. Democracy or Dictatorship?

Are the centrifugal and Coriolis forces fictitious or real? There is nothing wrong with the traditional view that these forces are fictitious. But you are not obliged to subscribe to the dictatorship of the inertial observer. Instead you can join the democracy of free observers. From the relativistic point of view, all coordinate systems are equally valid. However, the rotating system comes along with these extra forces – which are a kinematic result of the rotating frame of reference. This may seem a rather contrived perspective until it is recalled that a similar situation applies to gravity.

General relativity teaches us that the existence of a gravitational field is a consequence of the adopted spacetime coordinate system. If we adopt a free-falling frame of reference then there will be no gravitational field in our immediate vicinity. The force of gravity arises because we are perverse enough to adopt a non-free-falling frame of reference. Is gravity a fictitious force, then?

The idea that gravity is an illusion would be most quickly dispelled by jumping off a high building. On the other hand, if you are clinging on to a fast spinning playground roundabout and decide to let go on the grounds that centrifugal force is fictitious you will also suffer as a consequence. Does this make centrifugal force real? Clearly not, since as soon as you leave your rotating frame of reference you will find the inertial frame suddenly comes into play once more – and your subsequent trajectory is tangential to the roundabout, not radially outwards as your intuition might previously have suggested. However it does mean that knee-grazing, or other forms of impact damage, are not reliable arbiters of reality. But there again, contending that gravity is fictitious might seem quite convincing if you were in a stable orbit around the planet. Your apparently weightless state would be maintained indefinitely – and, of course, when you look down it is not you who is orbiting the planet, but the planet which is spinning, is it not?

It might be fun to challenge undergraduates' belief in the fictitious nature of the centrifugal and Coriolis forces by introducing this analogy with gravitation, to see what emerges from the interpretational thicket. However, the situations are not as closely analogous as I would have you believe. In truth there *is* a preferred set of frames of reference, the inertial frames. The fictitious forces are not required in those frames in which the field of fixed stars are just that – not rotating. So cosmology has legislated a preferred frame. And whilst a gravitational force can always be made to vanish locally by free-fall, this does not eliminate the gravitational field further away – except in the (non-existent) case of a uniform gravitational field. The bottom line is that you can work in any frame of reference you please, but the formulation of the dynamics will change accordingly. In some cases working in a frame of reference which introduces fictitious forces will render a problem more tractable rather than more complicated. The Foucault pendulum is a case in point.

4. The Foucault Pendulum

How does a Foucault pendulum differ from any other pendulum? The answer is not at all, in principle. The only requirement is that a Foucault pendulum should keep swinging for at least a period of hours, and preferably days. To achieve this, the pendulum must be very long, with a light, thin string and heavy bob. It will then be found that the plane in which the pendulum swings rotates slowly. That is to say, the plane of a pendulum on planet Earth rotates with respect to the Earth's surface. This is a consequence of the Earth's rotation.

There are two places where it is easy to see what the rotation rate of the plane of the pendulum must be. At the Earth's north and south poles it is readily understood that it is not really the plane of the pendulum which is rotating, but rather it is the Earth which is rotating beneath it. So, at the poles, the pendulum's precession rate will be one complete 360^{o} rotation in time T (which means, of course, that the initial plane of rotation is regained in time $T/2$). Here T is the period of the Earth's rotation with respect to the fixed stars, the stellar day, which is about 23 hours 56 minutes and 4 seconds. The angular precession rate at the poles is thus $\omega_F\left(\theta=\pm\pi/2\right)=\mp 2\pi/T$, where θ is the latitude. The sign is different at the two poles because the pendulum will appear to precess clockwise at the north pole but anticlockwise at the south pole, merely because we look down on the pendulum in both case – and the direction of "down" is reversed. (To remember the signs of these rotations you need only recall that the sun rises in the east and sets in the west. The Earth therefore rotates anticlockwise as seen from above the north pole).

The second place where things are simple is at the equator. Here there is no reason to prefer rotation in one sense to the other, and so there can be no precession. Hence, $\omega_F(\theta = 0) = 0$. Clearly there must be a continuous variation of precession rate with latitude so it is a fair bet that the precession rate at any latitude will be given by $\omega_F(\theta) = -(2\pi / T)\sin\theta$. So, at the latitude of my home town (51.7^o north) the precession period is $T / \sin 51.7^o = 30$ hours 30 minutes. That this is the right guess for the precession rate will now be proved. (Strictly the angle θ used should be corrected for the displacement of the centre of gravity by the centrifugal acceleration, as discussed in §2, but we have already seen that this is a small effect).

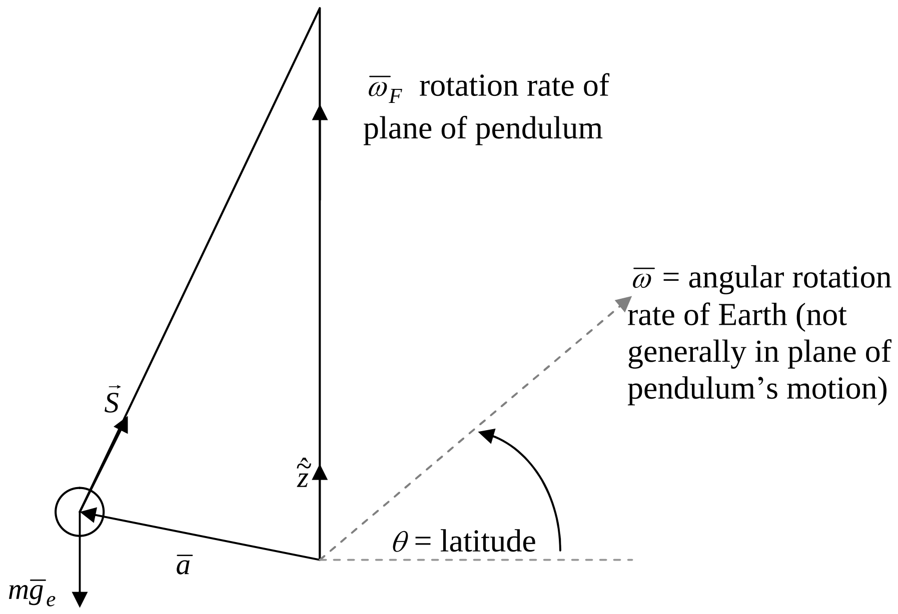
Inspired by having guessed the answer we consider a transformation to yet another coordinate system, namely one rotating about the vertical axis through the point from which the pendulum is suspended. By "vertical" we mean the direction of the effective acceleration due to gravity at the point in question on the Earth's surface, after allowance for the centrifugal force due to the Earth's rotation, as discussed in §2. The coordinates of the bob in the frame of reference which is comoving with the Earth are $\tilde{a}_i$, and the coordinates with respect to the new coordinate system which also rotates around the vertical are a_i'. Both correspond to the same vector, $\bar{a} = \tilde{a}_i \hat{n}_i = a_i' \hat{n}_i'$. The relationship between the two gives the equivalent of (6) and (8),

$$\frac{\tilde{d}\bar{a}}{dt} \equiv \frac{d'}{dt}\bar{a} + \bar{\omega}_F \times \bar{a} \tag{15}$$

$$\frac{\tilde{d}^2\bar{a}}{dt^2} = \frac{d'^2\bar{a}}{dt^2} + 2\bar{\omega}_F \times \frac{d'\bar{a}}{dt} + \bar{\omega}_F \times (\bar{\omega}_F \times \bar{a}) \tag{16}$$

where $\bar{\omega}_F$ is vertically upwards, i.e., anti-parallel to the effective acceleration due to gravity, $\bar{g}_e$, and the magnitude ω_F is the precession rate of the pendulum. Consequently, with the right choice for ω_F we expect to find that the motion of the pendulum lies in a fixed plane in the rotating system a_i'. The various vectors are illustrated in Figure 4 where vertically-up is denoted $\hat{\tilde{z}} \equiv \hat{\tilde{n}}_3$, so that $\bar{g}_e = -g_e \hat{\tilde{z}}$ and $\bar{\omega}_F = \omega_F \hat{\tilde{z}}$.

Figure 4 The Foucault Pendulum



The equation of motion for the bob wrt the Earth system is,

$$m\frac{\tilde{d}^2\bar{a}}{dt^2} = \bar{S} + m\bar{g}_e - 2m\bar{\varpi}\times\frac{\tilde{d}\bar{a}}{dt} \qquad (17)$$

where $\bar{S}$ is the string tension and $\bar{\varpi}$ is the Earth's angular rotation rate vector, whose direction is along the Earth's axis. Whilst the first two terms on the RHS of (17) lie in the plane of the pendulum's motion, the last term may have an out-of-plane component, thus causing the plane of motion to change with respect to the Earth's frame of reference. This is the origin of the Foucault pendulum effect.

Now substitute (15) and (16) into (17) to get the equation of motion expressed in the precessing system,

$$\begin{aligned} m\frac{d'^2\bar{a}}{dt^2} &= \bar{S} + m\bar{g}_e - 2m\bar{\varpi}\times\left(\frac{d'}{dt}\bar{a} + \bar{\varpi}_F\times\bar{a}\right) - 2m\bar{\varpi}_F\times\frac{d'\bar{a}}{dt} - m\bar{\varpi}_F\times(\bar{\varpi}_F\times\bar{a}) \\ &= \bar{S} + m\bar{g}_e - 2m\bar{\varpi}\times(\bar{\varpi}_F\times\bar{a}) - m\bar{\varpi}_F\times(\bar{\varpi}_F\times\bar{a}) - 2m(\bar{\varpi}+\bar{\varpi}_F)\times\frac{d'\bar{a}}{dt} \end{aligned} \qquad (18)$$

The first four terms on the RHS of (18) all lie in the instantaneous plane of the pendulum's motion. This is obvious for the first two terms (see Figure 4). The third and fourth terms involve cross products with $\bar{\varpi}_F\times\bar{a}$, but this vector is normal to the plane of motion and hence a cross product with $\bar{\varpi}_F\times\bar{a}$ must lie *in* the plane of motion. Consequently only the last term in (18) could drive the pendulum out of its current plane of motion in the precessing system. The pendulum will therefore remain in the same plane in the precessing system if this term is always zero. The vector $\bar{\varpi}+\bar{\varpi}_F$ cannot be zero for arbitrary latitude because these two vectors are not in the same direction. However the last term *will* be zero if $\bar{\varpi}+\bar{\varpi}_F$ is orthogonal to the velocity vector in the precessing system, $\frac{d'\bar{a}}{dt}$. Provided that the angular amplitude of the pendulum is small, this velocity vector will be approximately horizontal. Consequently the last term in (18) is zero if,

$$\left(\overline{\omega} + \overline{\omega}_F\right) \cdot \hat{\hat{z}} = \omega \sin\theta + \omega_F = 0 \qquad (19)$$

This establishes that the plane of the pendulum precesses at a angular rate of $\omega_R = -\omega \sin\theta$, as anticipated. The minus sign is because the precession is in the opposite sense to the Earth's rotation (which is obvious at the poles).

5. Coriolis Force and Weather

We see from (17) that it is the Coriolis force which causes the plane of the Foucault pendulum to rotate. But the most important effect of the Coriolis force on Earth is its role in large scale wind patterns. The example of the Foucault pendulum shows that the Coriolis force will tend to cause clockwise rotation with respect to the Earth in the northern hemisphere. That this is the correct sense of rotation can be understood from Figure 5 which shows the direction of the Coriolis force for various directions of motion when viewed above the north pole. The situation is summarised in the following Limerick (which applies only in the northern hemisphere),

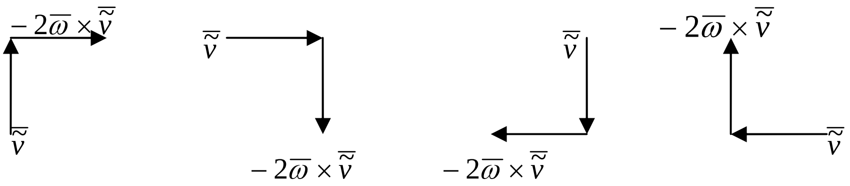
On a merry-go-round in the night,
Coriolis was shaken with fright.
Despite how he walked,
'Twas like he was stalked,
By some fiend always pushing him right.
Attributed to APS in Ref.[1]

Noting that this verse relates only to the northern hemisphere I offer a second verse,

Coriolis was feeling bereft
The theory he'd imagined so deft
In southerly parts
Regardless of starts
The fiend always pushed to the left

Figure 5 Coriolis Pushes to the Right in the Northern Hemisphere

The vector of the Earth's rotation $\overline{\varpi}$ is acting upwards, out of the paper

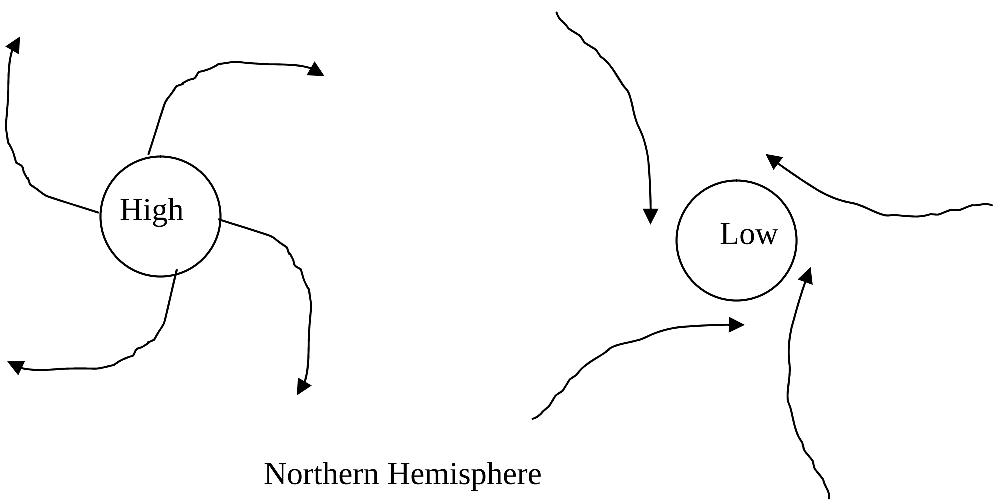


The Coriolis force always acts to the right of the motion (clockwise)

This immediately explains the direction of anticyclones, i.e., clockwise in the northern hemisphere and anticlockwise in the southern hemisphere. Any circular wind pattern requires forces acting on the atmosphere towards the centre of the motion. In the case of anticyclones the Coriolis force is the dominant force and hence the direction of anticyclones is as stated above. However, in general there will also be forces acting on large air masses due to large scale pressure gradients. In the case of anticyclones the higher pressure is found at the centre of the motion. The pressure gradient force is therefore acting outwards from the centre. This alone could not support the circular motion, and anticyclones are possible only because the Coriolis force is greater than the pressure gradient force and acts towards the centre for the direction of rotation indicated above.

The case of cyclones is the reverse of that for anticyclones. Cyclones rotate about centres of low pressure in which the pressure gradient force exceeds the Coriolis force and permits rotation in the anticlockwise sense in the northern hemisphere and clockwise in the southern hemisphere. The total force towards the centre in this case is the pressure gradient force minus the Coriolis force. It is perhaps not immediately obvious why there could not be a rotation about a low pressure centre in the anticyclonic direction. The pressure gradient force and the Coriolis force would then be in the same sense and would support a spectacularly violent wind. The reason that this does not occur is due to the nature of the radial flows which initiate the motion. The situation for anticyclones and cyclones is depicted in Figure 6.

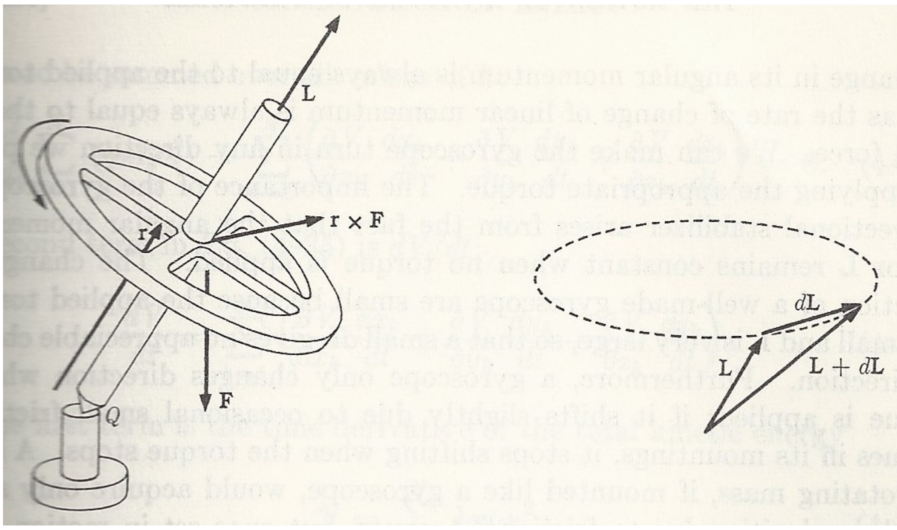
Figure 6 Influence of Radial Flow on Possible Cyclones



6. Why Are Gyroscopes Counter-Intuitive?

Set a gyroscope spinning and place it down at an angle to the vertical, as in Figure 7. It fails to topple over. Instead it precesses about the vertical axis. Understanding this in terms of mechanics is perfectly straight forward. The angular momentum, $\overline{L}$, lies along the axis of the gyroscope, at least when the precession rate is slow. If the gyroscope is in the (x, z) plane, with z vertical, the moment acting on it is in the y direction, i.e., $\overline{M} = M\hat{y}$. But the equation of motion is that the rate of change of angular momentum equals the applied moment, $\frac{d\overline{L}}{dt} = \overline{M}$. This is just the rotational equivalent of $\frac{d\overline{p}}{dt} = \overline{F}$. Consequently the small change in angular momentum in time interval dt is in the y direction, $d\overline{L} = Mdt \cdot \hat{y}$. The angular momentum vector, $\overline{L}$, therefore precesses about the support point, as illustrated in Figure 7.

Figure 7 The Gyroscope



This also explains why a gyroscope initially precesses slowly, when it is still spinning fast. If the angle of the axis to the vertical is θ, the horizontal component of its angular momentum is $L\sin\theta$. Since the vertical component of its angular momentum is constant, at least initially, the rate of change of the magnitude of its angular momentum is $L\cos\theta \cdot \dot{\theta}$. But this must equal the applied moment, M , which is constant in magnitude so long as θ is constant. Hence the rate of precession is just $\dot{\theta} = M / L\cos\theta$. Consequently as the gyroscope slows it precesses faster, both because L is reducing and also because it begins to drop to a larger angle θ . More detailed analysis reveals more complex motions, but that is beyond our purpose here.

The analysis is simple enough – but why does a gyroscope's behaviour confound us so? The answer is partly that a spinning gyroscope looks like a stationary object but actually possesses a substantial amount of momentum. Our intuition is fooled by this, probably because rotational motion associated with large amounts of angular momentum are rare or non-existent in nature. Consequently we have not evolved or developed to appreciate intuitively their behaviour. We do understand intuitively that a fast moving object will not behave like a stationary one. A fast cricket ball will not be deflected to one side of its trajectory by the slight force that would suffice for a stationary ball. So we have no right to be surprised that a gyroscope does not respond to a moment as a stationary object would respond. It appears to resist our attempts to change its orientation only because we fail to understand that it is a moving object. Actually it is not hard to cause it to re-orient itself. This is proved by its natural precessional motion. We merely have to remember that the direction in which we must apply the moment is not as it would be for a stationary object. So, to move the axis slowly in a direction $\hat{n}$ the applied moment must be parallel to $\hat{n}$ (for a fast spinning gyroscope).

A professor of engineering, who should have known better, once toured the country fooling people into believing in anti-gravity. His party trick involved a very large and heavy gyroscope on a shaft a couple of metres long. With one end resting on the ground he got a volunteer from the audience to hold the other end. While the gyroscope was stationary he asked them to try to lift the gyroscope (and recall he had them hold only one end of the shaft). They could not do so – apparently demonstrating its great weight. The gyroscope was then spun up using a drill motor. The volunteer then found it easy to lift the gyroscope, even one-handed. The professor then claimed that this was due to a remarkable anti-gravity effect of the gyroscope. The truth, of course, is that the weight of the gyroscope was not so great. It would have been easy to lift, even one-handed, from beneath its centre of gravity. The difficulty the volunteer experienced in doing so was because he was obliged to hold only the end of the shaft. The difficulty in lifting the stationary gyroscope was due to the difficulty in providing a sufficient *moment* with a pair of hands to overcome its weight-times-length. But, once spinning, it is not necessary to do so because the gyroscope is happy to remain at its non-zero angle to the vertical. To put it another way, the moment due to the weight-times-length is balanced by a slight precession. But the use of an electric motor to spin it up ensured that L was huge and the precession small enough not to ruin the demonstration.

References

[1] University of Aberdeen, http://www.abdn.ac.uk/~wpe001/meteo/metoh8.pdf