Chapter 13

Stationary Action: Is It Minimum?

Is the action minimum or maximum? Since action is defined as the time integral of (KE-PE), and because potential energy is a minimum in the static case, this suggests that the action should be a maximum. But of course it is not (usually). Sometimes texts refer to the Principle of Least Action, but in truth the action is not always a minimum either. What is going on here?

Last Update: 22/3/12

1. The Ubiquity of the Principle of Stationary Action and a Confession

In discussing unification within physics, Feynman (1964) has noted that it is trivial to express all the laws of physics as just one equation. It is only necessary to express each law in the form $F_i(\{x_j\})=0$ where i refers to the i^{th} law, F_i is arranged to be real valued, and $\{x_j\}$ are all the independent variables you might wish for. The Grand Unified Theory of Everything is then simply,

$$\sum_i F_i^2 = 0 \tag{1}$$

The point that Feynman was making is that this does not constitute physical unification in any meaningful way. It is merely an alternative way of writing the N different 'laws' $F_i(\{x_j\})=0$. In contrast, the principle of stationary action is a true unifying principle in physics. All the fundamental theories of physics can be expressed in this form. Whilst (1) is devoid of true content, the principle of stationary action is a real constraint upon physical theories. Thus, almost the whole of physics can be written,

$$\delta\int \mathcal{L}\cdot d^n x = 0 \tag{2}$$

where n is the dimensionality of spacetime. This is usually 4 if you are not a string theorist or into Kaluza-Klein theories or m-branes. Equ.(2) is written in the form suitable for field theories, whether classical or quantum fields. If we were dealing instead with the mechanics of discrete particles, the principle of stationary action would be,

$$\delta\int L\cdot dt = 0 \tag{3}$$

The Lagrangian is then defined as the difference between the kinetic energy of the system and its potential energy,

$$L = KE - PE \tag{4}$$

In the case of fields, the Lagrange density, $\mathcal{L}$, is defined in a bespoke manner for the fields in question, but can also often be interpreted as a kinetic energy density minus a potential energy density (where the latter may include mass terms in relativistic cases). For both discrete particles and fields the action is defined as the integral which is stationary,

$$S = \int \mathcal{L}\cdot d^n x \quad \text{or} \quad S = \int L\cdot dt \tag{5}$$

The variations indicated in (2) and (3), and with respect to which the action is stationary, are variations in *functions*. In the case of discrete particles, the functions in question are the particles' coordinates as functions of time, $\bar{r}_j(t)$. In the case of field theories, the functions in question are the fields themselves, which are functions of the n spacetime coordinates, $\phi_j(x_\mu)$. Hence, for example, for non-relativistic discrete particles, (3) is equivalent to Newton's equations of motion for the particles. For field theories, (2) is equivalent to the field equations, be it Maxwell's equations of classical electromagnetism, the Einstein equations of the general relativistic gravitational field, or the field equations of particle physics. We give a few illustrations in §3.

The theme of this Chapter is whether the action, being stationary, is a maximum or a minimum – or neither? The strict answer is – in general – neither. However, in the case of discrete particles the action really *is* a minimum so long as we do not consider time periods which are too long (more of this nicety later).

Now in the case of statics there is a general principle that an equilibrium configuration corresponds to the minimum of potential energy. Since forces are given by the spatial derivatives of the potential energy, and since equilibrium occurs when the net force is zero, it is clear that potential energy will be stationary at equilibrium. That it is a minimum follows simply in this case from the fact that forces act in the opposite direction from the gradient of the potential, $\bar{F} = -\bar{\nabla}(PE)$: which brings me to my confession.

In my youth I was want to write the definition of the Lagrangian, (4), with the sign reversed, $L = PE - KE$. My reasoning was that surely this dynamical principle should reduce in the static case to the principle of minimum potential energy. But on this basis it would seem that the definition $L = KE - PE$ would lead to action being maximised, not minimised. Since the texts referred to "the principle of least action" I concluded that they must actually mean $L = PE - KE$. Admittedly this strikingly arrogant conclusion, that I was right and all the texts were wrong, was accompanied by a degree of unease: justifiably so for my conclusion was completely fallacious. However it seems I can salvage some pride. In general the action is neither minimised nor maximised, but merely stationary. This Chapter is intended to elucidate when the action really is a minimum, and when not. Those familiar with the calculus of variations and the derivation of the Euler-Lagrange equations can skip §2 and §3.

2. Calculus of Variations in the Static Case

Consider the integral over x of a function f which depends both explicitly upon x and also upon a function $y(x)$ and its first derivative, $y' = \frac{dy}{dx}$, thus,

$$P = \int_{x_1}^{x_2} f(x, y, y')dx \tag{6}$$

The integral is defined over the fixed interval $[x_1, x_2]$. P is said to be a functional of y. Considering small changes in the *function*, $y(x) \to y(x) + \delta y(x)$, leads to small changes in the *functional*, P, and hence to the concept of a functional derivative. Suppose that we wish to determine the functions $y(x)$ for which the functional P is stationary, subject to the requirement that the values of y are fixed at the ends of the interval: $y(x_1) = y_1$, $y(x_2) = y_2$. The change in P is,

$$\delta P = \int_{x_1}^{x_2} \left[f\left(x, y + \delta y, y' + \delta y'\right) - f\left(x, y, y'\right) \right] dx = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right] dx \tag{7}$$

where we have used a Taylor expansion. The second term on the RHS can be transformed by integrating by parts as follows,

$$\begin{aligned} \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y'} \delta y' \right] dx &= \left[\frac{\partial f}{\partial y'} \int [\delta y'] dx \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \left(\int [\delta y'] dx \right) \cdot \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \cdot dx \\ &= \left[\frac{\partial f}{\partial y'} \delta y \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \delta y \cdot \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \cdot dx \\ &= - \int_{x_1}^{x_2} \delta y \cdot \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \cdot dx \end{aligned} \tag{8}$$

noting that $\delta y = 0$ at the ends of the interval due to the requirement that $y(x_1) = y_1$ and $y(x_2) = y_2$ be fixed. Substituting (8) into (7) gives,

$$\delta P = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] \delta y dx \tag{9}$$

The condition for a stationary value of P is that $\delta P = 0$ for any variations δy in the function y (subject to its fixed end points). This can only be fulfilled if the integrand of (9) vanished identically, i.e.,

$$\frac{\partial f}{\partial y} = \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \tag{10}$$

This is known as the Euler-Lagrange equation for the variational problem. As an example of its utility, consider the catenary problem. A heavy rope or chain, of uniform weight per unit length, is suspended in a uniform gravitational field from two fixed points, (x_1, y_1) and (x_2, y_2), where x is horizontal and y points vertically upwards. We wish to find the shape of the curve adopted by the rope, $y(x)$. The potential energy of a small element ds of the length of the rope is proportional to its mass, hence to ds, and also proportional to its vertical height from some arbitrary datum. So setting $g = 1$ and for a rope with unit mass per unit length, we have,

$$P = \int_{x_1}^{x_2} y ds \tag{11}$$

However, the length of the rope is also a fixed constraint, i.e., $S = \int_{x_1}^{x_2} ds$ is a given.

Imposing this constraint by the method of Lagrange multipliers, we therefore wish to find $y(x)$ such that,

$$\tilde{P} = \int_{x_1}^{x_2} (y - \lambda) ds \tag{12}$$

is stationary, where λ is some constant. The element of length is given by,

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx = \sqrt{1 + y'^2} \cdot dx \tag{13}$$

So the unconstrained functional whose extremum is sought, $\tilde{P}$, is,

$$\tilde{P} = \int_{x_1}^{x_2} (y - \lambda)\sqrt{1 + y'^2} \cdot dx \tag{14}$$

Now apply the Euler-Lagrange equation, (10), to this integrand and simplify to give,

$$(y - \lambda)y'' - y'^2 - 1 = 0 \tag{15}$$

It is readily checked by substitution that the solution to (15) is,

$$y(x) = \lambda + \Lambda \cosh \frac{x - x_0}{\Lambda} \tag{16}$$

where the constants λ, Λ, x_0 are found by imposing the end conditions $y(x_1) = y_1$ and $y(x_2) = y_2$, and the specified rope length, $S = \int_{x_1}^{x_2} \sqrt{1 + y'^2} \cdot dx$. Equ.(16) shows that the catenary is a hyperbolic cosine.

3. Dynamic and Field Equations Derived from Stationary Action

The Lagrangian formulation of dynamics has a fearsome reputation amongst undergraduates. It really should not have. The essence of the formulation is really quite elementary (though, like virtually any subject in physics, full mathematical rigour will certainly make things more difficult). For the motion of a single particle in one dimension we merely reinterpret x in (6) as time, t , and reinterpret the potential energy as the Lagrangian, i.e., the difference $L = KE - PE$. Hence for a single particle moving along the y axis in a potential given by $V(y)$ the action is,

$$S = \int_{t_1}^{t_2} \left[\frac{1}{2} m\dot{y}^2 - V(y)\right] dt \tag{17}$$

To follow the usual convention we have denoted the time derivative by $\dot{y}$, but it essentially means the same as y' in §2. The Euler-Lagrange equation is thus,

$$m\ddot{y} = -\frac{\partial V}{\partial y} \tag{18}$$

which is just Newton’s law of motion because the RHS is the force acting on the particle. Now consider N particles moving in three spatial dimensions, the jth particle having mass m_j and position vector $\bar{r}_j$. The potential function is assumed to be $V(\{\bar{r}_j\})$. Note that this might be a combination of an externally applied force field together with an interaction between the particles provided that this depends only upon their distance apart (e.g., the electrostatic force in the non-relativistic limit). The action is,

$$S = \int_{t_1}^{t_2} \left[\frac{1}{2} \sum m_j \left|\dot{\bar{r}}_j\right|^2 - V(\{\bar{r}_j\})\right] dt \tag{19}$$

Each of the $3N$ degrees of freedom has its own Euler-Lagrange equation, yielding,

$$m_j \ddot{r}_{jk} = -\frac{\partial V}{\partial r_{jk}} \tag{20}$$

This assumes that the coordinates of all the particles are specified at the two times t_1 and t_2, but of course the same equations of motion, (20), will apply if we choose alternative boundary conditions, such as specifying the coordinates and velocities at time t_1 only.

The stationary action principle for fields is essentially similar but the single independent variable (x or t) is replaced by n independent variables, x_μ, generally the four coordinates of familiar spacetime. The dependent functions are thus $\phi_j(x_\mu)$, and there may be any number, N, of independent fields, each defined over the n-dimensional spacetime. As in the preceding examples, the Lagrange density is assumed to depend on x_μ, $\phi_j(x_\mu)$ and the first derivatives of the fields, $\phi_{j,}{}^{\mu} \equiv \frac{\partial \phi_j}{\partial x_\mu}$, only. Hence we write,

$$S = \int_\Gamma \mathcal{L}\left(x_\mu, \phi_j, \phi_{j,}{}^{\mu}\right) \cdot d^4x \tag{21}$$

Here Γ is some specified, and fixed, region of spacetime on which the fields take specified, and fixed, values. If Γ is chosen to be the whole of spacetime, as it often is, then there will be some criteria to ensure convergence of (21), such as the fields reducing to zero sufficiently quickly at infinity. The principle of stationary action is the requirement that (21) be stationary, $\delta S = 0$, with respect to variations $\phi_j \to \phi_j + \delta\phi_j$ obeying the boundary conditions on Γ, i.e., such that $\delta\phi_j(\Gamma) = 0$. The Euler-Lagrange equations are derived as before,

$$\begin{aligned} \delta S &= \left[\int_\Gamma \mathcal{L}\left(x_\mu, \phi_j + \delta\phi_j, \phi_{j,}{}^{\mu} + \delta\phi_{j,}{}^{\mu}\right) - \mathcal{L}\left(x_\mu, \phi_j, \phi_{j,}{}^{\mu}\right) \right] \cdot d^4x \\ &= \int_\Gamma \left[\frac{\partial L}{\partial \phi_j} \delta\phi_j + \frac{\partial L}{\partial \phi_{j,}{}^{\mu}} \delta\phi_{j,}{}^{\mu} \right] d^4x \end{aligned} \tag{22}$$

where, in the last line, we are using the usual convention that repeated indices are summed. The second term can be evaluated by parts, as before,

$$\int_\Gamma \frac{\partial L}{\partial \phi_{j,}{}^{\mu}} \delta\phi_{j,}{}^{\mu} d^4x = -\int_\Gamma \frac{\partial}{\partial x_\mu} \left(\frac{\partial L}{\partial \phi_{j,}{}^{\mu}} \right) \delta\phi_j d^4x \tag{23}$$

by virtue of $\delta\phi_j(\Gamma) = 0$. Hence, substituting (23) into (22) and equating the factor of $\delta\phi_j$ to zero, since the latter are arbitrary variations, gives the Euler-Lagrange equations,

$$\frac{\partial L}{\partial \phi_j} = \frac{\partial}{\partial x_\mu} \left(\frac{\partial L}{\partial \phi_{j,}{}^{\mu}} \right) \tag{24}$$

So there is one such equation for each of the N fields, ϕ_j, and the RHS involves the sum over n terms. An example is the scalar field, for which the Lagrange density is,

$$\mathcal{L} = \partial^\mu \phi^* \cdot \partial_\mu \phi - m^2 |\phi|^2 \tag{25}$$

For which the Euler-Lagrange equations are the Klein-Gordon equation for the field and its conjugate,

$$\left(\partial^2 + m^2\right)\phi = 0 \quad \text{and} \qquad \left(\partial^2 + m^2\right)\phi^* = 0 \tag{26}$$

Similarly the reader may like to show that the Dirac equation, and its adjoint, follow from the Lagrange density,

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \tag{27}$$

Maxwell's equations for a 4-vector current source J^μ follow from the Lagrange density,

$$\mathcal{L} = -\frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - J^\mu A_\mu \tag{28}$$

where the action is stationary with respect to variations in the four-vector potential A_μ and the electromagnetic field is defined subsequently via $F_{\nu\mu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that in (27) and (28) we have used $\hbar = c = 1$.

4. A Simple Example in Classical Dynamics

We now return to the issue of whether the stationary action can be regarded as *least* action. We shall firstly consider classical, simple harmonic motion in one dimension. The Lagrangian is,

$$L = \frac{1}{2}\left(\dot{x}^2 - x^2\right) \tag{29}$$

where we have assumed unit mass and also a 'spring constant' of unity, i.e., unit force per unit displacement. The Euler-Lagrange equation of motion is, as we would expect,

$$\ddot{x} = -x \tag{30}$$

If released from rest at unit displacement at time zero the solution is therefore,

$$x = \cos t \tag{31}$$

As far as the formulation of the variational problem is concerned, we can suppose that we specified unit displacement at time zero together with zero displacement at time $t = \pi/2$, which gives the same solution. So the action is,

$$S = \int_0^{\pi/2} \frac{1}{2}\left(\dot{x}^2 - x^2\right)dt \tag{32}$$

Substitution of (31) into (32) shows that the action in this case is zero. As a simple investigation of whether variations produce an increase or a decrease in the action, consider $x(t)$ to be varied thus,

$$x(t) = \cos t + \lambda \sin 2t \tag{33}$$

This continues to respect the boundary conditions since the sine term vanishes at both $t = 0$ and $t = \pi / 2$. Substitution of (33) into (32) and carrying out the integrations gives,

$$S = \frac{3\pi}{8}\lambda^2 \tag{34}$$

The terms linear in λ cancel. Hence, irrespective of what sign we choose for the amplitude of the variation, λ, the action is increased by the variation (33), recalling that the solution, (30), has zero action. This is consistent with the claim that the action is a minimum. Of course, this is just an illustration and does not prove that *any* variation in $x(t)$ will result in an increased action. This is considered next.

5. Least Action in Discrete Classical Particle Dynamics

5.1 A Necessary Criterion for Least Action

To begin with consider a single particle moving in one dimension so that the action is $S = \int_{t_1}^{t_2} L(t, x(t), \dot{x}(t)) \cdot dt$. Considering a variation in $x(t)$ induces a variation in the action of,

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} \right] \cdot dt \tag{35}$$

Setting this to zero will lead to the Euler-Lagrange equation for an extremum, as we have seen. From our knowledge of ordinary calculus we expect minima and maxima to be distinguished by the sign of the second derivative. Specifically a minimum is expected to be indicated by $\delta^2 S > 0$. So consider the second functional derivative,

$$\delta^2 S = \int_{t_1}^{t_2} \left[\frac{\partial^2 L}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 L}{\partial x \partial \dot{x}}\delta x \delta \dot{x} + \frac{\partial^2 L}{\partial \dot{x}^2}(\delta \dot{x})^2 \right] \cdot dt \tag{36}$$

Noting that $\delta x \delta \dot{x} = \frac{1}{2}\frac{d}{dt}(\delta x)^2$ the second term in (36) can be transformed by integrating by parts as follows,

$$\int_{t_1}^{t_2} \left[2\frac{\partial^2 L}{\partial x \partial \dot{x}}\delta x \delta \dot{x} \right] \cdot dt = \left[\frac{\partial^2 L}{\partial x \partial \dot{x}} \cdot (\delta x)^2 \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}} \right) \cdot (\delta x)^2 \cdot dt \tag{37}$$

The first term on the RHS is zero due to the boundary conditions. So (36) becomes,

$$\delta^2 S = \int_{t_1}^{t_2} \left[Q(x)(\delta x)^2 + \frac{\partial^2 L}{\partial \dot{x}^2}(\delta \dot{x})^2 \right] \cdot dt \tag{38}$$

where,

$$Q(x) = \frac{\partial^2 L}{\partial x^2} - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}} \right) \tag{39}$$

It might be tempting to conclude that Q is identically zero at an extremum in view of the Euler-Lagrange equation $\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right)$, but this would suppose that we could commute the time derivative with the partial derivative with respect to x, which is not true. However even for non-zero Q a necessary condition that $\delta^2 S \geq 0$ for any variation $\delta x(t)$ obeying the boundary conditions is that the coefficient of $(\delta\dot{x})^2$ in (38) be non-negative at all points over the integration range,

$$\delta^2 S \geq 0 \Rightarrow \frac{\partial^2 L}{\partial \dot{x}^2} \geq 0 \text{ for all } t \in [t_1, t_2] \tag{40}$$

The proof can be found in Gelfand and Fomin (1963), §25. Heuristically the reason for this can be seen as follows. Suppose that $\frac{\partial^2 L}{\partial \dot{x}^2} < 0$ over some region of the integration range. We need only contrive a variation $\delta x(t)$ which has a large gradient over this region compared to elsewhere, and a small value elsewhere, to conclude that the integral (38) would be negative. This can only be avoided if $\frac{\partial^2 L}{\partial \dot{x}^2} \geq 0$ at all integration points. QED. Take careful note that (40) is only claimed to be a *necessary* criterion for the action to be a minimum. We shall come to the sufficient criterion shortly.

For a single particle, defining the Lagrangian as $L = KE - PE$ and since $KE = \frac{1}{2} m\dot{x}^2$, the criterion (40) is simply that the mass be non-negative, $m \geq 0$. So, consistent with the illustration in §4, things are looking favourable for demonstrating that the conventional definition of the action, via $L = KE - PE$, does lead to action being a minimum, not a maximum as I erroneously thought.

The criterion (40) generalises to the case of N particles in three dimensional space. We now have a matrix of second derivatives with respect to the velocities,

$$M_{JK} = \frac{\partial^2 L}{\partial \dot{r}_{ja} \partial \dot{r}_{kb}} \tag{41}$$

where the capital subscripts range over the $3N$ values of particle-plus-direction, i.e., $J \equiv ja$, $K \equiv kb$, where $j, k \in [1, N]$ and $a, b \in [1,3]$. The generalisation of criterion (40) is that the determinant of this matrix be non-zero when evaluated at any point along the trajectories determined by the Euler-Lagrange equations of motion, i.e.,

$$\delta^2 S \geq 0 \Rightarrow \|M\| \geq 0 \text{ along the extremum trajectories} \tag{42}$$

For non-relativistic dynamics in a potential field and in Cartesian coordinates, the action is given by (19) and so M is a diagonal matrix whose elements are the particle masses. Hence $\|M\| \geq 0$ is assured. So it is looking rather like action is a minimum for discrete particle dynamics. But there is a catch.

5.2 Sufficient Conditions for Least Action in Discrete Particle Dynamics

The rigorous conditions which are sufficient to ensure that the extremum given by a solution of the Euler-Lagrange equations is actually a minimum of the action are complex in general and the reader is referred to Gelfand and Fomin (1963) or a similar text on the calculus of variations. However the essence of the matter is captured by three conditions,

(i) The Euler-Lagrange equations are obeyed (and hence the action is stationary);

(ii) The determinant of the matrix of second derivatives wrt the gradients is strictly positive definite when evaluated on the extremals, $\|M\| > 0$;

(iii) The integration region contains no points conjugate to the end points.

The last criterion introduces the concept of a 'conjugate point'. Roughly speaking two points (times) are conjugate if their use as ends points of the action integral does not uniquely determine the solution. (This is my rough paraphrasing. I hope mathematicians are not shuddering too much). For example, the geodesic on the surface of a sphere, which minimises the path length between two points, is uniquely determined by the two points *except* in the case of points which are diametrically opposite. These are conjugate points since there is a multiplicity of different great arcs connecting them. Another example is provided by simple harmonic motion when the displacement is specified as zero at times differing by an integral multiple of the half-period. This clearly does not suffice to determine the amplitude of the motion.

In practical applications to discrete particle dynamics this subtlety may matter little. Adoption of a different integration range may establish a solution to be a strict minimum of the action. In the last example, adoption of any range of times less than the half-period will suffice.

However we see in this the beginnings of a problem for continua, for the dynamics of fields. The spectrum of continua will usually have no finite upper bound frequency. Consequently we may anticipate that the problem posed by criterion (iii) might be insuperable for fields because any time period, however short, will exceed the half-period for some frequencies. So we may also anticipate that, in the case of fields, the principle of stationary action may fail to be a principle of least action. This is correct and is illustrated next.

6. The Massless Real Scalar Field

In the case of just one field, the variation of the action, (22), becomes,

$$\delta S = \int_\Gamma \left[\frac{\partial L}{\partial \phi} \delta\phi + \frac{\partial L}{\partial \phi^{,\mu}} \delta\phi^{,\mu} \right] d^4 x \qquad (43)$$

The second derivative is,

$$\delta^2 S = \int_\Gamma \left[\frac{\partial^2 L}{\partial \phi^2} (\delta\phi)^2 + 2\frac{\partial^2 L}{\partial \phi^{,\mu} \partial \phi} \delta\phi^{,\mu} \delta\phi + \frac{\partial^2 L}{\partial \phi^{,\mu} \partial \phi^{,\nu}} \delta\phi^{,\mu} \delta\phi^{,\nu} \right] d^4 x \qquad (44)$$

Consider a real scalar field with zero mass whose Lagrange density is,

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \cdot \partial_\mu \phi = \frac{1}{2} \left[\dot{\phi}^2 - \left| \overline{\nabla} \phi \right|^2 \right] \qquad (45)$$

The corresponding Euler-Lagrange equation is just the wave equation, the massless limit of the Klein-Gordon equation,

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial t^2} \qquad (46)$$

Because this Lagrange density does not depend upon the field, ϕ, but only upon its derivatives, the first two terms in (44) are zero. In the third term we have $\frac{\partial^2 L}{\partial \dot{\phi}^2} = 1$

whereas for the spatial derivatives $\frac{\partial^2 L}{\partial \phi_{,i}^{\ 2}} = -1$. Hence the second derivative of the action is,

$$\delta^2 S = \int_\Gamma \left[\left(\delta\dot{\phi}\right)^2 - \left|\overline{\nabla}\delta\phi\right|^2 \right] d^4 x \tag{47}$$

So it follows that, since $\delta\phi$ is arbitrary, and since the two terms in (47) contribute to the integral with opposite signs, $\delta^2 S$ may be of either sign. Consequently we are not entitled to claim that the wave equation, (46), ensures that the action corresponding to (45) is a minimum. We can only refer to the wave equation as an extremum of this action.

It is clear from (47) that the origin of the problem is the contribution to the 'potential energy' from the spatial derivatives of the field. This is a generic property of field Lagrangians. In fact it is unavoidable in Lorentz invariant (i.e., relativistic) theories, since $\dot{\phi}$ cannot occur except as part of $\phi^{,\mu}$, and this must be converted to a Lorentz scalar by contraction with some other 4-vector, e.g., $\phi_{,\mu}$ itself. It is thus the Minkowski metric which results in the minus sign in (47) which is the source of the problem. Hence we cannot guarantee the sign of $\delta^2 S$ in field theory and so we should speak only of a stationary action principle for fields, not least action.

In the competition between the temporal and spatial derivatives in (47) as regards the sign of $\delta^2 S$ we can see, at last, the explanation for my earlier difficulty regarding whether the correct definition of action should use $KE - PE$ or $PE - KE$. To see this we return to the catenary, but now allow it to vibrate so as to introduce dynamics into the problem.

7. The Vibrating Catenary

Firstly, note that in the static problem of the catenary, we can now see that potential energy is minimised, as we knew it must be. The action, (14) is $\tilde{P} = \int_{x_1}^{x_2} (y - \lambda)\sqrt{1 + y'^2} \cdot dx$. The second derivative of the integrand with respect to y' is $\frac{y - \lambda}{\left(1 + y'^2\right)^{3/2}}$ which is positive by virtue of the solution (16), which must have $\Lambda > 0$.

Now consider a catenary which is allowed to vibrate in the vertical direction. Each element of length has kinetic energy $\frac{1}{2}\dot{y}^2 \delta m = \frac{1}{2}\dot{y}^2 \delta s = \frac{1}{2}\dot{y}^2 \sqrt{1 + y'^2} \cdot \delta x$. So, adopting a Lagrangian $KE - PE$ gives the action,

$$S = \int_{t_1}^{t_2}\int_{x_1}^{x_2} \left[\frac{1}{2}\dot{y}^2 - (y - \lambda) \right] \sqrt{1 + y'^2} \cdot dxdt \tag{48}$$

By adopting the sign $KE - PE$ for the integrand, its second derivative wrt $\dot{y}$, which is just $\sqrt{1 + y'^2}$, is positive and hence fulfils (this part of) the requirement for the action to be a minimum. In contrast, the second derivative wrt y', which is $\frac{0.5\dot{y}^2 - (y - \lambda)}{(1 + y'^2)^{3/2}}$,

is of indeterminate sign, and will be negative for small velocities. But we should no longer be surprised by this outcome. We have already seen in §6 that for dynamic continuum problems, the sign of the quadratic time derivative and the quadratic spatial derivatives will generally be opposite, and this results in the extremum being neither a minimum nor a maximum in general.

8. Conclusion

Whilst my youthful presumption that $PE - KE$ was to be preferred over $KE - PE$ was certainly wrong, I am partly vindicated by the failure of either to define a principle of least action. In general, for continuum (field) theories, action is only stationary. However, barring certain subtleties discussed in §5.2, the case of discrete particles does provide a principle of least action.

References

R.P.Feynman, R.BLeighton and M.Sands (1964) “The Feynman Lectures on Physics”, Volume II, §25-6.

I.M.Gelfand and S.V.Fomin (1963) “Calculus of Variations”, translated R.A.Silverman, Dover Publications, NY.