Chapter 9b – Freeze-Out Of Hydrogen Recombination?

Last Update: 24 June 2007

1. Introduction

In Chapter 8 we evaluated the temperatures (times) at which the formation of neutral atoms was complete up to an arbitrary percentage [100(1 – y)]. To perform these calculations it was assumed that the reactions, e.g. $e^- + p^+ \rightarrow H + N\gamma$, were in thermodynamic equilibrium, so the concentration of the remaining free protons and electrons were found to reduce continuously. However, this does not take account of the possibility that these reactions could freeze out due to cosmic expansion. We shall show in this Chapter that freeze-out of the hydrogen recombination reaction does occur, though not until recombination is just over 99.99% complete. Consequently, the freeze-out makes no significant difference to the abundance of neutral atomic hydrogen at these times. Nevertheless, this freeze-out is extremely important. It determines the non-zero residual density of free electrons and protons. Whilst only around 1 in 10^4 of the initial electrons and protons remain free, this is believed to play an important role in the formation of the first stars.

2. A Classical Model For The Proton-Electron Capture Cross-Section

To examine if, and when, the recombination reaction freezes out, we need to know its rate, or equivalently its cross section. We use a crude classical model in this Section. Imagine a non-relativistic electron with kinetic energy $mv^2/2$ moving towards a stationary proton with collision parameter 'b' (i.e. 'b' is the projected distance of closest approach). The mechanism by which the electron can be captured, becoming bound to the proton, is to radiate an amount of energy at least equal to its initial kinetic energy. (The electron can actually radiate more than its initial kinetic energy, because, by the time it reaches the proton, it would have even greater kinetic energy due to Coulomb attraction). It if does so, the electron must be bound since it cannot escape to large distances again. At the projected position of closest approach, the Coulomb force acting on the electron is $F = e^2/b^2$ (cgs units). Hence, its acceleration at this time is $a = e^2/mb^2$. The power radiated by a charge 'e' undergoing an acceleration of 'a' is, using standard classical electromagnetism (Maxwell's equations),

$$\text{Power radiated by accelerating charge} = \frac{2}{3} \cdot \frac{e^2 a^2}{c^3} \qquad (1)$$

which in this case becomes,

$$P = \frac{2}{3} \cdot \frac{e^2}{c^3} \left(\frac{e^2}{mb^2} \right)^2 = \frac{2e^6}{3m^2 c^3 b^4} \qquad (2)$$

To convert this to a total energy radiated we require a time over which acceleration at roughly this rate is occurring, i.e. how long is the electron close to the proton? In the case of capture of an electron we may imagine the electron being turned completely around, i.e. through a half-circle. Ignoring the change of speed of the electron, the time for the electron to perform a half-rotation at a distance 'b' and speed 'v' is $\pi b / v$. Thus, the energy radiated may be estimated as,

$$\text{Energy radiated} = \frac{2e^6}{3m^2 c^3 b^4} \cdot \frac{\pi b}{v} = \frac{2\pi e^6}{3m^2 c^3 b^3 v} \qquad (3)$$

For capture, this must exceed $mv^2/2$, hence,

$$\frac{2\pi e^6}{3m^2c^3b^3v} \geq \frac{mv^2}{2} \;\Rightarrow\; b \leq \left(\frac{4\pi}{3}\right)^{1/3} \frac{e^2}{mvc} \qquad (4)$$

Since this gives the smallest 'b' for which capture will occur, the cross section is clearly $\sigma = \pi b^2$, i.e.,

$$\sigma_{pe\to H\gamma} = \pi\left(\frac{4\pi}{3}\right)^{2/3}\left(\frac{e^2}{mvc}\right)^2 = \pi\left(\frac{4\pi}{3}\right)^{2/3}\frac{e^4}{2Emc^2} \qquad (5)$$

where E is the (non-relativistic) kinetic energy. The dependence on e^4 is as expected from the lowest order contributing Feyman diagram (i.e. $\propto \alpha^2$, but note that Equ.5 does not depend upon Planck's constant – nor does the Thompson cross-section for electron-photon scattering, as used in Chapter 9). Note that the factor $\pi\left(\frac{4\pi}{3}\right)^{2/3} = 8.16$ is almost the same as the factor of $8\pi/3 = 8.38$ occurring in the Thompson cross-section. Equ.(5) for electron-proton capture differs from the latter in that 2E occurs in the denominator in place of a second mc^2 factor. Hence, the electron-proton capture cross section is predicted to be a factor $mc^2/2E$ larger than the electron-photon scattering cross section. At the energies of interest (~0.4eV at ~3000°K) this is a factor of about 6 x 10^5. This is consistent with recombination occurring whilst the universe becomes transparent.

3. Electron Capture Times

The flux of incoming electrons is roughly $\rho_e^N v/4$ where ρ_e^N is the electron number density and v is their typical speed. The number of electron captures per second per proton (i.e. the reaction rate) is thus,

$$\frac{1}{T_{ep}} = \sigma\rho_e^N v/4 = \frac{\rho_e^N v}{4}\pi\left(\frac{4\pi}{3}\right)^{2/3}\frac{e^4}{2Emc^2} = 1.44\rho_e^N \frac{e^4 c}{\sqrt{\left[E(mc^2)^3\right]}} \qquad (6)$$

Where T_{ep} is the ep reaction time. The last version has been derived using $v = \sqrt{2E/m}$. Strictly, v should be the relative velocity of the electron and the proton, but the speed of the latter will be small in comparison (at the same energy).

Following Chapters 8 and 9 the residual electron density can be expressed in terms of the fraction (y) left after hydrogen recombination (given in Chapter 8) by,

$$\rho_e^N = 0.875y\frac{0.2436}{\xi_{\gamma N} = 1.9x10^9}\left(\frac{kT}{\hbar c}\right)^3 \qquad (7)$$

Combining (6) and (7) gives,

$$\frac{1}{T_{ep}} = 1.32\text{x}10^{-10}\, y \frac{e^4 (kT)^{5/2}}{m^{3/2} c^5 \hbar^3} = 2.0\text{x}10^{-18}\, yT^{5/2} \qquad (8)$$

(the last version giving the reaction rate in sec^{-1} for temperature in ^oK). To derive Equ.(8) we have replaced the kinetic energy with the typical thermal energy of $^3/_2kT$. The following Table gives the results of using Equ.(8) with the y values from Chapter 8:-

Electron-Proton Capture Reaction Times T_{ep}

T (^{o}K)	**y**	**1.5$t_{universe}$ (years)** [(1)]	**T_{ep} (years)**
3070	0.0088	526,000	3,400
3000	0.005	551,000	6,400
2900	0.002	580,000	17,400
2800	0.00078	611,000	48,700
2700	0.00029	645,000	144,000
2600	9×10^{-5}	683,000	509,000
2586	8×10^{-5}	688,500	580,000
2574	7×10^{-5}	693,000	670,000
2562	6.1×10^{-5}	698,000	778,000
2500	3×10^{-5}	724,000	1,680,000
2400	8×10^{-6}	770,000	7×10^{6}
2320	2.6×10^{-6}	810,000	2×10^{7}
2200	$\sim 10^{-6}$	877,000	7×10^{7}

[(1)]Freeze-out of the capture reaction will occur if the reaction rate is less than the universe's expansion rate, H. But $t_{universe} = 2 / 3H$ in the matter dominated era, whereas $t_{universe} = 1 / 2H$ in the radiation era. This time marks the transition between the two. For sake of argument the reaction time, T_{ep}, is compared with $1/H = 1.5t_{universe}$. It is clear from the above results that using $1/H = 2t$ as the freeze-out criterion would reduce the associated value of 'y' by less than a factor of 2. This Table has assumed a time-temperature relation $T\sqrt{t} = 1.02 \times 10^{10}$ k$\sqrt{s}$ at earlier times, but giving way to a matter dominated expression at later times.

The line in red indicated where freeze-out of the capture reaction occurs, at about a temperature of 2574^oK. Somewhat less than 1 in 10^4 of the initial electrons remain free at this point. This result implies that recombination of hydrogen actually ceases at this temperature, so the number of free electrons (and protons) at this temperature will survive for a much longer period, taking an active role later in star formation.

From Equ.(7) the absolute density of remnant free electrons and protons is $\sim 10^4$ per m^3 at this time. Of course, their density will continue to reduce as the universe expands, but the actual number of free electrons and protons will remain constant after this time. Constant, that is, until our assumption of small-scale homogeneity breaks down, when the first stars or galaxies start to form by gravitational collapse.

4. Would The Universe Remain Opaque If $\xi_{\gamma N}$ Were Small Enough?

No. At first sight it seems that it might. If we mistakenly assume that the remnant fraction of free electrons is $\sim 10^{-4}$ for any value of $\xi_{\gamma N}$ then, for sufficiently small values of $\xi_{\gamma N}$, the absolute density of free electrons would indeed remain large enough to keep the universe opaque after freeze-out. But that is not what happens. Instead, following the methodology of Chapters 8 and 9, we find that the remnant fraction of free electrons at freeze-out varies in proportion to $\xi_{\gamma N}$. Thus, if we decrease $\xi_{\gamma N}$ by a factor of, say, 1000, then the remnant electron fraction also drops by a fraction of 100, to $\sim 10^{-7}$. This means that the absolute remnant free electron density is essentially the same, and hence implies transparency after freeze-out. This fidning is consistent with Peebles (1993), Equ.(6.119), which gives the remnant electron fraction to be,

$$x = 1.2x10^{-5} \frac{\sqrt{\Omega}}{h\Omega_b} \tag{9}$$

where Ω is the total matter density relative to the critical density (~0.3) and Ω_b is the baryon relative density (~0.04), and h is the Hubble parameter, conventionally normalised (h ~ 0.7). But $\Omega_b \propto 1/\xi$, so that the above relation implies $x \propto \xi$ (at least when dark matter dominates over baryonic matter, so that Ω is roughly constant), as we have also found.

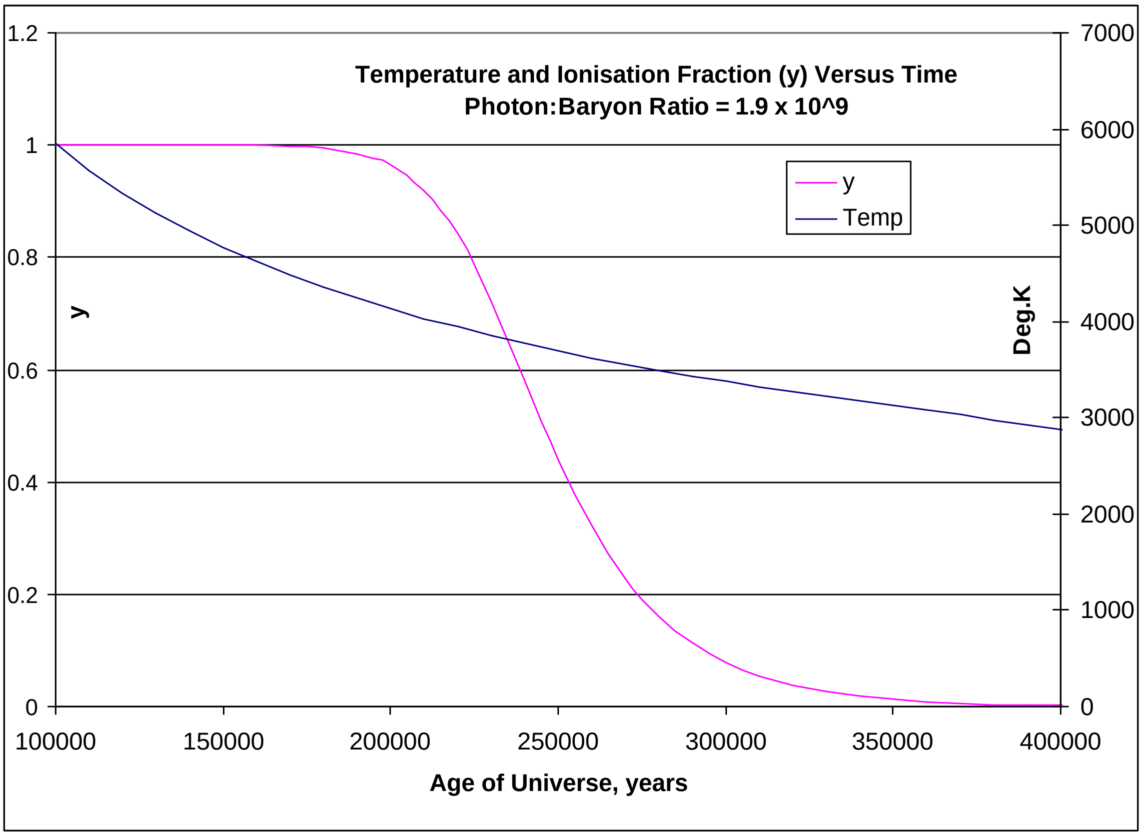
In fact, running the calculations for a range of ξ, varied by up to 4 orders of magnitude in both directions, shows that the universe becomes transparent roughly 150,000 years or so before recombination freeze-out in every case. Of course, for *very* large ξ (>10^{11}), the universe is transparent from the start (since there is then not enough matter around to ever make it opaque). The results of these calculations are illustrated in the graphs which follow

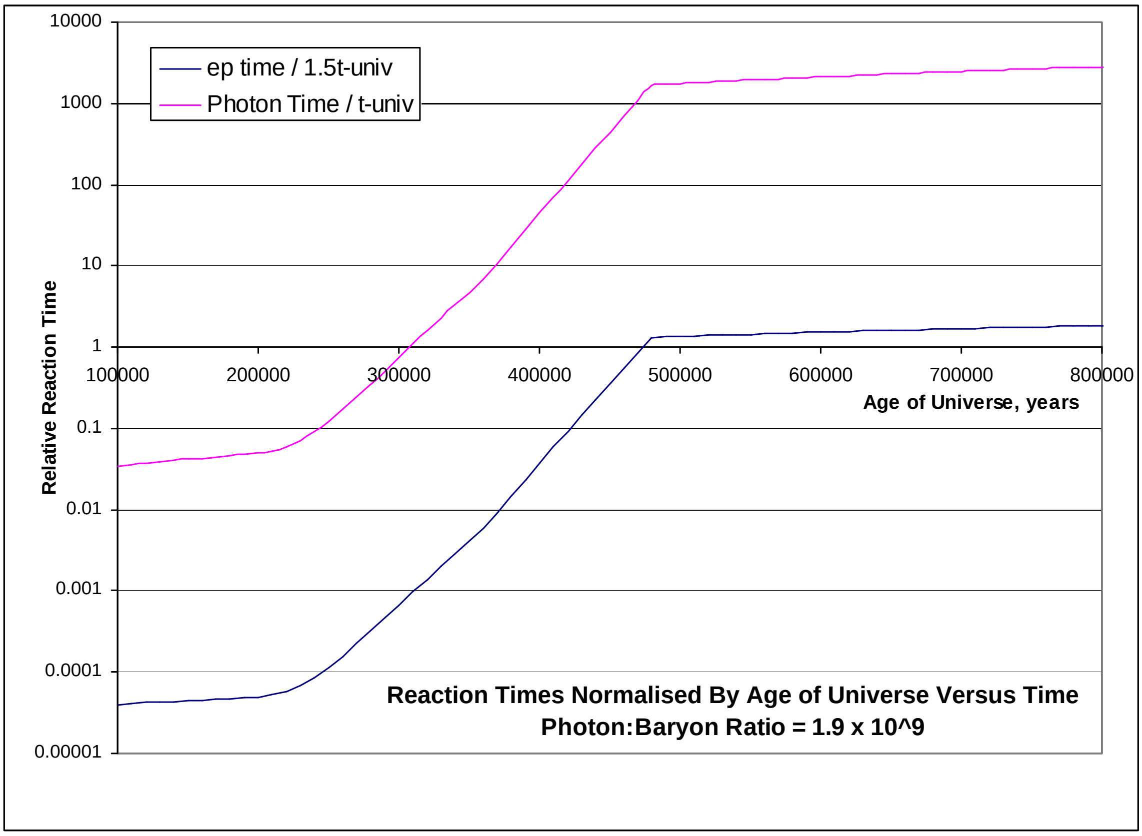
5. Conclusion

Hydrogen recombination ceases at a temperature of 2574^oK (at 462,000 years) when the numbers of remnant protons (and electrons) are a fraction 7 x 10^{-5} of their initial numbers.

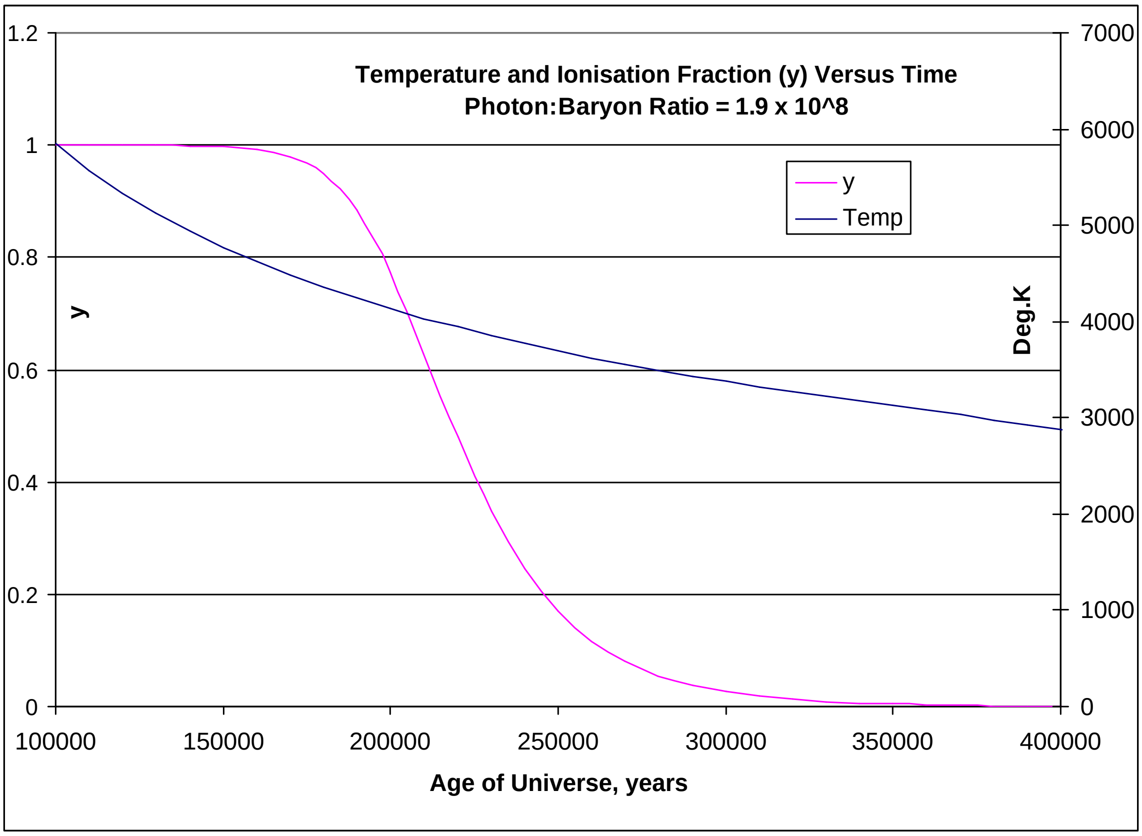
If the photon:baryon ratio were changed, even by up to 4 orders of magnitude, this qualitative picture is unchanged though the precise time and temperature at transparency and freeze-out vary slightly.

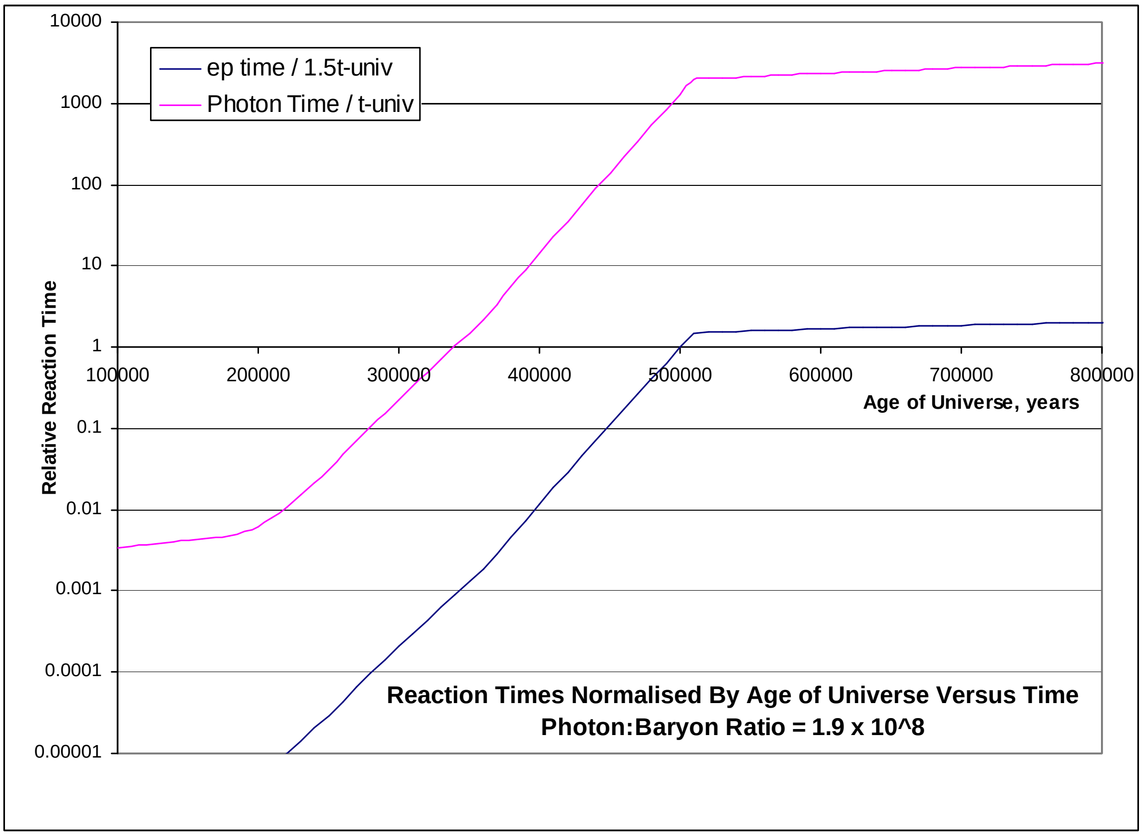
The definition of transparency employed is that the mean free path of a photon is greater than or comparable with the size of the universe. It is virtually obvious, therefore, that radiation pressure cannot prevent structure formation once the universe is transparent against this definition. The only doubt is that the vast numerical preponderance of photons means that only one in $\sim 10^9$ photons would need to interact to imply that most baryons would be affected. The doubt is allayed by Peebles (1993), Equ.(6.142), which implies that, for a residual ionisation given by (9), the characteristic time for deceleration of streaming gas particles by radiation drag is longer than the prevailing age of the universe for re-shifts less than ~8,000 (i.e. for times later than ~20,000 years).



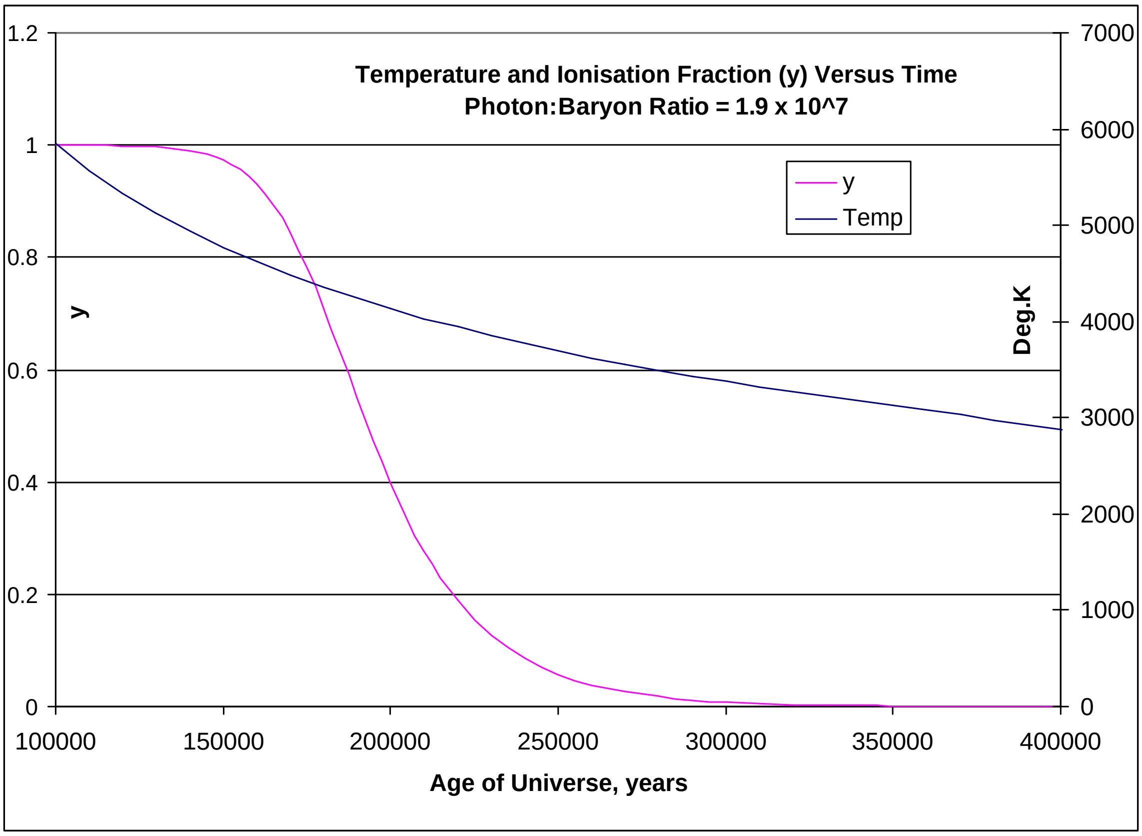


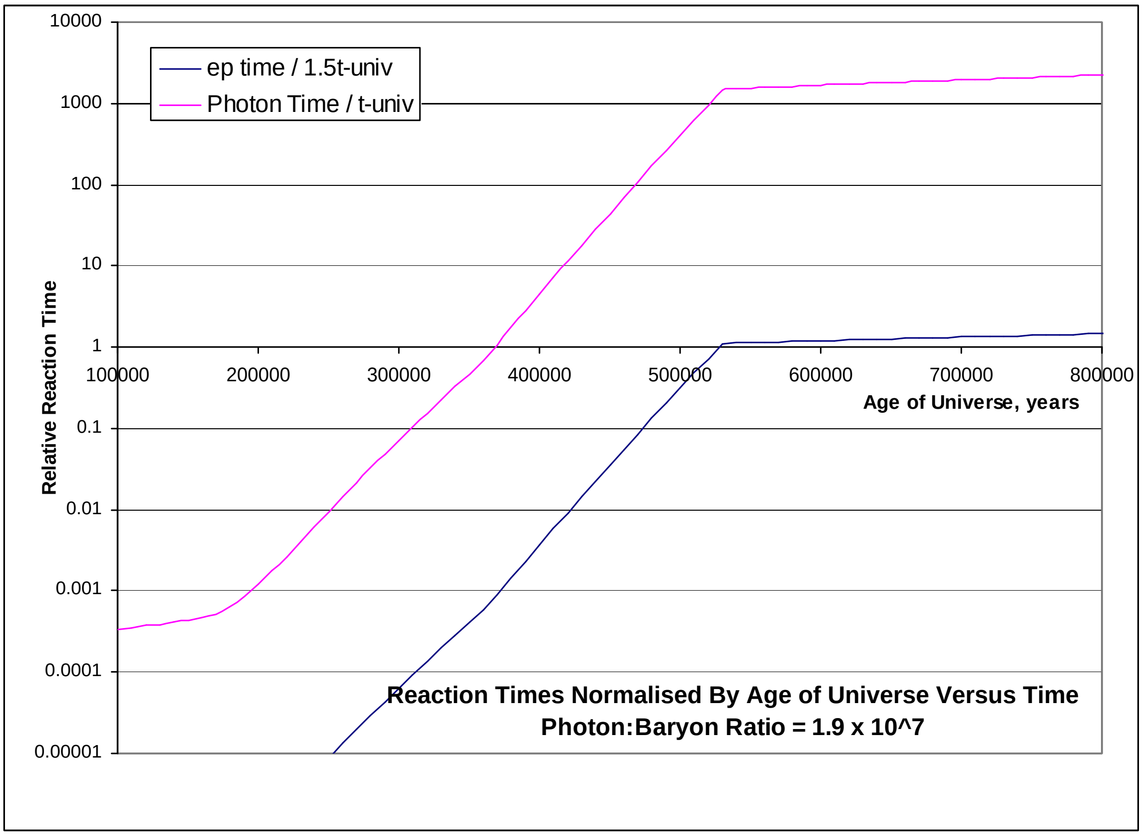
The above lines cross unity when recombination ceases (ep) and at transparency (photon).



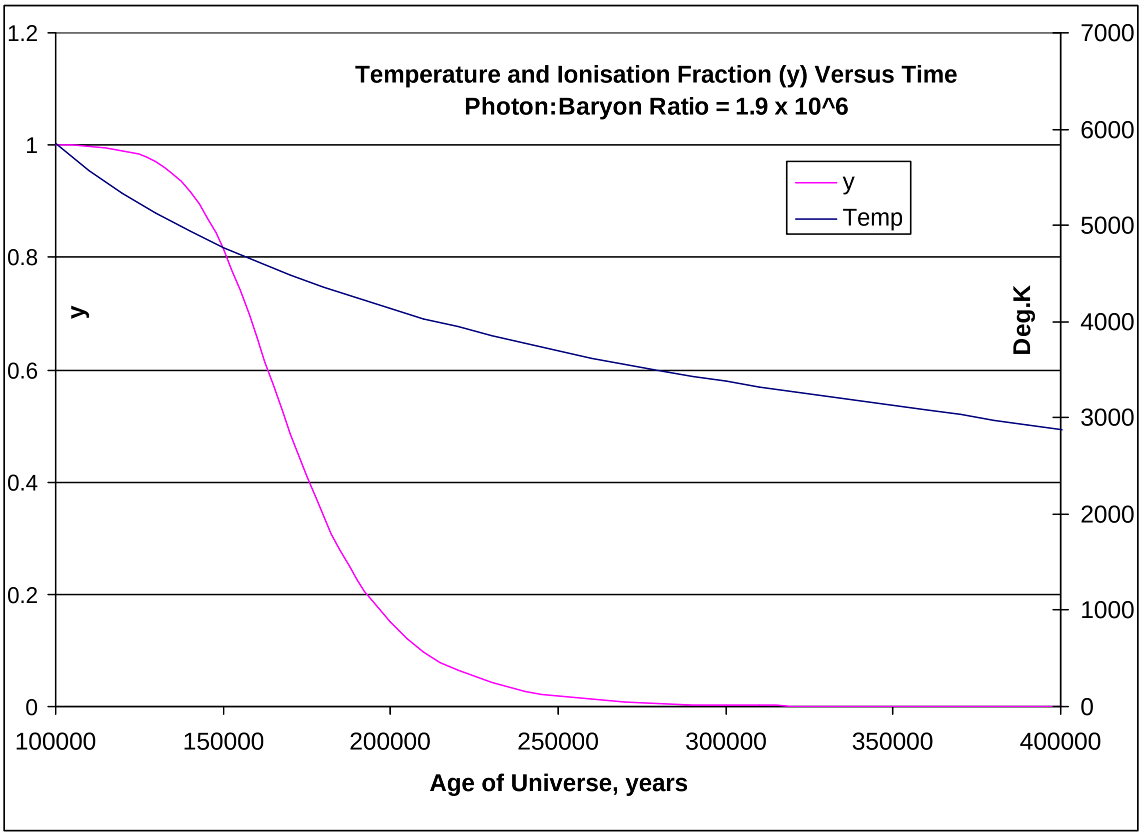


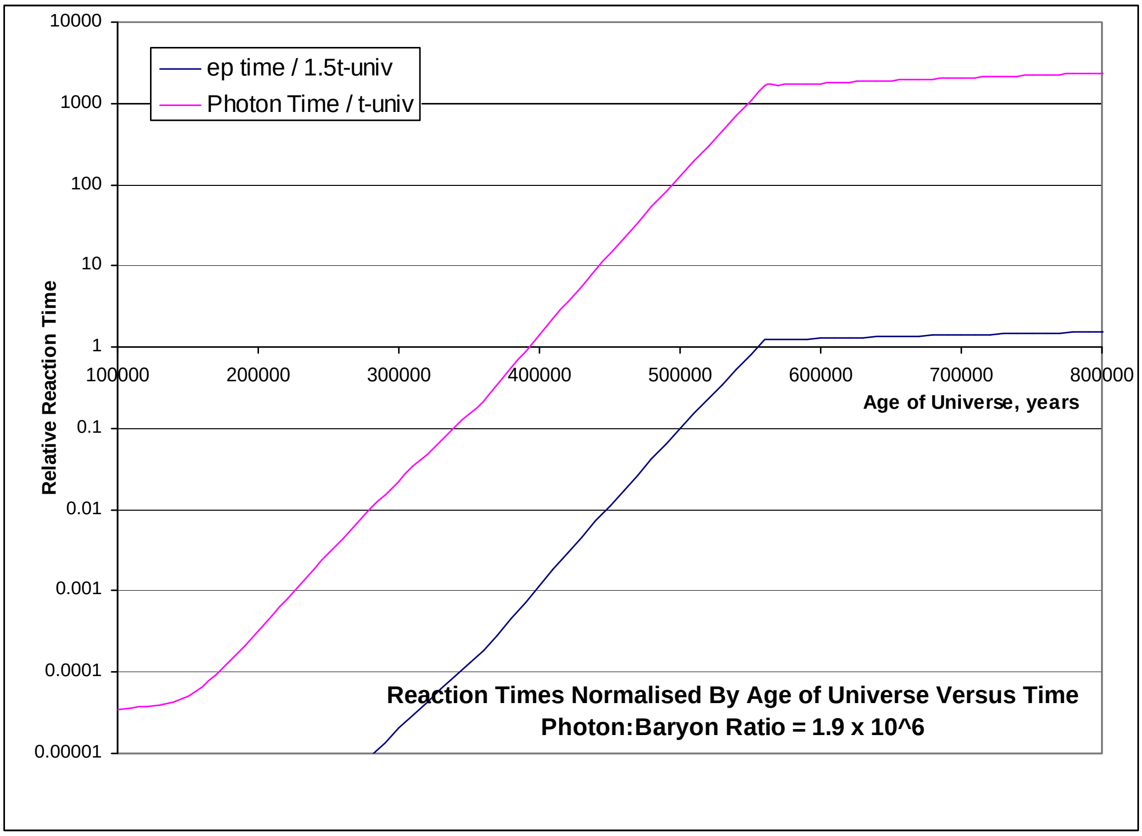
The above lines cross unity when recombination ceases (ep) and at transparency (photon)



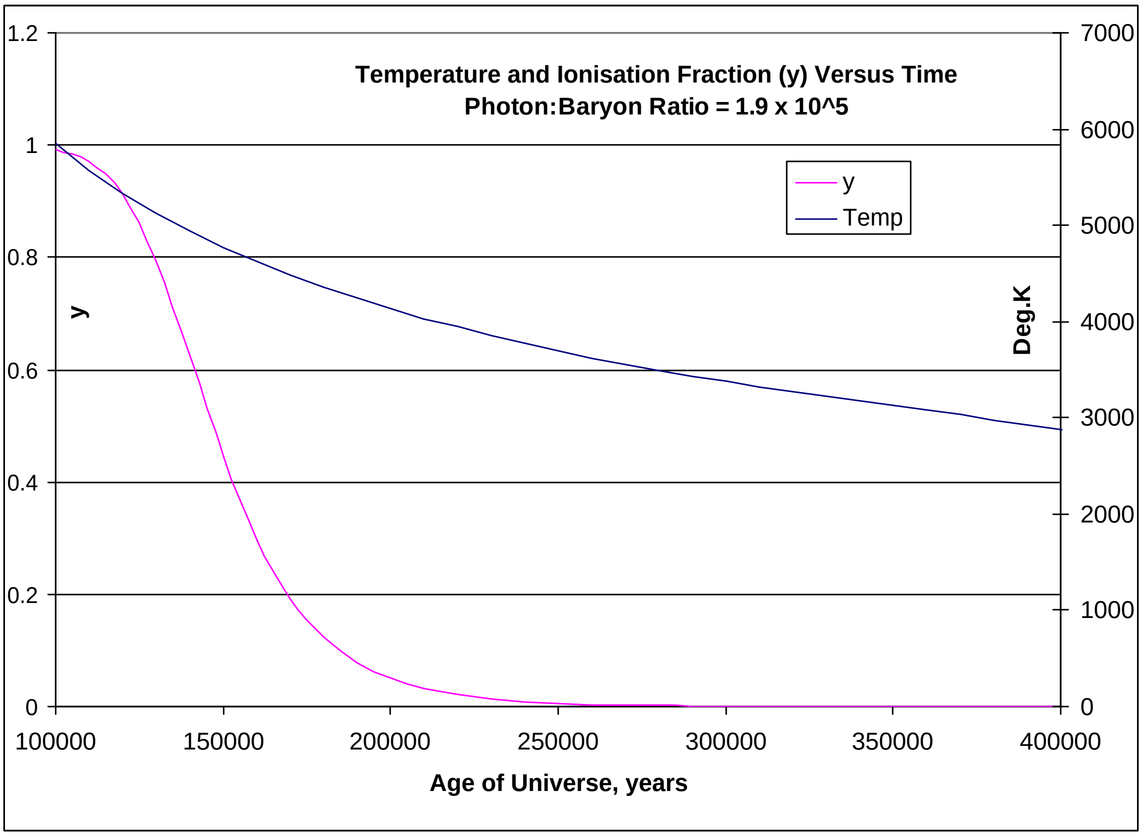


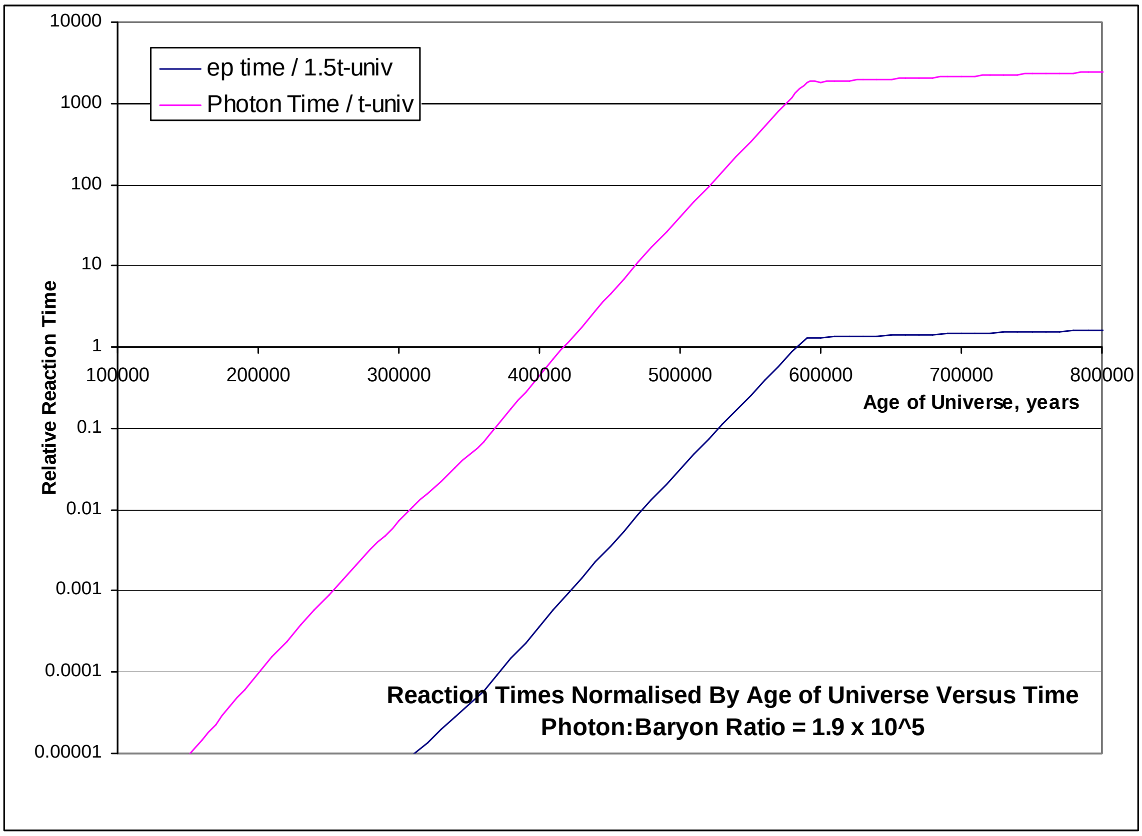
The above lines cross unity when recombination ceases (ep) and at transparency (photon)



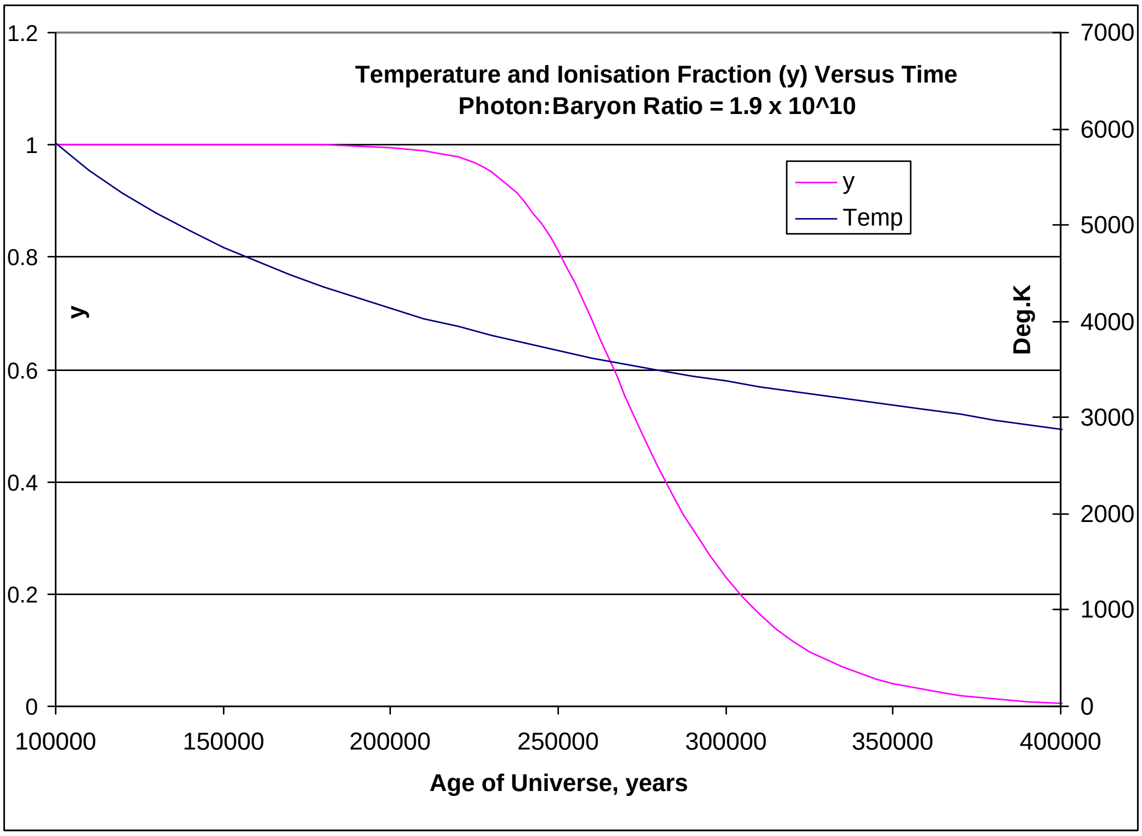


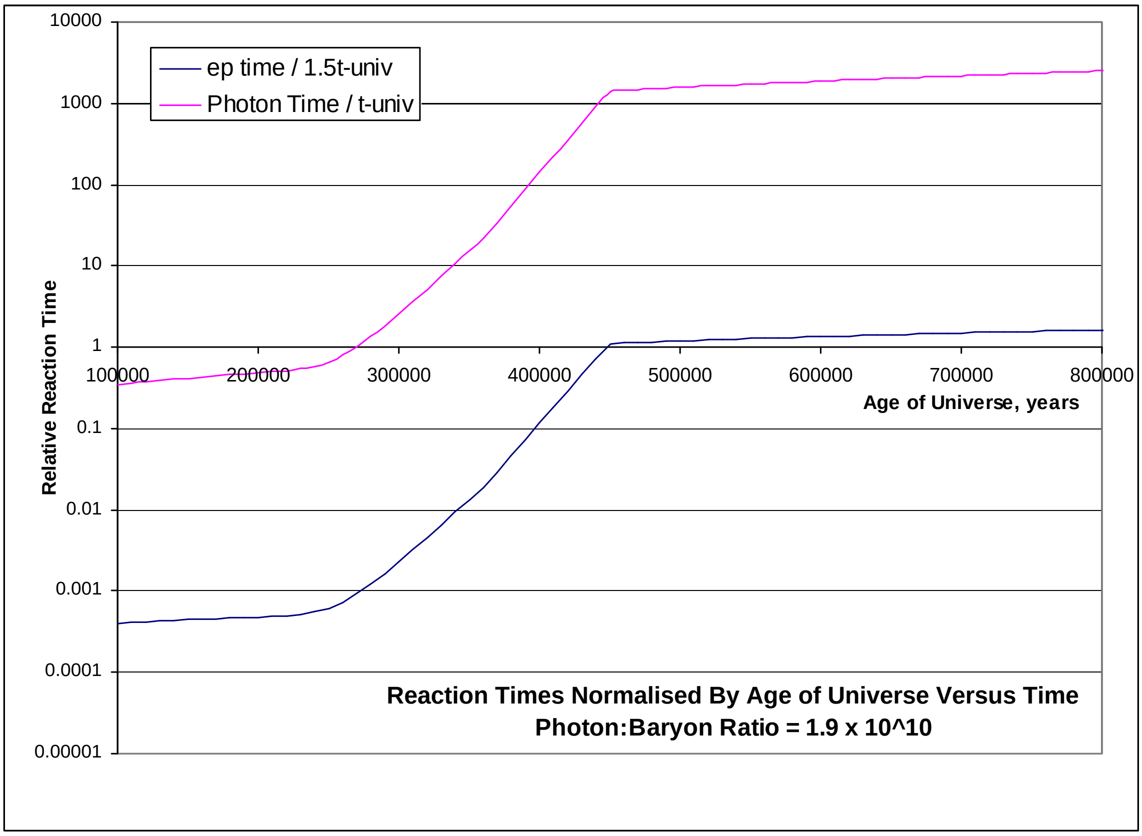
Note – Transparency still occurs (~600,000yrs) even with x1000 as much baryonic matter

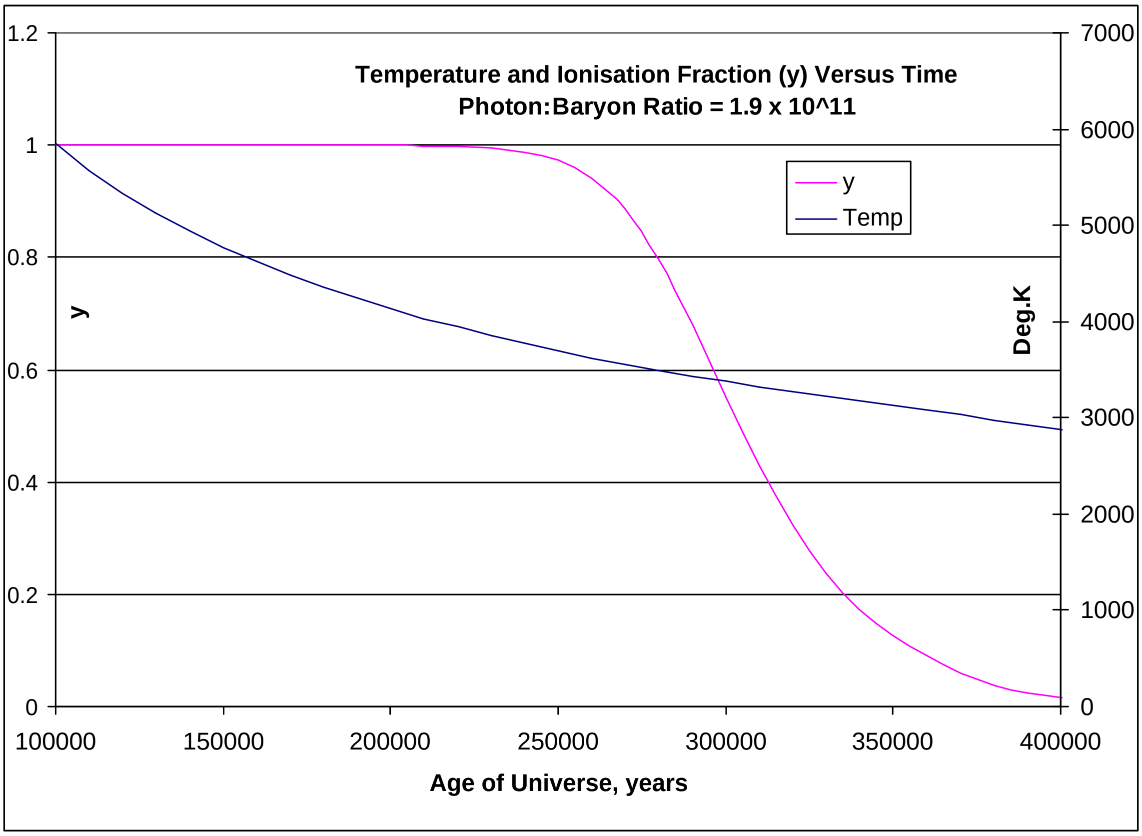

Temperature and Ionisation Fraction (y) Versus Time
Photon:Baryon Ratio = 1.9 x 10^5
y
Temp
y
Deg.K
1.2
1
0.8
0.6
0.4
0.2
0
7000
6000
5000
4000
3000
2000
1000
0
100000
150000
200000
250000
300000
350000
400000
Age of Universe, years

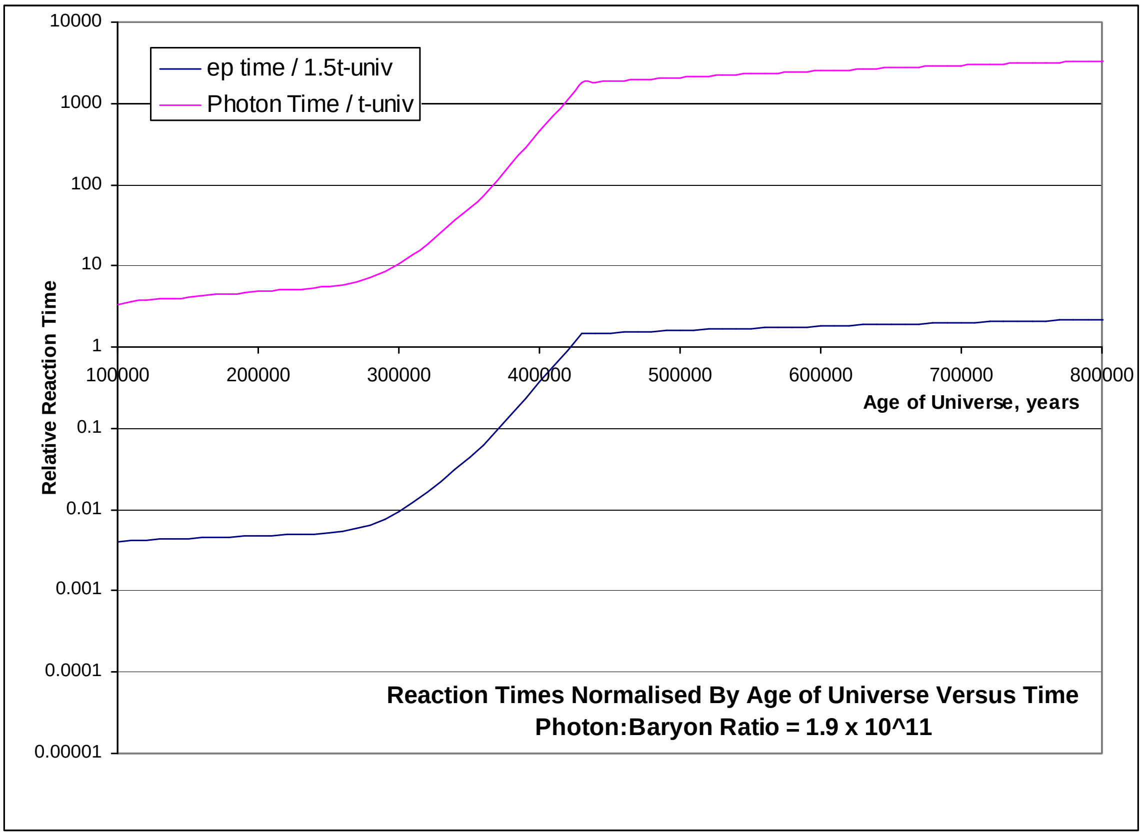

ep time / 1.5t-univ
Photon Time / t-univ
Relative Reaction Time
10000
1000
100
10
1
0.1
0.01
0.001
0.0001
0.00001
100000
200000
300000
400000
500000
600000
700000
800000
Age of Universe, years
Reaction Times Normalised By Age of Universe Versus Time
Photon:Baryon Ratio = 1.9 x 10^5

Increased photon:baryon ratio:-

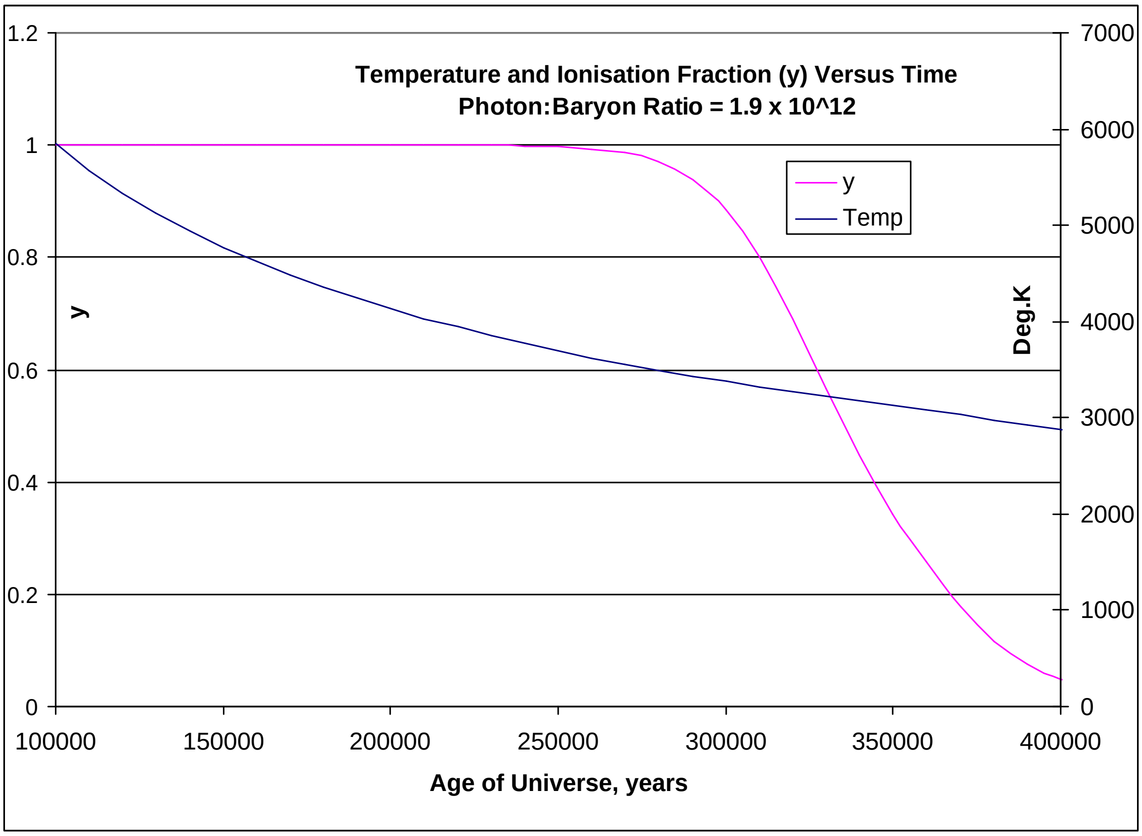


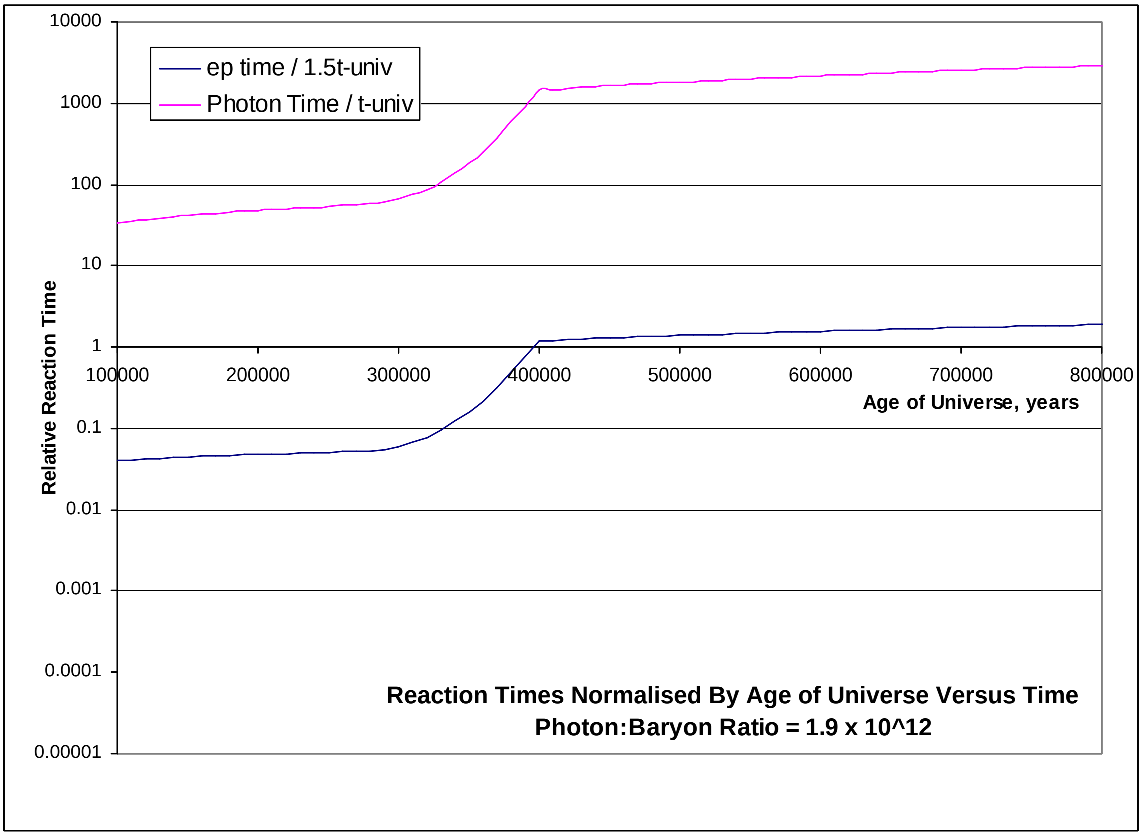


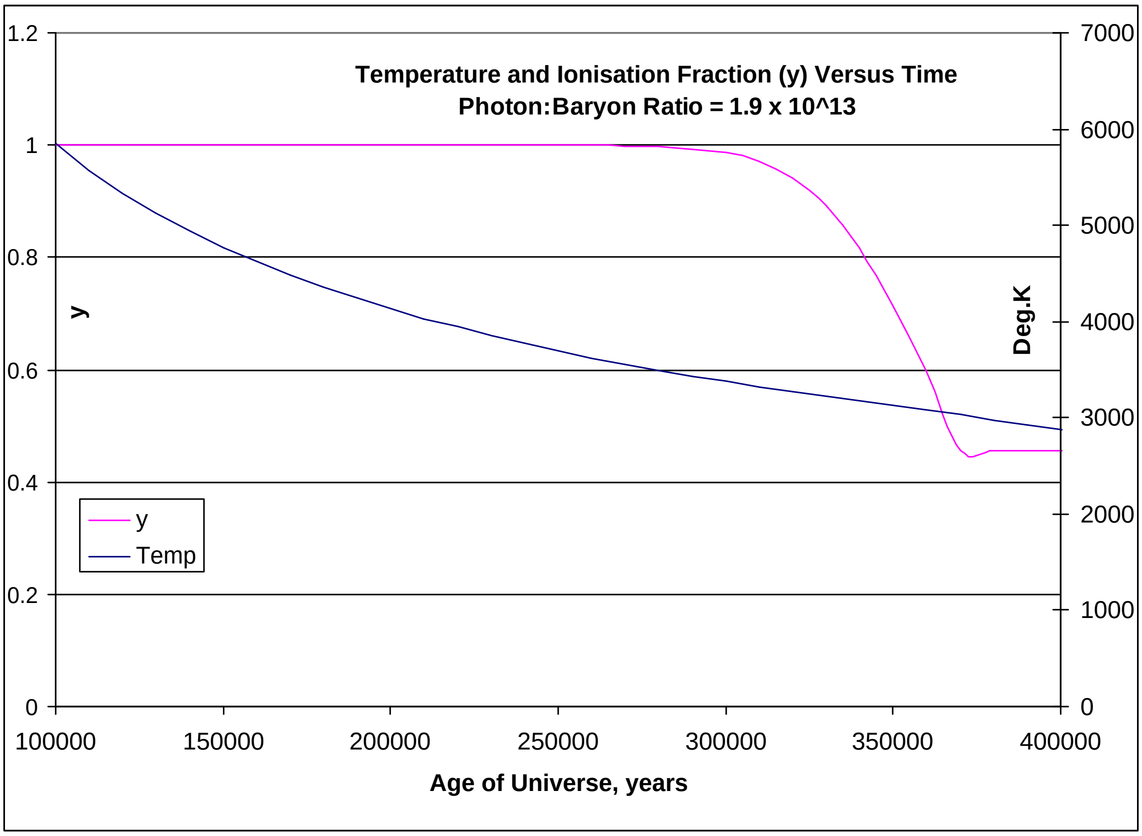


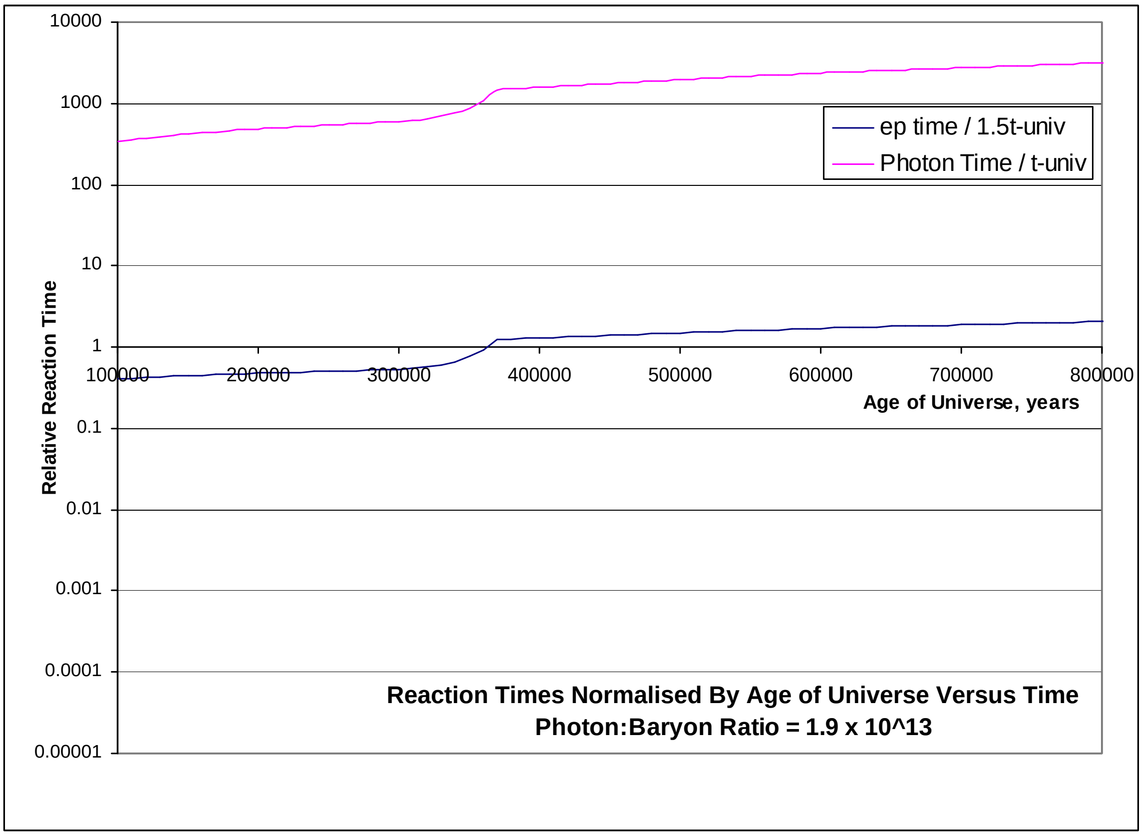


In this, and subsequent, cases the universe is always transparent (too little matter to make it opaque at any time).


Temperature and Ionisation Fraction (y) Versus Time
Photon:Baryon Ratio = 1.9 x 10^12
y
Temp
y
Deg.K
Age of Universe, years
1.2
1
0.8
0.6
0.4
0.2
0
7000
6000
5000
4000
3000
2000
1000
0
100000
150000
200000
250000
300000
350000
400000


ep time / 1.5t-univ
Photon Time / t-univ
Relative Reaction Time
Age of Universe, years
Reaction Times Normalised By Age of Universe Versus Time
Photon:Baryon Ratio = 1.9 x 10^12
10000
1000
100
10
1
0.1
0.01
0.001
0.0001
0.00001
100000
200000
300000
400000
500000
600000
700000
800000

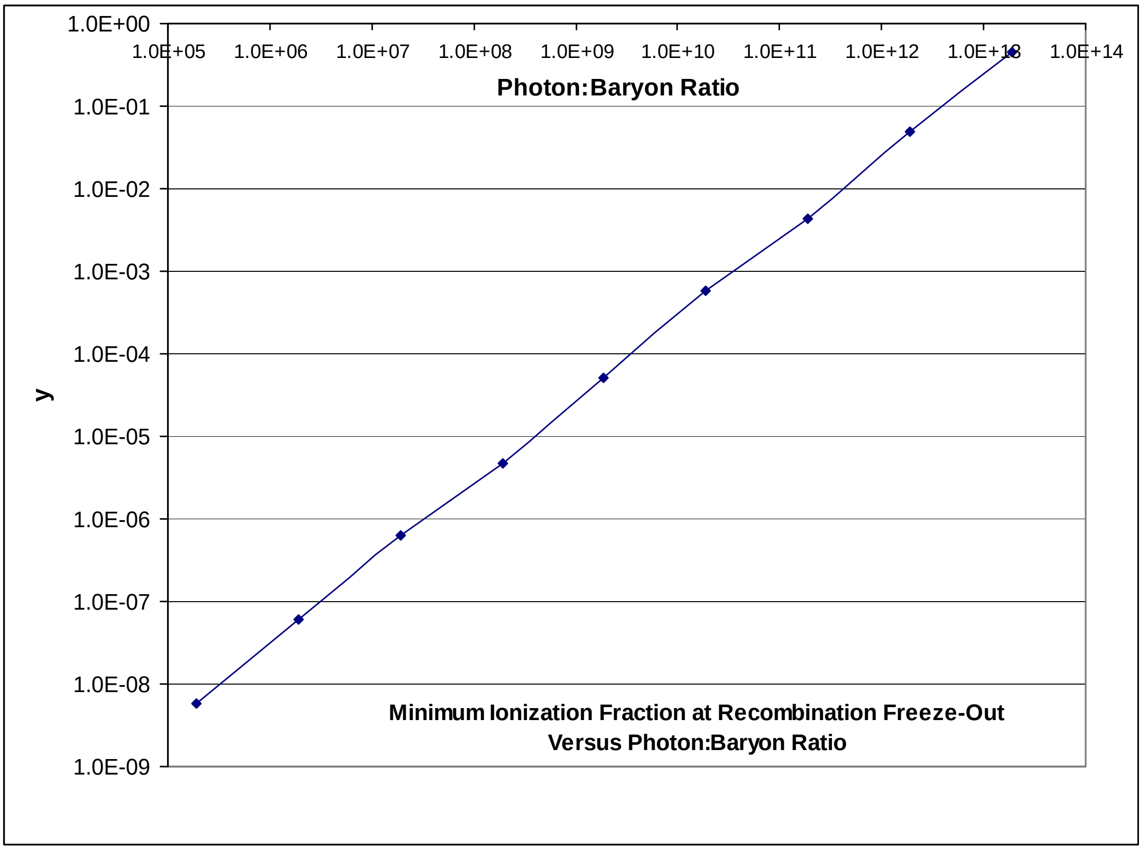

Temperature and Ionisation Fraction (y) Versus Time
Photon:Baryon Ratio = 1.9 x 10^13
y
Deg.K
y
Temp
Age of Universe, years
1.2
1
0.8
0.6
0.4
0.2
0
7000
6000
5000
4000
3000
2000
1000
0
100000
150000
200000
250000
300000
350000
400000


Relative Reaction Time
ep time / 1.5t-univ
Photon Time / t-univ
Age of Universe, years
Reaction Times Normalised By Age of Universe Versus Time
Photon:Baryon Ratio = 1.9 x 10^13
10000
1000
100
10
1
0.1
0.01
0.001
0.0001
0.00001
100000
200000
300000
400000
500000
600000
700000
800000

The minimum ionization fraction at recombination freeze-out (y) is found to be almost exactly proportional to the assumed photon:baryon ratio. Numerically,

Epsilon	ymin
1.90E+13	0.455863
1.90E+12	0.049403
1.90E+11	0.004278
1.90E+10	0.000572
1.90E+09	5.07E-05
1.90E+08	4.61E-06
1.90E+07	6.43E-07
1.90E+06	6.07E-08
1.90E+05	5.85E-09

Thus, ymin changes by very nearly a factor of 10 each time the photon:baryon ratio changes by a factor of 10. In graphical form,



(A straight line on this log-log plot implies a power law relationship. But the gradient is unity so the power is unity, i.e. they are proportional).