Chapter 7 – Details of Electron/Positron Annihilation Time

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1. Introduction

In Chapters 3 and 4 we have seen that the temperature before e^+/e^- annihilation is given by 0.73 x 10^{10} /√t(sec), and afterwards by 1.021 x 10^{10} /√t(sec). Using the former expression, the time at which kT equals mc^2 for the electron mass (0.511 MeV) is 1.52 sec. However, we cannot expect all the positrons to disappear at this time. Rather, the density of positrons will be that for thermal equilibrium at the prevailing temperature. However, we recall the discussion in Chapter 6 regarding the neutron density and the role of the nucleon-lepton reactions. Thus, the positron abundance will correspond to thermal equilibrium unless the reaction which provides the mechanism for establishing the thermal equilibrium, i.e. $e^+ + e^- \Leftrightarrow \gamma + \gamma$, becomes 'frozen out' due to the reaction rate falling below the universe's expansion rate. Thermal equilibrium and the possibility of $e^+ + e^- \Leftrightarrow \gamma + \gamma$ freeze-out are examined in turn below.

2. Thermal Equilibrium Density of Positrons

In general, whether the particles are relativistic or non-relativistic, or in between, the black body number density of Fermions with wavenumbers between k and k+dk is,

$$d\rho^N = \frac{N_s N_a}{2\pi^2} \cdot \frac{k^2 dk}{1 + e^{E_k / k_B T}} \tag{1}$$

(see the General Physics section of the web site for a derivation) . In (1), k_B is Boltzmann's constant, k is the wavenumber, and E_k is the energy of an orbital with wavenumber k. Thus, generally,

$$E_k = \sqrt{(\hbar c k)^2 + (mc^2)^2} \tag{2}$$

Substituting (2) into (1) gives,

$$d\rho^N = \frac{N_s N_a}{2\pi^2 (\hbar c)^3} \cdot \frac{\sqrt{E^2 - (mc^2)^2}\, EdE}{1 + e^{E / k_B T}} \tag{3}$$

We confine attention to the positrons ($N_a = 1$) but include both spin states ($N_s = 2$). The number density of positrons, integrating over all energies, is,

$$\rho^N = \frac{1}{\pi^2 (\hbar c)^3} \cdot \int_{mc^2}^{\infty} \frac{\left(\sqrt{E^2 - (mc^2)^2}\right) EdE}{1 + e^{E / k_B T}} = \frac{1}{\pi^2} \left(\frac{k_B T}{\hbar c}\right)^3 \int_{x_1}^{\infty} \frac{\left(\sqrt{x^2 - x_1^2}\right) xdx}{1 + e^x} \tag{4}$$

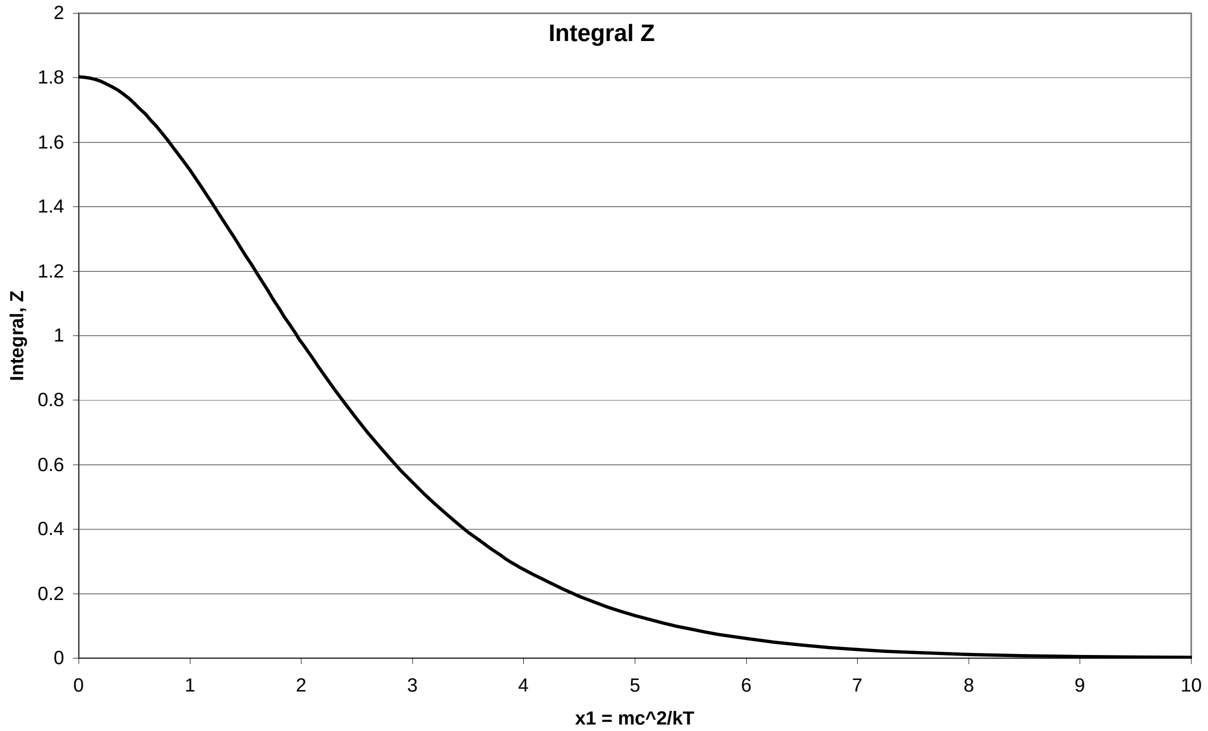
where, $x_1 = mc^2 / k_B T$. In the extreme relativistic case, x_1 is replaced by zero. The integral in (4) it that case is,

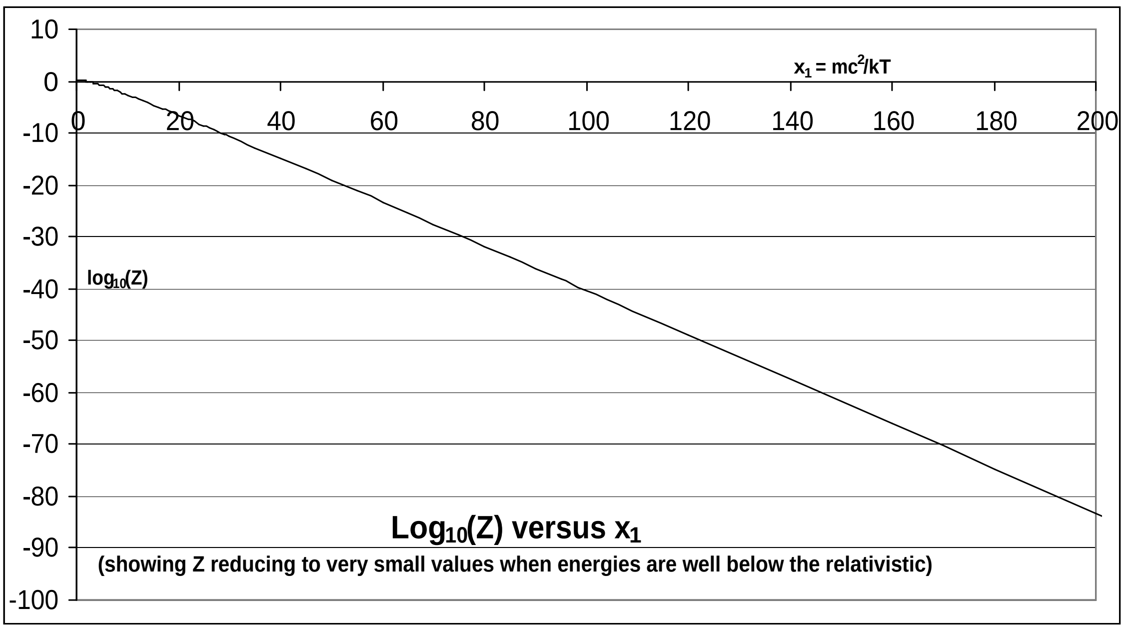
$$\int_0^\infty \frac{x^2 dx}{1+e^x} = \left(1-2^{-2}\right)\Gamma(3)\varsigma(3) = 0.75 \text{ x } 2 \text{ x } 1.202056 = 1.803084 \quad (5)$$

Generally, the integral in (4), i.e.,

$$Z(x_1) = \int_{x_1}^\infty \frac{\left(\sqrt{x^2 - x_1^2}\right) x dx}{1+e^x} \quad (6)$$

may be evaluated numerically, as given in the graphs below.





The number of photons at temperature T is,

$$\rho^N = \frac{2.404}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \qquad (7)$$

and the density of baryons is just 1 / 1.9 x 10^9 times the photon density (see Chapter 4). We may pick out the following salient times:-

t = 1.52 sec (T ~ 5.9 x 10^9 K, using 'before annihilation' time-temperature relation):
x_1 = 1.0: Z(1)= 1.51, i.e. reduction in positron density due to the positron having a finite mass is the factor Z(1)/Z(0) = 1.51/1.80 = 84%. There are still of the order of ~10^9 times more positrons than nucleons.

t = 3 sec (T ~ 4.2 x 10^9 K, using 'before annihilation' time-temperature relation):
x_1 = 1.4: Z(1.4)= 1.30, i.e. reduction in positron density due to the positron having a finite mass is the factor Z(1.4)/Z(0) = 1.30/1.80 = 72%. There are still of the order of ~10^9 times more positrons than nucleons.

t = 10 sec (T ~ 2.3 x 10^9 K, using 'before annihilation' time-temperature relation):
x_1 = 2.6: Z(2.6)= 0.70, i.e. reduction in positron density due to the positron having a finite mass is the factor Z(2.6)/Z(0) = 0.70/1.80 = 39%, but since this suggests that more than half the positrons have annihilated…

t = 10 sec (T ~ 3.2 x 10^9 K, using 'after annihilation' time-temperature relation):
x_1 = 1.8: Z(1.8)= 1.09, i.e. reduction in positron density due to the positron having a finite mass is the factor Z(1.8)/Z(0) = 1.09/1.80 = 60%. Averaging this and the previous estimate, probably ~50% of the positrons remain compared with the number if they had zero mass.

t = 15 sec (T ~ 2.6 x 10^9 K, using 'after annihilation' time-temperature relation):
x_1 = 2.25: Z(2.25)= 0.85, i.e. reduction in positron density due to the positron having a finite mass is the factor Z(2.25)/Z(0) = 0.85/1.80 = 47%, or a bit less. Hence, if the "positron annihilation time" is defined as the time at which there are only about half the number that would occur if they had zero mass, then this time is around 10 to 15 sec. Note, though, that there are still 0.85 positrons for every 2.404 / 1.9 x 10^9 baryons, i.e. a ratio of 7 x 10^8 times more positrons.

t = 63 sec (T ~ 1.3 x 10^9 K, using 'after annihilation' time-temperature relation):
x_1 = 4.6: Z(4.6)= 0.18, i.e. reduction in positron density due to the positron having a finite mass is the factor Z(4.6)/Z(0) = 0.18/1.80 = 10%. Hence, after about 1 minute there are about 10% of the positrons that there would be if they had zero mass. Note, though, that there are still 0.18 positrons for every 2.404 / 1.9 x 10^9 baryons, i.e. a ratio of ~10^8 times more positrons.

t = 167 sec (T ~ 8 x 10^8 K, using 'after annihilation' time-temperature relation):
x_1 = 7.5: Z(7.5)= 0.018, i.e. reduction in positron density due to the positron having a finite mass is the factor Z(7.5)/Z(0) = 0.18/1.80 = 1%. Hence, after about 3 minutes there are about ~1% of the positrons that there would be if they had zero mass. Note,

though, that there are still 0.018 positrons for every 2.404 / 1.9 x 10^9 baryons, i.e. a ratio of ~10^7 times more positrons.

t = 2000 sec (T ~ 2.3 x 10^8 K, using 'after annihilation' time-temperature relation):
$x_1 = 26$: Z(26)= 9 x 10^{-10}, i.e. reduction in positron density due to the positron having a finite mass is the factor Z(26)/Z(0) = 9 x 10^{-10}/1.80 = 5 x 10^{-10}. At this time there are just Z(26) = 9 x 10^{-10} positrons for every 2.404 / 1.9 x 10^9 = 13 x 10^{-10} baryons, i.e. the number of positrons is now about equal to the number of baryons. Note that this is well beyond the time usually taken as the positron annihilation time (~14 sec).

Hence, even by the end of the big bang nucleosynthesis period (3 – 30 minutes), there are still about the same number of positrons around as baryons. This also implies that there are about double the number of electrons in existence than will be the case later. The existence of these positrons and excess electrons would appear to have no implications for nucleosynthesis. This is because the nucleon/lepton reactions $n + \bar{e} \Leftrightarrow p + \bar{v}_e$, $n + v_e \Leftrightarrow p + e$ have been 'frozen out' by the universe's expansion since ~1 second in any case.

3. Does The Annihilation Reaction $e^+ + e^- \Leftrightarrow \gamma + \gamma$ Get Frozen Out?

We employ the same considerations here as we did in Chapter 6 when discussing the freeze-out of the nucleon-lepton reactions. Hence, we need to calculate the reaction rate for $e^+ + e^- \Leftrightarrow \gamma + \gamma$ per particle, and compare this to the universe's strain rate (H). Provided that a thermal equilibrium exists at all times (as assumed in Section 2) the rates of the forward and backward reactions will be the same. Evaluation of the rate of the forward (annihilation) reaction, $e^+ + e^- \to \gamma + \gamma$, requires the current number density of the electrons and positrons (since the reaction rate per particle is $\rho c \sigma$). The e^+/e^- number density is reducing rapidly, and the density derived in Section 2 would be required for this calculation. An alternative is to consider the reverse (pair production) reaction $\gamma + \gamma \to e^+ + e^-$. In this case the required number density is that of the photons, which is always given by the black body equations at the prevailing temperature. This reaction can be in balance against the forward reaction by virtue of its cross-section being reduced by a phase space factor.

Bjorken & Drell, Section 7.8, give the forward (annihilation) reaction cross section, for unpolarised particles, in the laboratory system (i.e. for a stationary electron target) to be:-

1) Non-Relativistic Limit:
$$\sigma_{lab}^{NR} = \frac{\pi\alpha^2}{m^2}\left(\frac{E}{p}\right) \qquad (8)$$

2) Extreme-Relativistic Limit:
$$\sigma_{lab}^{ER} = \frac{\pi\alpha^2}{mE}\left[\log\left(\frac{2E}{m}\right) - 1\right] \qquad (9)$$

where α is the electromagnetic fine-structure constant, $\alpha = e^2/\hbar c$ = 1/137, m is the electron mass, and p and E are the incoming positron's momentum and energy in the laboratory system.

If the electron and positron energy and momenta in the centre-of-mass (CoM) system are E* and +/-p*, we can readily derive,

$$E^* = \sqrt{\frac{m(E+m)}{2}}\ ,\ p^* = \sqrt{\frac{m(E-m)}{2}}\ ,\ E = \frac{2E^{*2} - m^2}{m}\ ,\ p = \frac{2E^*}{m}\sqrt{E^{*2} - m^2} \qquad (10)$$

The phase-space suppression factor for the reverse (pair production) reaction can be obtained from Perl, Equ.6-39, or Mandl & Shaw, Equ.8.18, i.e., for any 2-body-to-2-body reaction, and in the CoM system,

$$\sigma_{CoM}(a + b \to c + d) = \frac{1}{64\pi^2 s}.\frac{p_f^*}{p_i^*}\int |T_{if}|^2 d\Omega \qquad (11)$$

where, $p_i^* = |\bar{p}_a^*| = |\bar{p}_b^*|$ is the momentum of the incoming particles in the CoM system, and $p_f^* = |\bar{p}_c^*| = |\bar{p}_d^*|$ is the momentum of the produced particles in the CoM system, and $s = (E_a^* + E_b^*)^2 = (E_c^* + E_d^*)^2$ in the CoM system, and T is the matrix element given by the relevant Feyman diagram (modulo possible normalisation factors involving the particle masses and normalisation volumes only). We note that writing Equ.11 for the reverse reaction, the value of s is unchanged and the (modulus of the) matrix element is also the same. The only change is that the ratio of the final and initial momenta is inverted. Hence, we have,

$$\sigma_{CoM}(\gamma + \gamma \to e^+ + e^-) = \left(\frac{p_e^*}{p_\gamma^*}\right)^2 \sigma_{CoM}(e^+ + e^- \to \gamma + \gamma) \qquad (12)$$

Hence, as must be the case, the pair production reaction is switched off when the momentum of the produced electron (or positron) reduces to zero, i.e. when the energy per photon reduces to m. Noting that $p_\gamma^* = E_\gamma^* = E_e^*$ in the CoM system, we have, using Equs.8, 9, 10, 12, for the pair production reaction, $\gamma + \gamma \to e^+ + e^-$,

1) Non-Relativistic Limit:
$$\sigma_{CoM}^{NR} = \frac{\pi\alpha^2}{2m^2}\left(\frac{p_e^*}{m}\right) \qquad (13)$$

2) Extreme-Relativistic Limit:
$$\sigma_{CoM}^{ER} = \frac{\pi\alpha^2}{m^2}\left(\frac{m}{E_e^*}\right)^2\left[\log\left(\frac{2E_e^*}{m}\right) - \frac{1}{2}\right] \qquad (14)$$

Note that, in both limits, the cross section is much less than $\frac{\pi\alpha^2}{2m^2}$ since, for the non-relativistic case p/m << 1, whereas for the extreme relativistic case, m/E << 1. The maximum value is found approximately as follows,

p/m	$2\left(\frac{m}{E_e^*}\right)^2\left[\log\left(\frac{2E_e^*}{m}\right)-\frac{1}{2}\right]$
0.1	0.39
0.3	0.43
0.5	0.49
0.6	**0.51**
0.8	**0.54**
1.0	**0.54**
1.5	**0.48**
2.0	**0.40**
10.0	**0.05**

The figures in bold indicate where the non-relativistic approximation gives way to the relativistic one (albeit an approximate extrapolation of both). The figures are what $\frac{\pi\alpha^2}{2m^2}$ must be multiplied by to give the pair production cross-section. Hence, the maximum value of this cross section is,

$$\text{Maximum of } \sigma_{CoM}^{NR}(\gamma+\gamma\to e^+ + e^-) \approx \frac{\pi\alpha^2}{4m^2} = 6.4\,\text{x}\,10^{-30}\ m^2 \qquad (15)$$

[Noting that we are taking particle physicists' liberties with units again in Equ.(15), and that 1 MeV^{-1} corresponds to a length of ~2 x 10^{-13} m]. We shall use the cross-section given by (15) for now and comment on the significance of using a maximum value later.

The reaction rate, i.e. the number of pair productions per photon per second, is,

$$\text{Reaction rate} = \frac{1}{T_I} = \rho_\gamma c\sigma_{CoM} \qquad (16)$$

and the density of photons is,

$$\rho_\gamma = 0.2436\left(\frac{k_B T}{\hbar c}\right)^3 \qquad (17)$$

Hence, putting Equs.15, 16, 17 together,

$$\text{Reaction rate} = \frac{1}{T_I} \le 3.89\,x\,10^{-14}\,T^3 \text{ per second} \qquad (18)$$

This is an upper bound to the reaction rate because we have used the maximum cross section at any energy. This occurs for p ~ m/2. After the first few seconds the majority of electrons and positrons will have less energy than this. Since the cross section, and hence the reaction rate, is proportional to the momentum of the electrons (positrons), we can improve the estimate of the reaction rate by scaling by √E or √T, or equivalently by $t^{1/4}$ after, say, 15 seconds. Thus, our rough estimate is,

T < 14 sec Reaction rate = $\frac{1}{T_I} = 3.89\,x\,10^{-14}\ \mathrm{T}^3$ per second (18a)

T > 14 sec Reaction rate = $\frac{1}{T_I} = 3.89\,x\,10^{-14}\left(\frac{1}{\mathrm{t}}\right)^{1/4}\mathrm{T}^3$ per second (18b)

This reaction rate is to be compared with the universe's expansion rate, i.e. H = 1/2t. We use the previously derived temperature-time relations (see Chapters 3 to 5), i.e., T = 1.021 x 10^{10}/√t after e^+/e^- annihilation, and T = 0.73 x 10^{10}/√t before. From Section 2 we take the transition time between the two as ~15 seconds, since this is when there are just 50% of the number of e^+/e^- pairs that would occur for zero mass particles. Hence, we can now draw up a Table giving the time, temperature and hence reaction rate from (18a,b) at any given time. For comparison we also include the reaction rate for the nucleon/lepton reactions derived in Chapter 6:-

t (sec)	**T (K)**	**H (1/sec) = 2/3t**	**Reaction rate for** $\gamma + \gamma \rightarrow e^+ + e^-$ **(1)**	**Reaction rate for** $n + \overline{\nu} \rightarrow p^+ + e^-$ **(2)**
0.1	2.3 x 10^{10}	5	~5 x 10^{17}	35.6
0.25	1.5 x 10^{10}	2	~1 x 10^{17}	4.2
0.6	9.4 x 10^{9}	0.83	~3 x 10^{16}	0.4
1	7.3 x 10^{9}	0.5	~1.5 x 10^{16}	0.12
3	4.2 x 10^{9}	0.166	~3 x 10^{15}	0.007
15	1.9 x 10^{9}	0.033	~3 x 10^{14}	0.00014
15	2.6 x 10^{9}	0.033	~3 x 10^{14}	
63	1.3 x 10^{9}	0.008	~3 x 10^{13}	
167	7.9 x 10^{8}	0.003	~5 x 10^{12}	
2000	2.3 x 10^{8}	0.00025	~1 x 10^{10}	
10^{6}	1.0 x 10^{7}	0.5 x 10^{-6}	~1 x 10^{6}	
10^{13}	~3,200	0.5 x 10^{-13}	7 x 10^{-7}	

(1) Per photon per sec (2) Per neutron per sec

Hence, over the period to ~3 minutes, the pair production reaction rate exceeds the universe expansion rate by about 15 orders of magnitude. There is therefore no question of the expansion of the universe causing freeze-out of the pair production reactions over this period, despite the fact that we have used only a crude estimate for the reaction rate. Even at ~2000 seconds the reaction rate based on the maximum cross-section is 14 orders of magnitude larger than the universal expansion rate. The Table shows that the annihilation reaction is not frozen out by cosmic expansion even at about 1 million years, and beyond. This suggests that essentially no primordial positrons would be expected to survive by the present epoch (or anytime close!).

As a check, consider firstly the following argument that equilibrium conditions will indeed prevail. Consider time 2000 seconds. The number density of black body photons at this time (temperature 2.3 x 10^8K) would be ~10^{32} /m^3, and hence, although the number density of positrons at this time is x10^{-9} smaller (see Section 2), this is still a large number, ~ 10^{23} /m^3. However, the annihilation reaction rate at this time is around 10^{10} per second per positron. Consequently, despite the large absolute numbers of positrons, they are each reacting with an electron about every 10^{-10} secs. They would therefore all disappear within a very small fraction of a second if they were not being continually re-created from the photons. In other words, the positron density is due to thermal equilibrium with the photons.

Consequently, we can estimate the positron density from equilibrium considerations. As an example, at ~8 hours the temperature is ~6 x 10^7K and hence $x_1 = mc^2/k_BT$ ~ 100. Hence, from the graph in section w, the positron density compared with that of zero rest mass particles , i.e. Z / 1.80, is of order 10^{-40}. Hence, at this time the thermal equilibrium number of positrons is entirely negligible.

The Table also gives the nucleon/lepton reaction rates and confirms the finding in Chapter 6 that the reaction rates exceeds the universal expansion rate before ~0.6 seconds, at which time the nucleon/lepton reactions become frozen out. Note that there is a slight difference from Chapter 6 in that we have here used the time-temperature relation $T\sqrt{t} = 0.73 \times 10^{10}$ together with H=1/2t. In Chapter 6 we derived H from a black body density, including the electrons and positrons. The Chapter 6 formulation is equivalent to a time-temperature relation $T\sqrt{t} = 1.15 \times 10^{10}$.

4. Summary

Section 3 has confirmed that freezing out of the electron-positron pair production / annihilation reactions, due to universal expansion, does not occur. This is in stark contrast to the nucleon-lepton reactions which freeze out due to universal expansion before ~1 second. The reason for this difference may be traced to the dramatically different magnitudes of the weak and electromagnetic cross-sections. The ratio of Equ.6b in Chapter 6 to Equ.15 here, assuming photon or neutrino energies (k_BT) are of the order of 1 MeV, gives,

$$\frac{\sigma_{weak}}{\sigma_{electromagnetic}} = \frac{32}{\pi^2}.\left(\frac{10^{-5}}{\alpha}\right)^2\left(\frac{m_e}{M_p}\right)^2\left(\frac{E}{M_p}\right)^2 \approx 3\left(137x10^{-5}\right)^2\left(\frac{1}{1930}\right)^2 10^{-6} \approx 10^{-18} \qquad (19)$$

This is consistent with the above Table, which shows that the e^+/e^- annihilation reaction is indeed about 18 orders of magnitude faster than the weak leptonic reaction.

Because there is no freeze-out, the positron density is given at all times by thermal equilibrium with the photon field.

In Section 2 we have shown how to take account of the finite mass of the electrons and positrons as regards the black-body number density for the general case of particles which may be in the intermediate regime between non-relativistic and extreme relativistic.

At ~1.5 seconds the temperature is such that k_BT ~ mc^2, but the positron number density is little different from that for zero mass particles (~84%).

At ~15 seconds (which is often quoted as the electron-positron annihilation time) the positron number density is about half that which would occur for a zero mass particle.

At ~63 seconds the positron number density is about 10% of that which would occur for a zero mass particle.

At ~167 seconds the positron number density is about 1% that which would occur for a zero mass particle.

At ~2000 seconds (~30 minutes) the positron number density is comparable with that of the baryons, i.e. of the order of 1 per 10^9 photons.

After a few hours the number of positrons has become negligible.