Chapter 5B: Cosmic Geometry: General Relativity

Last Update: 14 August 2006

1. Introduction

Contrary to popular imagination, general relativity does not play a particularly large role in developing the physical evolution of the universe. We have already been able to derive, albeit rather heuristically, such key results as the variation in time of the density, temperature and size scale of the universe without any recourse to relativity theory. We did, however, have to assume the Hubble relation between R and dR/dt. How do we know that this is true? This is a question about the dynamics of the universe, which is addressed by general relativity. Also, our heuristic derivations of Chapters 1-4 rather fudged the distinction between the relativistic and non-relativistic values for energy and density. In this Chapter we set these derivations on a firmer, relativistic, footing.

However, the main benefit of general relativity is in providing a language or conceptual framework within which to understand the geometry of space-time. This is not merely a philosophical nicety but provides clear definitions of numerical quantities which might otherwise be ill-defined. Most importantly, the concept of curved space provides an answer to the question every child asks: "If the universe does not go on forever then what lies beyond it?" This question is answered very elegantly by the concept of space with positive intrinsic curvature. Such a space may be finite but with no boundary. There is no outside. On the other hand, space may actually be infinite.

In this Chapter we shall consider only cosmologies which are spatially isotropic and homogeneous. This is traditional, and appears to be a good description of the universe on sufficiently large scales[1]. Ironically, many of the hard won lessons of special relativity are disrespected in these cosmological solutions. For one thing, there *is* a preferred time co-ordinate – cosmic time. There are also preferred spatial co-ordinate systems – namely systems which are co-moving with respect to the 'Hubble flow'. Finally, and most disturbingly, it is possible for the distance between two material objects to increase faster than the speed of light. This causes some conceptual difficulties which we will try to dispel in this Chapter. The irony is that these conceptual difficulties would not arise if we had not learnt our special relativity so well!

The topics dealt with in this Chapter plus Chapter 5C are listed below,

- Newtonian Cosmologies (§2);
- The formulation of general relativity (§3);
- Cosmological models (Friedman-Robertson-Walker space-times) (§4);
- The surprising observation that Newtonian cosmologies encompass the general relativistic cosmologies provided that pressure and the cosmological constant are zero (§3 & §5);
- Precisely what is meant by an "expanding universe" (§4 + Ch.5C);

[1] i.e. larger than about 100 Mps, the size scale of galactic superclusters. There have been suggestions that the CMB shows indications of a preferred direction (the "axis of evil"), which might indicate universal rotation.

- Precisely what is meant by an infinite universe, and why an infinite universe would always have been infinite, even at the Big Bang (§4.3);
- What is meant by the curvature of space and how it is related to whether the universe is finite or infinite (§4);
- Explanation of why I keep referring to the "size scale" of the universe, rather than simply "the size of the universe" (§4);
- What is the "Cosmological Constant" and how does it affect the possible cosmological models (§3 & §5);
- How the relationship between the age of the universe and the Hubble time depends upon the cosmological model (§6);
- The current consensus model (based on observational data) (§7);
- The cosmology of a universe filled with a pressurised fluid – how pressure causes gravity (Ch.5C);
- Demonstration that an expanding universe leads to a redshift, and that this redshift obeys Hubble's Law, at least to first order? or is it exact? (Ch.5C);
- The definitions of several different length measures and derivation of their relation to the redshift for different cosmological models (Ch.5C);
- In particular, the definition of the length scale in Hubble's Law v = HR which permits the "velocity" v = dR/dt to exceed c (Ch.5C);
- The size of the observable universe – and why this is greater than ct, but depends upon the cosmological model (Ch.5C);
- Why the horizon encompasses a decreasing amount of matter as we look further back in time – the Horizon Problem (Ch.5C);
- The topology of the universe – how a non-trivial topology could solve the "finite but unbounded" problem but with zero curvature (Ch.5C).

2. Newtonian Cosmologies

The Newtonian view was of a Euclidean space. Space itself was conceived of as infinite, though this might not necessarily mean that matter extended to infinity also. We shall see later that one of the starting points of general relativistic cosmologies is the assumption that the universe is (and always was) spatially isotropic and homogeneous (noting that isotropy everywhere implies homogeneity). Of course, this can only apply on a sufficiently large scale, large enough to average out the lumpiness cased by stars and galaxies and clusters of galaxies. Applying this "Cosmological Principle" to the Newtonian view of things necessarily implies that the matter density would be, in the large, uniform and extend to infinity. The reason is that any finite distribution of matter density would violate homogeneity because points near the edge of the matter distribution would be distinguished as such. This would appear to have been Newton's view of the universe. In his opinion it had an additional benefit. Newton recognised that if matter occupied only a finite region in an infinite Euclidean space then gravity would tend to cause the universe to collapse inwards towards the universe's centre of gravity[2]. Newton's chosen solution appears (Ref?) to have been the assumption that matter extended uniformly to infinity, thus providing no centre

[2] Why did Newton not imagine that all material would orbit about the universe's c.g.? Given his solution to the problem of why the planets do not fall towards the sun under gravity, this would surely have seemed the obvious solution to him. It would involve orbital (tangential) velocities increasing linearly with distance r from the c.g. – in other words, rigid body rotation. An overall rotation of the universe could be avoided by having half the material orbit in the opposite direction. Newton was free to imagine such things since, unlike us, he had not the observational evidence to contradict it.

about which to contract. This would, of course, require the matter density to be uniform to very high precision. One is inclined to think that this would be a very unstable state of affairs. But small fluctuations in the original density would lead to local gravitational collapse, presumably analogous to structure formation in the same manner as envisaged in modern cosmology.

In this Section we consider a Newtonian cosmology. But we relax the usual requirement for strict isotropy and homogeneity. It is replaced with the weaker condition that the matter density be uniform within a sphere of radius R(t), which might vary with time, and that the density for r > R be zero. This is isotropic only about the centre of the sphere, not elsewhere. Note that we are making the assumption that the density is uniform at all times, albeit varying in time and only within a time-varying radius. The space in question is, of course, infinite and Euclidean, although the matter within this space is of finite extent. In this sense the model may be described as "Newtonian" despite not being Newton's!

To derive the motion of a particle of mass m at radius r we appeal to the fact that the gravity inside a uniform spherical shell is zero, and hence only the matter at radii < r contributes to the motion of m. The gravity due to the matter at radii less than r is the same as if all the mass were concentrated at the centre, hence,

$$m\ddot{r} = -\frac{Gm}{r^2}\cdot\frac{4\pi}{3}\rho r^3 \Rightarrow \ddot{r} = -\frac{4}{3}\pi G\rho r \qquad (2.1)$$

But, the assumption that density is always uniform can be written,

$$\rho(t)R^3(t) = \rho_0 R_o^3 = \text{constant} \qquad (2.2)$$

where the subscript $_0$ denotes some arbitrary time. Moreover, if we write the radial location of the mass m at this time as r_0, it is clear that the whole mass distribution is expanding/contracting so that $r / r_0 = R / R_0$. Hence,

$$\rho(t)r^3(t) = \rho_0 r_o^3 = \text{constant} \qquad (2.3)$$

Now, since $\ddot{r} \equiv \dot{r}d\dot{r}/dr$ we can integrate (2.1) explicitly to give,

(2.4):-

$$\int \dot{r}d\dot{r} = -\frac{4\pi}{3}G\int\frac{\rho_0 r_0^3}{r^2}dr \Rightarrow \frac{\dot{r}^2}{2} = \frac{4\pi}{3}G\frac{\rho_0 r_0^3}{r} + \text{constant} \Rightarrow \dot{r}^2 = \frac{8\pi}{3}G\frac{\rho_0 r_0^3}{r} - kc^2 = \frac{8\pi}{3}G\rho r^2 - kc^2$$

where we have written the constant of integration as $-kc^2$ for reasons which will be clear later. We note that if the constant of integration is zero, (2.4) is just a Hubble relation with $H^2 = \frac{8\pi}{3}G\rho$ in agreement with Chapters 1 and 2 for the case of a universe of critical density, i.e. flat. A crucial assumption in (2.1-4) is that the matter of this universe moves only along the radial line, but not tangentially. In particular this means that the universe has been assumed not to be rotating. If it were, the gravitational force may be partially, or completely, balanced by the centrifugal force.

2.1 Solutions of the Newtonian Cosmology Equation, (2.4)
The integration constant, k, is unknown. It would require a boundary condition to be given to determine k (such as specifying the velocity, dr/dt, at a given radius, r, and at a given density, ρ). We note that the qualitative behaviour of this universe depends only upon whether k is zero, negative or positive – rather than on its absolute magnitude. This is because, for non-zero k we can re-scale length thus,

$$r \to s = \frac{r}{c\sqrt{|k|}} \qquad (2.5)$$

(k is dimensionless, so s has the dimension of time). So that Equ.(2.4) becomes,

$$\text{for } k > 0: \ \dot{s}^2 = \frac{8\pi}{3}G\rho s^2 - 1; \qquad \text{for } k < 0: \ \dot{s}^2 = \frac{8\pi}{3}G\rho s^2 + 1 \qquad (2.6)$$

independent of k. It suffices therefore to consider k = 0, k < 0 or k > 0, as follows:-

<u>**(A) k = 0**</u>: Equ.(2.4) becomes,

$$\dot{r} = A/\sqrt{r}\text{, where } A = \sqrt{8\pi G\rho_0 r_0^3/3} = \sqrt{2GM_{univ}} \qquad (2.7)$$

Integration of which yields,

$$r = \left[\frac{3}{2}At + \text{constant}\right]^{2/3} \qquad (2.8)$$

Whatever the value of the integration constant there is some time at which the […] in (2.8) must be zero. We can shift the origin of time to that time, giving, without loss of generality,

$$r = \tilde{A}t^{2/3} \qquad \text{where, } \tilde{A} = \left(6\pi G\rho_0 r_0^3\right)^{1/3} \qquad (2.9)$$

So, immediately with our first (and Newtonian!) cosmology we see that there is a predicted state of infinite density (finite mass within zero radius) at a finite time in the past. In other words, there is a Big Bang. The universe then expands outwards monotonically, but with decreasing velocity ($v = dr/dt \propto 1/t^{1/3}$). The initial velocity, at the Big Bang, is infinite. The result $r \propto t^{2/3}$ agrees with Chapter 2 for the case of a matter dominated universe (as in this Newtonian model).

[Aside: We can also deduce the result that $r \propto \sqrt{t}$ for a radiation dominated universe. To do so we assume the density is due to a fixed number of photons, each of whose energy is proportional to 1/R. The density is then proportional to $1/r^4$, and hence the RHS of (2.4) is proportional to $1/r^2$. This integrates to $r \propto \sqrt{t}$, as required].

<u>(B) k < 0</u>: Equ.(2.4) becomes,

$$\dot{r} = \sqrt{\frac{A^2}{r} + |k|c^2} \tag{2.10a}$$

where A is given in (2.7). The qualitative behaviour of r(t) is clear without explicit integration. The RHS is positive definite (since r, being a radial co-ordinate, is ≥ 0). hence, r increases monotonically. For sufficiently large r the velocity becomes constant ($|k|c$), and hence r increases linearly with time. At smaller r the velocity is larger. Hence the curve of r versus t is one of monotonically reducing velocity, i.e., the curve is convex from above. It is clear that the curve must intersect the t-axis at a finite time. In other words, the radius, r, must become zero at some finite time in the past. Once again there is a Big Bang! Once again the velocity is divergent at the Big Bang, and the universe then expands monotonically with decreasing velocity (in this case asymptotic to $|k|c$).

Note that, near the Big Bang (r ~ 0), (2.10a) approximates to (2.7), so that (2.9) is the approximate solution, i.e. the behaviour for small times is $r \propto t^{2/3}$.

Note that we have assumed the positive root in (2.10), but the negative root is also possible, i.e.,

$$\dot{r} = -\sqrt{\frac{A^2}{r} + |k|c^2} \tag{2.10b}$$

This gives the same picture but with time reversed, i.e. a Big Crunch.

<u>(C) k > 0</u>: Equ.(2.4) becomes,

$$\dot{r} = \pm\sqrt{\frac{A^2}{r} - |k|c^2} \tag{2.11}$$

In this case there exists a radius, $r_{max} = A^2/|k|c^2$, at which the velocity becomes zero. Only radii smaller than r_{max} are permitted since larger radii would have imaginary velocities. Hence, r_{max} is indeed a maximum possible radius. Thus, for times just after achieving the maximum radius we must assume the minus root in (2.11), to ensure that the radius then *decreases* below r_{max}. Similarly, for times just before r_{max}, we must assume the positive root in (2.11) for the same reason. Hence, considering the curve of r versus t, and going backwards in time from r_{max}, the velocity is positive and steadily increasing as we consider reducing t. Thus, once again the curve of r versus t is convex from above and must intersect the t-axis at a finite time. Once again there is a Big Bang at a finite time in the past. The velocity is again divergent at the Big Bang. The universe then expands with decreasing velocity. Unlike the previous cases, the velocity decreases to zero in a finite time and then reverses sign, so that the matter filled region of the universe contracts back to zero size in a finite time. This is a universe with a finite lifespan.

Note that, near the Big Bang (r ~ 0), (2.10b) again approximates to (2.7), so that (2.9) is again the approximate solution. Thus, the behaviour for small times is always $r \propto t^{2/3}$ whatever the value of k.

The key qualitative difference between the cases $k \le 0$ and $k > 0$ is that, for $k \le 0$ the matter-filled region expands without limit, becoming infinitely dilute as $t \to \infty$, whereas for $k > 0$ the matter-filled region is always of finite extent, does not fall below a certain minimum density, and the universe has only a finite life.

<u>Explicit Solutions for r(t)</u>

For non-zero k, we have deduced the qualitative form of the solutions to Equ.(2.4) without explicit integration. However, explicit integration is simple. If we choose some time, say the present, and denote it by the sub-script 0, then the conservation of mass can be written as,

$$\rho r^3 = \rho_0 r_0^3 \qquad (2.12)$$

and hence Equ.(2.4) can be written as,

$$\dot{r}^2 = \frac{A^2}{r} - kc^2 \qquad (2.13)$$

where,

$$A \equiv \left(\frac{8\pi}{3} G\rho_0 r_0^3\right)^{1/2} \qquad (2.14)$$

We will see below that , when there is no cosmological constant, the coefficient A can also be written as,

$$\mathbf{\underline{k = 1}}\text{:}\quad A^2 = \frac{c^3 \tau_0 \Omega_0}{(\Omega_0 - 1)^{3/2}} \qquad \mathbf{\underline{k = -1}}\text{:}\quad A^2 = \frac{c^3 \tau_0 \Omega_0}{(1 - \Omega_0)^{3/2}} \qquad (2.15)$$

where τ_0 is the reciprocal of the Hubble parameter at epoch $_0$, and Ω_0 is the matter energy density ratio, $\rho_{matter} / \rho_{critical}$, at the same time[3]. The solutions to (2.4,13) for non-zero k can be written in parametric form as,

$$\mathbf{\underline{k = 1}}\text{:}\quad r = \frac{A^2}{2c^2}(1 - \cos 2\psi), \quad t = \frac{A^2}{2c^3}(2\psi - \sin 2\psi) \qquad (2.16a)$$

$$\mathbf{\underline{k = -1}}\text{:}\quad r = \frac{A^2}{2c^2}(\cosh 2\psi - 1), \quad t = \frac{A^2}{2c^3}(\sinh 2\psi - 2\psi) \qquad (2.16b)$$

where ψ can take any real value. These solutions are illustrated graphically below, on scales with c = A = 1. See the Annex for a discussion of subtleties.

[3] This Ω includes dark matter, but excludes dark energy (see below). In (2.15) it is crucial that the strict definition of the critical density $\rho_{crit} = 3H^2 / 8\pi G$ is used, <u>not</u> an expression $\propto 1/Gt^2$.

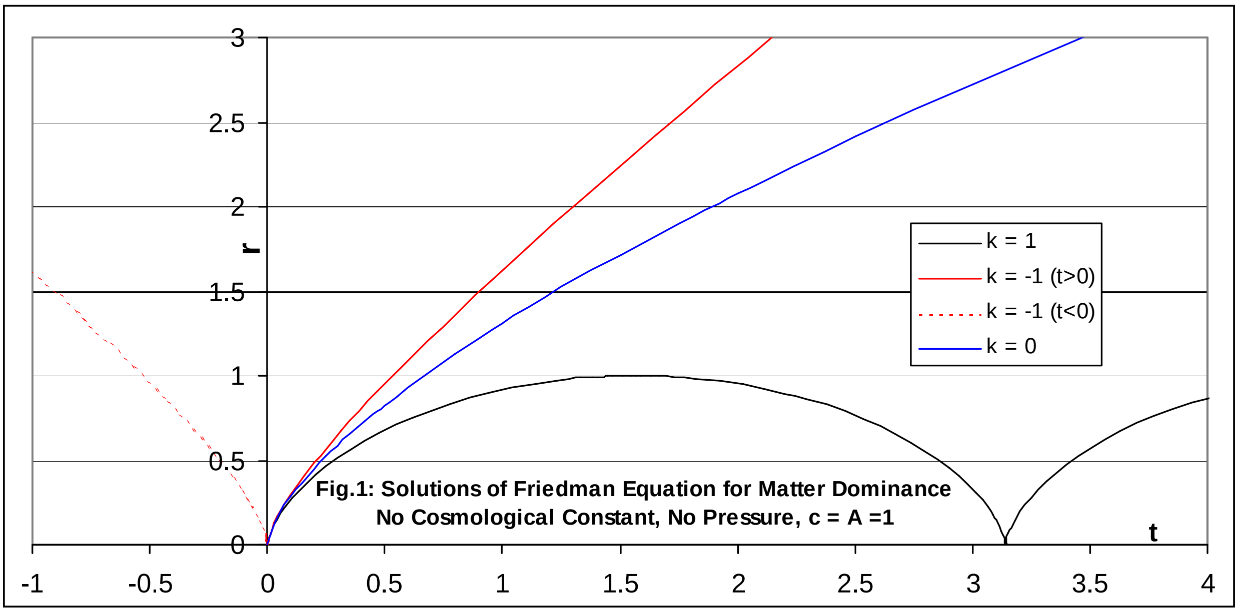


Fig.1: Solutions of Friedman Equation for Matter Dominance
No Cosmological Constant, No Pressure, c = A =1

2.2 Approximate Newtonian Treatment Of a Radiation Dominated Universe

In the case k = 0 we have alluded above to the solution for a radiation dominated universe as being $r \propto \sqrt{t}$, consistent with Chapter 2. The proper treatment of the radiation dominated case should include the effects of radiation pressure, as well as the radiation energy (mass) density. But the effects of pressure on gravitation are included only in relativistic formulations. In Newtonian physics, pressure does not act as a source of gravity. Thus, the correct handling of the radiation dominated case must await the full relativistic treatment in Section 5. Nevertheless, it is instructive to look at the non-relativistic approximation which assumes radiation dominance of the energy (mass) density, but neglecting pressure. We envisage a fixed number of photons whose individual energy will vary with the size scale 'r' proportionally as 1/r. Hence, the total energy density is proportional to $1/r^4$, in contrast to the $1/r^3$ for the matter dominated case. Specifically, we choose to write the density in terms of a constant B, where,

$$\rho_{radiation} = \frac{B^2c^2}{\frac{4\pi}{3}Gr^4} \qquad (2.17)$$

So Equ.(2.1) integrates to give, instead of (2.4),

$$\dot{r}^2 = \frac{B^2c^2}{r^2} - kc^2 \qquad (2.18)$$

The solution for k = 0 is clearly,

$$\underline{\mathbf{k = 0}}: \quad r = \sqrt{2Bt} \qquad (2.19)$$

It may be checked by substitution that the solutions for non-zero k are,

k = 1: $(ct - B)^2 + r^2 = B^2$ (2.20a)

k = -1: $(ct \pm B)^2 - r^2 = B^2$ (2.20b)

Both these universes have a Big Bang at t = 0 (i.e. r = 0 at t = 0 and subsequently r increases rapidly).

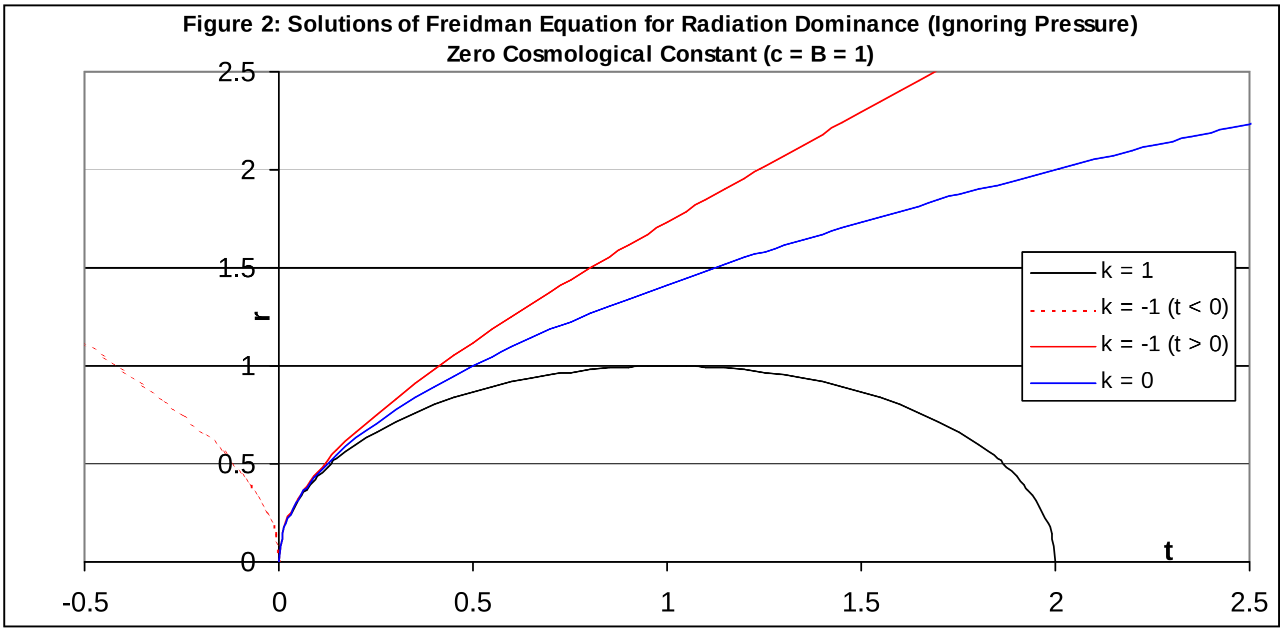
For k = 1 the graph of the radius r versus time t is a semicircle. This universe survives for a time 2B/c, at which time there is a Big Crunch (r reduces back to zero). At time B/c the universe achieves its maximum size, r = B.

Conversely, for k = -1, consider the solution with the + sign in the bracket. This represents a universe whose graph of r versus t is (half a) hyperbola. r starts at zero (the Big Bang) and then increases monotonically with t. Asymptotically, r tends to ct. If the – sign is chosen in the bracket, then r is real only for negative t. This solution is the other half of the hyperbola. This solution is a continually contracting universe which ends in a Big Crunch at datum time zero.

Note that in both cases the solutions can be extended to negative r. Interpreting a Big Bang in our universe (with r > 0) as originating from the Big Crunch of a precursor universe with r < 0 also implies that the direction of the flow of time is reversed at this 'point'.

Note that the k = 1 solution can be written $r^2 = 2Bct - c^2t^2$ and the k = -1 solution for an expanding universe can be written $r^2 = 2Bct + c^2t^2$. For early times, t << B/c, and hence both these approximate to $r \approx \sqrt{2Bt}$, and hence are indistinguishable from the k = 0 case, Equ.(2.19).

These solutions are illustrated in Figure 2 below.



3. Formulation of General Relativity

We make no attempt to explain general relativity here. That would be a big job. Fortunately very little knowledge of general relativity is required to address the issues of interest in this chapter, beyond an appreciation of what the metric means.

Newtonian gravity is based on a single scalar potential which obeys a Poisson equation, $\nabla^2\phi = 4\pi G\rho$ (though Newton would not have recognised this description). In this equation, the mass density, ρ, is the source of the gravitational potential, ϕ. Thus, a point source $\rho = M\delta^3(r)$ gives the usual $\phi = -GM/r$. In general relativity, the single scalar potential is replaced by ten potential functions which form the components of a symmetrical tensor in the four dimensions of space-time, $g_{\alpha\beta}$. This tensor is the metric tensor of space-time, defined so that the line element ds between any two neighbouring points in space-time is a co-ordinate system invariant quadratic form in the co-ordinate differences between the points, i.e.,

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \tag{3.1}$$

This line element with respect to any curvilinear coordinate system in spacetime is the generalisation of the Minkowski line element in special relativity, $ds^2 = c^2dt^2 - |d\bar{r}|^2$. In place of the linear Poisson equation we have the non-linear Einstein field equations. There are ten of these, which suffices to determine the ten unknown potentials, $g_{\alpha\beta}$. They can be written,

$$E_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{3.2}$$

where, $$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}; \qquad R \equiv R^a{}_\alpha \equiv g^{\alpha\beta}R_{\alpha\beta} \tag{3.3}$$

and, $$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu} \equiv g^{\alpha\beta} R_{\beta\mu\alpha\nu} \tag{3.4}$$

and, $$R^{\alpha}{}_{\mu\beta\nu} \equiv \Gamma^{\alpha}{}_{\mu\nu,\beta} - \Gamma^{\alpha}{}_{\beta\mu,\nu} - \Gamma^{\alpha}{}_{\tau\beta}\Gamma^{\tau}{}_{\nu\mu} + \Gamma^{\alpha}{}_{\tau\nu}\Gamma^{\tau}{}_{\beta\mu} \tag{3.5}$$

and, $$\Gamma^{\alpha}{}_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\tau}\left[g_{\beta\gamma,\tau} - g_{\beta\tau,\gamma} - g_{\gamma\tau,\beta}\right] \tag{3.6}$$

and we have used the convention that repeated indices are summed, and a comma before an index implies ordinary partial differentiation, i.e. $X_{\alpha,\beta} \equiv \frac{\partial X_{\alpha}}{\partial x^{\beta}}$. The metric tensor can be used to lower or raise indices to convert between covariant and contravariant forms. (A more detailed explanation of these matters is beyond the scope of this work).

In the Einstein field equations, (3.2), the Einstein tensor is given the symbol E rather than the more common notation G, so as to avoid confusion with the universal gravitation constant, G (= 6.67 x 10^{-11} $m^3kg^{-1}s^{-2}$). From (3.3)-(3.6) the field equations are seen to be linear in the second derivatives of the metric tensor, but quadratic in the first derivatives. However there are also various factors of g around. In short, the field equations are nastily non-linear.

The LHS of (3.2) is the most general expression which is linear in the second derivatives of the metric and also generally covariant (i.e. a second rank tensor with respect to arbitrary curvilinear coordinate transformations). It was these two requirements which led Einstein to these equations. An additional requirement is that permissible spacetimes must be locally Lorentzian. This means that within any small region the spacetime can be approximated by a Minkowski coordinate system. Is this correct?

On the RHS of the field equations, (3.2), the energy-momentum tensor occurs in place of what was just the mass density in the Newtonian Poisson equation. The $_{00}$ component of the energy-momentum tensor is the nearest equivalent to the density, just as the $_{00}$ component of the metric tensor is the nearest equivalent to the Newtonian potential.

Finally, the scalar Λ on the LHS of (3.2) is the infamous Cosmological Constant, more of which later.

Units: The RHS of (3.2) has units L^{-2}. This matches the units of E, noting that g is dimensionless and that the temporal co-ordinate, x^0, is made spatial, e,g, x^0 = ct. This means that Λ has dimensions L^{-2} in this convention. Thus, the cosmological constant has units consistent with an interpretation as G/c^2 times a mass density or G/c^4 times an energy density.

4. The Friedman-Robertson-Walker Space-Times

The general formulation of the field equations, (3.2), is rather forbidding. Fortunately the assumptions that underlie most general relativistic cosmological models lead to a dramatic simplification. These assumptions are,

- There is a universal (cosmic) time co-ordinate, t;
- The spatial 3-spaces defined by t = constant are isotropic and homogeneous.

The idea is that the universe is filled with a uniform substratum of material, energy, etc. A so-called “fundamental observer” is one who moves along with this material substratum. They are said to be “co-moving” observers. The centre of gravity of sufficiently large clusters of galaxies are expected to be co-moving. Conversely, individual galaxies in general have a degree of individual (‘peculiar’) motion, e.g. around the cluster centre, superimposed on the substratum motion (which is also called the “Hubble flow”).

The assumption of a universal time co-ordinate marks an abandonment of the full covariance which was the hallmark of the complete field equations, (3.2). It allows a dramatic simplification. In fact, the most general metric respecting the above two assumptions can be derived (see Appendix ? for the proof). It is known as the Friedman-Robertson-Walker metric. It can be written in many forms. The first we shall use is,

$$ds^2 = c^2dt^2 - a^2(t)R_0^2\left\{\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta.d\phi^2\right\} \qquad (4.1)$$

We have written (4.1) so that both a(t) and r can be considered as dimensionless, and the dimensions of length are carried by the constant R_0. This means that k is dimensionless. We will also write,

$$R(t) \equiv a(t)R_0 \qquad (4.2)$$

Firstly we note that the magnitude of k is irrelevant. The only thing that matters is whether k is zero, positive or negative. This is because, for non-zero k, we can redefine the co-ordinate r as follows,

$$r \to \tilde{r} = r\sqrt{|k|} \qquad (4.3)$$

So that (4.1) comes in three distinct forms, according to whether k = 0, k < 0 or k > 0, thus,

k = 0: $$ds^2 = c^2dt^2 - a^2(t)R_0^2\left\{dr^2 + r^2d\theta^2 + r^2\sin^2\theta.d\phi^2\right\} \qquad (4.1a)$$

k < 0: $$ds^2 = c^2dt^2 - a^2(t)R_0^2\left\{\frac{d\tilde{r}^2}{1+\tilde{r}^2} + \tilde{r}^2d\theta^2 + \tilde{r}^2\sin^2\theta.d\phi^2\right\} \qquad (4.1b)$$

k > 0: $$ds^2 = c^2dt^2 - a^2(t)R_0^2\left\{\frac{d\tilde{r}^2}{1-\tilde{r}^2} + \tilde{r}^2d\theta^2 + \tilde{r}^2\sin^2\theta.d\phi^2\right\} \qquad (4.1c)$$

Whichever of these applies, we see that the space-time has the simple interpretation of being a set of 3-dimensional sub-spaces (with space-like intervals ds) orthogonal to the global time co-ordinate t. The metric of these 3D spatial sub-spaces is given by the {…}, except that the whole space is also subject to a time dependent scaling by the factor a(t).

For k = 0 we see that the sub-spaces are Euclidean, but that 3D space is still subject to the time dependent scaling by a(t). Hence this space-time is not Minkowskian. It is similar to Minkowski spacetime, but its spatial part is subject to an expanding size scale.

4.1 Interpretation of (4.1c) [k > 0]

We shall now show that the 3D space-like subspaces of (4.1c), i.e. for k > 0, can be interpreted as a 3-sphere embedded within a Euclidean 4D space. If co-ordinates in this hypothetical 4D Euclidean space are denoted x_1, x_2, x_3, x_4, then a 3-sphere in this space is defined by,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2 \qquad (4.4)$$

Note that x_4, nor any of the other Euclidean co-ordinates, have anything whatsoever to do with the time co-ordinate, t, in our true space-time. The line element in this 4D Euclidean space will be denoted ds_4, whereas a line element within the 3D sub-space spanned by x_1, x_2, x_3 will be denoted ds_3. In terms of spherical polar co-ordinates in this latter 3D space we can write,

$$ds_3^2 = d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right) \qquad (4.5)$$

Noting the usual convention that $d\rho^2$ means $(d\rho)^2$, not $d(\rho^2)$. Where the latter is meant it will be written explicitly.

We have, $\quad x_1 = \rho\sin\theta\cos\phi, \quad x_2 = \rho\sin\theta\sin\phi, \quad x_3 = \rho\cos\theta \qquad (4.6)$

i.e., $$\rho^2 = x_1^2 + x_2^2 + x_3^2 \qquad (4.7)$$

and the 4D line element is given by $ds_4^2 = ds_3^2 + dx_4^2 \qquad (4.8)$

Now by taking the derivative of (4.4) we can find dx_4 as follows,

$$x_1dx_1 + x_2dx_2 + x_3dx_3 + x_4dx_4 = 0 \Rightarrow dx_4 = -\frac{\sum_{i=1}^{3} x_i dx_i}{x_4} \qquad (4.9)$$

But, $$\sum_{i=1}^{3} x_i dx_i = \frac{1}{2}\cdot d\left(x_1^2 + x_2^2 + x_3^2\right) = \frac{1}{2}d(\rho^2) \qquad (4.10)$$

and from (4.4) and (4.7) $x_4^2 = R^2 - \rho^2 \qquad (4.11)$

Hence, (4.9) becomes, $(dx_4)^2 = \frac{[d(\rho^2)]^2}{4(R^2 - \rho^2)} = \frac{\rho^2 d\rho^2}{(R^2 - \rho^2)}$ (4.12)

Hence, (4.8) with (4.5) and (4.12) give,

$$\begin{aligned} ds_4^2 &= d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta \cdot d\phi^2) + \frac{\rho^2 d\rho^2}{(R^2 - \rho^2)} \\ &= \frac{R^2 d\rho^2}{(R^2 - \rho^2)} + \rho^2(d\theta^2 + \sin^2\theta \cdot d\phi^2) \end{aligned} \quad (4.13)$$

Both ρ and R have dimensions of length. Finally we make the substitution,

$$\tilde{r} = \rho / R \quad (4.14)$$

So that (4.13) becomes,

$$ds_4^2 = R^2 \left[\frac{d\tilde{r}^2}{(1 - \tilde{r}^2)} + \tilde{r}^2 (d\theta^2 + \sin^2\theta \cdot d\phi^2) \right] \quad (4.15)$$

So that we can identify (4.15) with the spatial part of the metric in (4.1c) for k > 0, with R = R(t) = R_0a(t).

Define positive & negative curvature. It is clear from the identification of the k > 0 space with an embedded Euclidean hypersphere that it is a space of positive curvature.

We note that the origin of the minus sign in the denominator of the first term (i.e. that corresponds to k > 0) is the positive sign in front of x_4^2 in the equation for a 3-sphere, (4.4).

4.2 But (4.1b) [k < 0] Cannot Be Interpreted In An Analogous Manner

We now attempt a similar interpretation of the case with k < 0. Initially, following the above observation, we consider in place of a 3-sphere the hyperbolic sub-space given by,

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = \rho^2 - x_4^2 = R^2 \quad (4.4b)$$

The spherical polars for the subspace x_1, x_2, x_3 are employed unchanged from before.

Now by taking the derivative of (4.4b) we can find dx_4 as follows,

$$x_1 dx_1 + x_2 dx_2 + x_3 dx_3 - x_4 dx_4 = 0 \Rightarrow dx_4 = + \frac{\sum_{i=1}^{3} x_i dx_i}{x_4} \quad (4.9b)$$

But, $$\sum_{i=1}^{3} x_i dx_i = \frac{1}{2}\cdot d\left(x_1^2 + x_2^2 + x_3^2\right) = \frac{1}{2} d(\rho^2) \qquad (4.10)$$

and from (4.4b) and (4.7) $x_4^2 = -R^2 + \rho^2$ (4.11b)

Hence, (4.9b) becomes, $$(dx_4)^2 = \frac{\left[d(\rho^2)\right]^2}{4\left(-R^2 + \rho^2\right)} = \frac{\rho^2 d\rho^2}{\left(-R^2 + \rho^2\right)} \qquad (4.12b)$$

Note that the 4D embedding space is again being assumed to be Euclidean, i.e. (4.8) holds. (4.12b) leads to,

$$\begin{aligned} ds_4^2 &= d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta\cdot d\phi^2\right) + \frac{\rho^2 d\rho^2}{\left(\rho^2 - R^2\right)} \\ &= \frac{(2\rho^2 - R^2)d\rho^2}{\left(\rho^2 - R^2\right)} + \rho^2\left(d\theta^2 + \sin^2\theta\cdot d\phi^2\right) \end{aligned} \qquad (4.13b)$$

which is NOT the same as (4.1b). Hence the spatial geometry of (4.1b), i.e. with $k < 0$, is NOT that of a 3-surface embedded in a Euclidean space in the manner specified by (4.4b).

Instead of (4.4b) we now consider,

$$-\left(x_1^2 + x_2^2 + x_3^2\right) + x_4^2 = -\rho^2 + x_4^2 = R^2 \qquad (4.4c)$$

which gives, $$x_4^2 = \rho^2 + R^2 \qquad (4.11c)$$

and $$-\left(x_1 dx_1 + x_2 dx_2 + x_3 dx_3\right) + x_4 dx_4 = 0 \Rightarrow dx_4 = +\frac{\sum_{i=1}^{3} x_i dx_i}{x_4} \qquad (4.9c)$$

But, $$\sum_{i=1}^{3} x_i dx_i = \frac{1}{2}\cdot d\left(x_1^2 + x_2^2 + x_3^2\right) = \frac{1}{2} d(\rho^2) \qquad (4.10)$$

Hence, (4.9c) becomes, $$(dx_4)^2 = \frac{\left[d(\rho^2)\right]^2}{4\left(R^2 + \rho^2\right)} = \frac{\rho^2 d\rho^2}{\left(R^2 + \rho^2\right)} \qquad (4.12c)$$

And hence, $$\begin{aligned} ds_4^2 &= d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta\cdot d\phi^2\right) + \frac{\rho^2 d\rho^2}{\left(\rho^2 + R^2\right)} \\ &= \frac{(2\rho^2 + R^2)d\rho^2}{\left(\rho^2 + R^2\right)} + \rho^2\left(d\theta^2 + \sin^2\theta\cdot d\phi^2\right) \end{aligned} \qquad (4.13c)$$

So, again we find that the curved 3-surface embedded in a 4D Euclidean space as given by (4.4c) is NOT equivalent to (4.1b) with $k < 0$.

In fact, it can be shown that the spatial part of (4.1b) is not interpretable as any 3-surface embedded in a higher dimensional Euclidean space. (Is this true? Or is it only true that it cannot be embedded in a 4D Euclidean space?).

Demonstrate that the k < 0 space has negative curvature (and define what that means!).

4.3 Alternative Algebraic Forms of the Friedman-Robertson-Walker Metric

4.3.1 k > 0

For k > 0, we have tacitly assumed that the term in {…} in (4.1c) is a spacelike subspace of the space-time. However, this is only true provided the {…} is positive, since only then is it a spacelike interval. But we observe that if $|\tilde{r}| > 1$ then the interval $d\tilde{r}$ with $dt = d\theta = d\phi = 0$ will be timelike. Thus, in (4.1c), we must restrict $\tilde{r}$ to lie in the interval [-1, +1]. This can be seen from the interpretation of this 3D subspace as a hypersphere embedded in 4D Euclidean space since this requires, from (4.11) and (4.14),

$$R^2\tilde{r}^2 + x_4^2 = R^2 \tag{4.16}$$

and hence the maximum possible magnitude of $\tilde{r}$, when $x_4 = 0$, is 1. This suggests an alternative co-ordinate system in which, using (4.6),

$$\tilde{r} = \frac{\rho}{R} = \sin\psi ; \qquad x_4 = R\cos\psi \tag{4.17}$$

$$x_1 = R\sin\psi\sin\theta\cos\phi ; \quad x_2 = R\sin\psi\sin\theta\sin\phi ; \quad x_3 = R\sin\psi\cos\theta \tag{4.18}$$

where ψ is a hyperspherical polar angle with respect to the x_4 axis. Whilst R is the radius of the hypersphere, ρ is the projection of R onto the subspace x_1, x_2, x_3 of the hypothetical 4D Euclidean space x_1, x_2, x_3, x_4. Thus the whole of space is covered by the angular ranges,

$$0 \le \psi \le \pi; \quad 0 \le \theta \le \pi; \quad 0 \le \phi \le 2\pi \tag{4.19}$$

In terms of ψ the space-time metric becomes,

$$ds^2 = c^2dt^2 - R^2(t)\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\cdot d\phi^2\right)\right] \tag{4.20}$$

It is tempting to conclude that because the co-ordinate ranges in (4.19) are all finite, this implies that the space in question is finite. However, that would not be a correct argument (though, as it happens, the answer is!). The reason is that a finite coordinate range can be made infinite by a suitable co-ordinate transformation, for example $\tan(\psi/2)$ will range from 0 to infinity for $\psi \in [0,\pi]$. To determine if space is finite we need to calculate its volume (noting that this is invariant under spatial co-ordinate transformations). Of course, we already know that the volume is finite since it equals the ‘surface area’ of the hypersphere (and hence is $2\pi^2R^3$). This can be evaluated explicitly as follows: an element of the hypersphere’s ‘area’ is given by,

$$dV = Rd\psi . dS_\rho \tag{4.21}$$

where dS_ρ is the element of area of an ordinary sphere of radius ρ, i.e.,

$$dS_\rho = \rho^2 d(\cos\theta)d\phi = R^2 \sin^2 \psi d(\cos\theta)d\phi \tag{4.22}$$

Hence,
$$V = R^3 \int_0^\pi \int_0^\pi \int_0^{2\pi} d\psi d\theta d\phi \sin^2\psi \sin\theta = 4\pi R^3 \int_0^\pi d\psi \sin^2\psi = 2\pi^2 R^3 \tag{4.23}$$

Thus, we have the result that the Friedman-Robertson-Walker (FRW) space-times with $k > 0$ consist of 3D spacelike subspaces which are finite in the sense of having finite volume. However, this volume will vary with time since R is time dependent. However, at any time, R is the true size of the universe, as measured by its hyperspherical radius. Hence for $k > 0$ cosmologies we need not be mealy-mouth and talk about the 'size scale, R' of the universe. For $k > 0$, R is simply the size of the universe. This means that $k > 0$ FRW cosmologies did start at a single point at $t = 0$, since R reduces to zero at $t = 0$ (see Section 5).

Finally, we mention another algebraic form of the $k > 0$ Robertson-Walker metric. This is obtained by the substitution,

$$\tilde{r} = \frac{u}{1 + \frac{u^2}{4}} \tag{4.24}$$

From which it is readily shown that,

$$d\tilde{r} = \frac{1 - \frac{u^2}{4}}{\left(1 + \frac{u^2}{4}\right)^2} \quad \text{and} \quad \frac{1}{1 - \tilde{r}^2} = \left(\frac{1 + \frac{u^2}{4}}{1 - \frac{u^2}{4}}\right)^2 \tag{4.25}$$

and hence we get the metric in space-time from (4.1c) to be,

$$ds^2 = c^2 dt^2 - R^2(t)\left[\frac{du^2 + u^2 \cdot d\theta^2 + u^2 \sin^2\theta \cdot d\phi^2}{\left(1 + \frac{u^2}{4}\right)^2}\right] \tag{4.26}$$

The numerator of […] looks like an ordinary Euclidean line element, but the magnitude of this line element is limited by the denominator at large 'u'. This form also illustrates that $\tilde{r}$ can only range up to a magnitude of unity since -1 and +1 are the minimum and maximum of the function in (4.24) (achieved at $u = -2$ and $+2$).

4.3.2 k < 0

Unlike the case for k > 0, for k < 0 the {…} in (4.1b) remains a spacelike interval (i.e. positive) for any real $\tilde{r}$. We may suspect that this implies that space is infinite in this case, but have already noted that this cannot be concluded from the co-ordinate ranges alone. We note that a substitution like (4.17) is not possible since this would restrict the possible range of $\tilde{r}$. However, the whole of the real line is available to $\tilde{r}$ if we make the substitution,

$$\tilde{r} = \sinh\psi \tag{4.27}$$

then since $d\tilde{r} = \cosh\psi d\psi$ and $1 + \tilde{r}^2 = 1 + \sinh^2\psi = \cosh^2\psi$ we see that the space-time interval is,

$$ds^2 = c^2dt^2 - R^2(t)\left[d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta \cdot d\phi^2\right)\right] \tag{4.28}$$

and hence is identical to (4.20) except for the replacement of sinψ by sinhψ. However, the co-ordinate ranges are also modified, thus,

$$0 \le \psi \le \infty; \quad 0 \le \theta \le \pi; \quad 0 \le \phi \le 2\pi \tag{4.29}$$

To determine if space is infinite we calculate the volume as follows: firstly note that the metric (4.28) tells us what the length is associated with the change in each co-ordinate in turn. Thus for a change $d\psi$ with other co-ordinates constant the length element is $Rd\psi$; for a change $d\theta$ with other co-ordinates constant the length element is $R\sinh\psi d\theta$; for a change $d\phi$ with other co-ordinates constant the length element is $R\sinh\psi\sin\theta d\phi$. The product of these three lengths gives the element of volume,

$$dV = R^3\sinh^2\psi\sin\theta d\psi d\theta d\phi \tag{4.30}$$

Attempting to integrate (4.30) over the full range of the co-ordinates as given by (4.29) reveals that this space has infinite volume. Whilst the integral over the 'normal' angular co-ordinates evaluates to 4π, the usual solid angle, the integral of $\sinh^2\psi$ over $\psi \in [0,\infty]$ is clearly divergent. Thus, we conclude that space is infinite in this k < 0 Friedman-Robertson-Walker (FRW) space-time.

Note that space is infinite at all times t > 0 for which R is non-zero. Thus, at the first instant after the Big Bang, when t is just one quantum of time, space is already infinite. Thus, for spacetimes obeying our two cosmological conditions (the existence of a universal time and isotropic everywhere), and therefore of FRW form, an infinite space must always have been infinite. It must have been created infinite. So it is ***not*** correct in the k < 0 case to think of the universe as springing from a single point at the Big Bang. Rather, all points in an already infinite space were created at the first instant. Nevertheless, if we consider any finite region at a finite time t, this region will shrink to zero size as we go backwards in time towards t = 0. This is true no matter how large a region we consider. Thus, the density must be infinite at t = 0.

5. General Relativistic Cosmological Models

In Section 4 we have presented the general solution for the metric under the usual simplifying cosmological assumptions that, (a)there exists a universal time coordinate, and, (b)space is, in the large, homogeneous and isotropic. The latter condition is essentially a generalised Copernican principle. However, this general form of metric, given by Equ.(4.1) or Equs.(4.1a-c), contains an unspecified size scale R(t) as a function of time. The form of the metric was deduced purely from the cosmological assumptions, (a) and (b) above. We have not yet imposed the requirement that the metric must obey Einstein's gravitational field equations, Equ.(3.2). These are the dynamical conditions analogous to Equ.(2.1) in the Newtonian model studied in Section 2. We now impose the Einstein field equations, and consider the case in which the source term (the energy-momentum tensor of matter, $T_{\mu\nu}$) contains not just the mass-energy density as a source of gravity ($T_{00} = \rho c^2$), but also pressure ($T_{11} = T_{22} = T_{33} = p$). This pressure could include the pressure of matter acting as a gas (although this will be negligible at the present epoch). More importantly, it also includes the radiation pressure. The latter is also negligible at the present epoch, but would have been significant in the early universe. Similarly, T_{00} is the sum of both the matter and radiation densities.

When this is done we find the field equations reduce to two (ordinary) differential equations which R(t) must obey, i.e.,

$$\ddot{R} = -4\pi G\left(\rho + \frac{3p}{c^2}\right)\frac{R}{3} + \frac{\Lambda c^2}{3}R \qquad (5.1)$$

$$\dot{R}^2 = \frac{8\pi}{3}G\rho R^2 - kc^2 + \frac{\Lambda c^2}{3}R^2 \qquad (5.2)$$

The second equation reduces, in the case of zero cosmological constant, to Equ.(2.4) which we used to generate the 'Newtonian' cosmologies in Section 2. By differentiating (5.2) and substituting from (5.1) we deduce the so-called *fluid equation*,

$$\frac{d}{dt}\left(\rho R^3\right) + \frac{p}{c^2}\frac{d}{dt}\left(R^3\right) = 0 \qquad (5.3)$$

The pair of equations (5.1)+(5.2) can be replaced by (5.2)+(5.3), i.e. (5.1) also follows from (5.2) and (5.3). Modulo a factor of $4\pi/3$, the first term in (5.3) is the rate of change of the amount of mass-energy in a sphere of radius R. The 'mass' part will not change with time, but the 'energy' part will – both that of the photons, whose individual energy reduces as 1/R, and also the kinetic energy of the matter particles. Including the factor of $4\pi/3$, the second term in (5.3) is pdV/c^2 per time interval dt, i.e. the work done by the radiation and matter in expanding during that period. Hence we recognise (5.3) as the thermodynamic statement that the change in internal energy balances against the work done by the fluid in expanding. For a given universe dynamic, i.e. given R(t), Equ.(5.3) relates the pressure and density of the fluid, i.e. it is an equation of state for the fluid.

For the remainder of this Section we shall ignore Equ.(5.3) and derive as much as we can about the universe's dynamic, R(t), from Equ.(5.2) alone. Since (5.2) differs from (2.4) only in the addition of the cosmological constant term it follows that any difference in the behaviour of R(t) deduced here compared with that of the 'Newtonian' cosmologies of Section 2 is due solely to the cosmological constant. We also note that, if pressure is neglected, Equ.(5.3) is the same as the conservation of mass, i.e. Equ.(2.2). Thus, we deduce the remarkable result that when pressure is negligible and there is no cosmological constant, the general relativistic cosmological model, which reduces to just Equ.(5.2), is identical to the Newtonian model we

Before looking at the behaviour of R(t), we define some parameters which are generally used in the literature:-

5.1 Cosmological Parameters

The first parameter has already been defined. It is the Hubble parameter, given by,

$$H(t) = \frac{\dot{R}}{R} \qquad (5.4)$$

The second we have also already defined in Chapter 2b: the matter density ratio,

$$\Omega(t) = \frac{\rho_{matter}}{\rho_{critical}} = \frac{8\pi G\rho}{3H^2} \qquad (5.5)$$

(NB: when we write Ω or ρ with no subscript they refer to 'matter' only. In this context 'matter' includes radiation as well as dark matter. It excludes only dark energy).

The next parameter is just a dimensionless form of the cosmological constant,

$$\lambda(t) = \frac{\Lambda c^2}{3H^2} \qquad (5.6)$$

Finally, the dimensionless deceleration parameter is defined as,

$$q(t) = -\frac{R\ddot{R}}{\dot{R}^2} \qquad (5.7)$$

The deceleration parameter was defined in an era when cosmologists still thought that the acceleration of the universe would be slowing down, i.e. $\ddot{R}$ was thought to be negative and hence q would be positive. It is currently believed that $\ddot{R}$ is positive and q negative.

Note that H, Ω, ρ, λ, q and R are all varying with time. Only Λ and k (and obviously, G) are constants in Equ.(5.2). Note that time variation of λ in (5.6) is due to that of H.

Relationships between these parameters are derived as follows. Firstly we express 'k' in terms of these parameters. Equ.(5.2) gives,

$$kc^2 = \frac{8\pi}{3}G\rho R^2 + \frac{\Lambda^2 c^2}{3}R^2 - \dot{R}^2 = \Omega H^2 R^2 + \lambda H^2 R^2 - H^2 R^2 = H^2 R^2 \{\Omega + \lambda - 1\} \quad (5.8)$$

If we regard the (dimensionless) cosmological constant as the contribution of dark energy to the total density parameter, i.e. putting $\lambda \equiv \Omega_{darkenergy} \equiv \Omega_\Lambda$ and $\Omega_{m+\Lambda} = \Omega + \Omega_\Lambda$, then (5.8) can be written simply as,

$$kc^2 = H^2 R^2 (\Omega_{m+\Lambda} - 1) \quad (5.9)$$

This shows that the geometry of the universe, i.e. whether it is open, closed or flat (i.e. whether k is <0, >0 or =0), is identical to whether the *total* density parameter, *including dark energy*, is <1, >1 or =1 respectively.

For convenience we define the curvature parameter, Ω_R,

$$\Omega_R = -\frac{kc^2}{H^2 R^2} = -\frac{kc^2}{\dot{R}^2} = 1 - \Omega_{m+\Lambda} \quad (5.9a)$$

and hence:

$$\Omega_R + \Omega_{m+\Lambda} = \Omega_R + \Omega + \Omega_\Lambda = 1 \quad (5.9b)$$

Note that (5.9b) is not a contingent condition on this particular universe. Actually it is merely a re-writing of the Friedmann equation, (5.2). So (5.9b) is the dynamic behaviour of the universe resulting from the Einstein field equations. This stands in stark contrast to the observational result from WMAP and elsewhere that $\Omega_{TOT} \approx 1$ to within about 2%. This *is* a contingent condition which just happens to be true for our particular universe. It carries the implication, derived from (5.9b), that $|\Omega_R| << 1$. In fact the curvature parameter appears to be less than ~0.02 in magnitude, and its sign is unknown. Consequently, whether the universe is open is an open question.

Using (5.1) and (5.2), the deceleration parameter can generally be written,

$$q = \frac{4\pi G\left(\rho + \frac{3p}{c^2}\right)\frac{R^2}{3} - \frac{\Lambda c^2}{3}R^2}{\frac{8\pi}{3}G\rho R^2 - kc^2 + \frac{\Lambda c^2}{3}R^2} \quad (5.10)$$

In the case p=0, dividing (5.10) by H^2 gives,

$$q = \frac{\frac{1}{2}\Omega R^2 - \lambda R^2}{\Omega R^2 + \lambda R^2 - kc^2/H^2} \quad (5.11)$$

Substituting for k from (5.8) gives,

For p = 0 $$q = \frac{1}{2}\Omega - \lambda \quad \text{or,}\ \Omega = 2(q + \lambda) \qquad (5.12)$$

Thus, only two of these three parameters are independent. (5.12) shows that we will have a universe with accelerating expansion ($q < 0$) if the dimensionless cosmological constant (λ) is greater than half the dimensionless density, Ω. This condition appears to be easily met in our universe at present since $\lambda \sim 0.7$ whereas $\Omega \sim 0.3$.

5.2 Qualitative Behaviour of R(t) for p = 0 but Non-Zero Λ

Just as was the case for the 'Newtonian' cosmologies of Section 2, we shall find that the qualitative behaviour of R(t) prescribed by Equ.(5.2) can be discovered without explicitly solving this differential equation. To recap, the equation we are considering is,

$$\dot{R}^2 = F(R), \quad \text{where,}\ F(R) = \frac{8\pi G\rho_0 R_0^3}{3R} - kc^2 + \frac{\Lambda c^2}{3}R^2 \qquad (5.2b)$$

Note that we have re-written ρR^2 as $\rho_0 R_0^3 / R$. This assumes that ρR^3 is constant. From Equ.(5.3) this is equivalent to assuming that the pressure is zero (p = 0).

5.2.1 $\underline{\Lambda < 0}$

The first term in F is positive and reduces monotonically from infinity, at R = 0, towards zero at large R. In the case of negative cosmological constant, the third term also reduces monotonically, starting at zero at R = 0 and diverging to minus infinity at large R. It follows that F itself reduces monotonically from plus infinity to minus infinity. Hence there is a unique radius, R_0, at which F is zero. For $R > R_0$, F is negative, and, since $\dot{R}^2$ cannot be negative, it follows that $R > R_0$ is not possible. Consequently the solution is confined to $R < R_0$.

For sufficiently small R the last term in F will be small compared with the first two terms. Hence, for small R the solution of (5.2b) approximates to the Newtonian cosmologies, i.e. (2.16a) or (2.16b). For k > 0 and sufficiently small Λ, the solution of (5.2b) may be little different from (2.16a).

On the other hand, for k < 0 and Λ < 0, the solution of (5.2b) is always qualitatively different from (2.16b) for large R, even when Λ is numerically small. This is because, for large R, approaching R_0, the third term will become important and causes the rate of expansion to reduce to zero. Thus R_0 is always the maximum size scale of the universe, irrespective of the sign of 'k'. The universe then contracts. There may be many such cycles ('oscillating universe').

The point is worth emphasising. In a universe with no cosmological constant, having k > 0 means two things: that the universe has positive curvature and also that it reaches a maximum size before contracting back to a Big Crunch. Similarly, in a universe with no cosmological constant, having k < 0 means that the universe has negative curvature and also that it goes on expanding indefinitely. We have seen in Section 4 that having k > 0 means a finite universe, i.e. a universe with a finite volume and a definite finite 'radius'. Conversely, having k < 0 means that the universe is infinite in the sense that its volume is infinite. [Since we are dealing with homogeneous spaces, if it has a non-zero density it follows that its mass will also be infinite]. But the identification of spaces of positive (negative) curvature, and hence finite (infinite) size, with spaces which reach a maximum size (expand indefinitely) only holds when $\Lambda = 0$. This is something of a relief since it was never physically clear why this identification should be true. We see that, when $\Lambda < 0$, we may be dealing with spaces with $k < 0$ or $k > 0$ or $k = 0$, but a maximum size scale, R_0, is reached in *all* these cases. Hence we can have negatively curved ($k < 0$) or flat ($k = 0$) spaces which expand to a maximum size-scale and then contract again. Note that such spaces are still infinite. Thus, the finiteness or infiniteness of space is still determined by k.

Physically, a negative cosmological constant produces an additional attractive gravitational force. This force increases with radius, as can be seen from Equ.(5.1). Consequently, the Λ-induced force is bound to dominate eventually and cause the universe to re-contract. This happens irrespective of the curvature. A negative curvature ($k < 0$) causes a positive offset to the 'velocity' in Equ.(5.2), but this will be dominated by the (negative) Λ term at sufficiently large R.

5.2.2 $\Lambda > 0$ and $k \leq 0$

In this case all three terms in (5.2b) are positive and hence it is clear that R is monotonically increasing. Moreover, at large R the third term is dominant and hence the solution will tend asymptotically to an exponential 'explosion', i.e.,

$$R \to \propto \exp\left\{\sqrt{\frac{\Lambda c^2}{3}} \cdot t\right\} \tag{5.13}$$

Physically, a positive cosmological constant represents a repulsive gravitational force. Since this repulsive force grows proportionally as R, see Equ.(5.1), it inevitably dominates eventually and we get the exponential growth.

A special case is when both $k = 0$ and $\rho = 0$, for which (5.13) is then the complete solution for all times. This is the de Sitter solution and has some interesting properties. However it is hardly physical since it represents an empty universe.

5.2.3 $\Lambda > 0$ and $k > 0$

In this case the first and third terms of F are both positive, but the second term is negative. The qualitative nature of the solution depends upon whether there is a radius at which F becomes zero. There is definitely a unique minimum of F. This is at radius,

$$R_{min}^3 = \frac{3A^2}{2\Lambda c^2} \tag{5.14}$$

where A is given by (2.7), (2.14) or (2.15). The minimum of F is thus,

$$F_{min} = \frac{1}{R_{min}}\left\{\frac{3}{2}A^2 - kc^2 R_{min}\right\} \tag{5.15}$$

Hence, the critical value of Λ (Λ_c) such that the minimum of F is zero is given by,

$$R_{min} = R_c = \frac{3A^2}{2kc^2} \tag{5.16}$$

Combining this with (5.14) gives,

$$\Lambda_c = k\left(\frac{2kc^2}{3A^2}\right)^2 = \frac{k}{R_c^2} \tag{5.17}$$

This can be also be written,

$$\Lambda_c c^2 = \frac{4}{9} \cdot \frac{H^2(\Omega_{TOT} - 1)^3}{\Omega^2} \quad \text{or} \quad \lambda_c = \frac{4}{27} \cdot \frac{(\Omega_{TOT} - 1)^3}{\Omega^2} \tag{5.17b}$$

The three possible cases are:-

$\underline{\Lambda > \Lambda_c}$

In this case F cannot be negative for any R, so R is monotonically increasing. For sufficiently large R the expansion becomes exponential, i.e. the asymptotic behaviour is once again given by (5.13). This is another instance of the cosmological term 'winning' over the curvature effect (since k > 0 tends to halt the expansion). If it is the case that k > 0, then this is almost certainly the correct model. This is because current estimates place Ω_{TOT} within about 0.02 of unity, whereas $\Omega \approx 0.3$, so (5.17b) gives the critical λ_c to be about 10^{-5}, compared with the current estimates of ~0.7. Hence, $\Lambda > \Lambda_c$ seems almost certain. Nevertheless, for completeness we consider the two final cases.

$\underline{\mathbf{0 < \Lambda < \Lambda_c}}$

In this case there are two real radii at which F = 0. Call them R_1 and R_2 with $R_1 < R_2$. The allowed regimes (where $\dot{R}^2 > 0$) are $[0, R_1]$ and $[R_2, \infty]$. These two disconnected regimes correspond to two distinct solutions. In the first, $[0, R_1]$, there is a Big Bang at R = 0 and the expansion reverses at a maximum radius of R_1. We can think of this solution as being one in which R is restricted to sufficiently small values that the cosmological term does not get a chance to dominate and cause exponential growth.

Conversely, the second solution, with R э $[R_2, \infty]$, can be thought of as one in which the radius is confined to a range where the cosmological constant is always dominant. The universe can be imagined to have been given an initial inward velocity at some large radius, thus making it move initially against the repulsive Λ-effect. Its speed of contraction slows, until the universe 'bounces' back at radius R_2, at which point the velocity is zero. This is the first instance of a universe with no Big Bang. [A Big Bang has, at some time designated as t = 0, R = 0, hence infinite density, and infinite initial speed. At the 'bounce' of this universe, R is non-zero and the speed is zero].

$\underline{\Lambda = \Lambda_c}$

This case is more subtle. One solution is the static Einstein universe. Putting $R = R_c$ we have $\dot{R} = 0$, by definition. Moreover, substitution of R_c and Λ_c into Equ.(5.1) shows that $\ddot{R}$ is also zero. Consequently, the universe can be balanced at constant R provided that its size and the cosmological constant are chosen appropriately. 'Balanced' is the correct word since this comes about because the attractive force of gravity is matched against the repulsive cosmological force. It is an unstable equilibrium.

More reasonably we have two solutions for which the radius is confined respectively to either $[0, R_c]$ or $[R_c, \infty]$. These differ from the solutions for $0 < \Lambda < \Lambda_c$ because, as R approaches R_c, not only does the speed approach zero, but the acceleration or deceleration approaches zero also. This means the solution 'gets stuck' at $R = R_c$. Thus, for the Einstein-Lemaitre model EL1, the universe only tends asymptotically to its maximum size of R_c, and never contracts. Conversely, the Einstein-Lemaitre model EL2 starts off infinitesimally larger than R_c and initially grows very slowly. Eventually, however, the cosmological repulsive force catches hold and the universe eventually expands exponentially.

The Einstein-Lemaitre model EL1 is unique. It is the only instance of a universe which has a maximum size, but which approaches it asymptotically and never contracts. EL1 does have a Big Bang, however. EL2 is unusual in having no Big Bang, but rather a non-zero initial size. It is similar to the 'bounce' model in this respect.

An interesting special case is worth mentioning. This is obtained for $\Lambda = (1 + \varepsilon)\Lambda_c$ where ε is very small, but positive. Of course this is actually a special case of $\Lambda > \Lambda_c$ and hence is a solution which starts at R = 0 and expands monotonically to infinity. However, its proximity to the EL1 and EL2 solutions gives it some characteristics of these. Specifically, it appears to be like an EL1 followed by an EL2. In other words this universe expands rapidly to near R_c. It then stays near R_c for a long time. Eventually, however, the cosmological repulsion wins and an exponential expansion takes over.

The Einstein-Lemaitre, Lemaitre and 'bounce' models are nicely illustrated graphically in Rowan-Robinson, Figures 4.12 to 4.14. We do not dwell further on these models here since they appear not to represent our universe. Have I said what the Lemaitre model is? It would be nice for completeness to include graphs of all the models.

The qualitative features of all the solutions we have discussed are summarised below:-

Λ	**k**	**Starting Radius (1)**	**Finite Maximum Radius? (2)**	**Re-Contraction?**
0	> 0	0	Yes	Yes
0	≤ 0	0	No	No
< 0	any	0	Yes	Yes
> 0	**≤ 0**	**0**	**No**	**No**
$> \Lambda_c$	**> 0**	**0**	**No**	**No**
$0 < \Lambda < \Lambda_c$	> 0	0	Yes	Yes
		≠ 0 ('bounce')	No	No
$\Lambda = \Lambda_c$	> 0	≠ 0 ('E')	Yes	No (Static)
		0 ('EL1')	Yes	No

		$\neq 0$ ('EL2')	No	No

(1)'Zero' also implies infinite initial speed, i.e. a Big Bang
(2)'No' implies the universe expands to $R \to \infty$

The models most representative of this universe are shown in bold blue. The cosmological constant appears to be positive and large. However, k is close to zero and hence its sign is uncertain. Inflation models require k = 0. A large positive Λ is enough to unambiguously favour a Big Bang model (with R = 0 initially) and destined to expand forever, at an exponentially increasing rate. However, whether the universe is spatially finite or infinite at present is unknown since this depends upon the sign of k (i.e. whether Ω_{TOT} is greater or smaller than unity). WMAP data current favours positive curvature slightly, i.e. a finite universe, but is also consistent with k = 0 (infinite universe). CHECK the latest position.

6. The Age of the Universe

In Chapter 2 we discovered that the age of the universe differs, in general, from the reciprocal of the Hubble parameter ($\tau_0 = 1/H$) by a numerical factor. In the early universe, when $R \propto t^{2/n}$, this numerical factor is just 2/n. Thus, in the early, radiation-dominated universe, with n = 4, $t = 0.5\tau_0$. In the matter dominated case, if this cosmological solution still applied, we would have $t = 0.667\tau_0$. However, we noted that standard references currently quote $t \approx \tau_0$. In this Section we derive the numerical factor for currently favoured cosmological parameters and show that it is indeed close to unity.

Because our cosmological models are based on a favoured time co-ordinate, we do not need to agonise over what the correct time measure might be. It is the same 't' which appears in R(t), i.e., in the FRW metric. Hence, we need look no further for a means to evaluate the age of the universe. When the 'size' of the universe is R_0 its age must be,

$$t_0 = \int_0^{R_0} \frac{dR}{\dot{R}} \tag{6.1}$$

But, $$\dot{R}^2 = \frac{8\pi G\rho_0 R_0^3}{3R} - kc^2 + \frac{\Lambda c^2}{3}R^2 = \Omega_0 H_0^2 \frac{R_0^3}{R} - kc^2 + \lambda_0 H_0^2 R^2 \tag{6.2}$$

[from the Friedman equation, (5.2)]. Hence, using (5.9),

$$\begin{aligned}\dot{R}^2 &= \Omega_0 H_0^2 \frac{R_0^3}{R} - H_0^2 R_0^2 [\Omega_0 + \lambda_0 - 1] + \lambda_0 H_0^2 R^2 \\ &= \frac{R_0^2}{\tau_0^2}\left[\frac{\Omega_0}{x} + (1 - \Omega_0 - \lambda_0) + \lambda_0 x^2\right] \\ &= \frac{R_0^2}{\tau_0^2}\left[\frac{\Omega_0}{x} + \left(1 - \frac{3}{2}\Omega_0 + q_0\right) + \left(\frac{\Omega_0}{2} - q_0\right)x^2\right]\end{aligned} \tag{6.3}$$

Where $x = R/R_0$ and the last step follows from (5.12). Substituting (6.3) into (6.1) gives,

$$t_0 = \tau_0 \int_0^1 \frac{dx}{\left[\frac{\Omega_0}{x} + \left(1 - \frac{3}{2}\Omega_0 + q_0\right) + \left(\frac{\Omega_0}{2} - q_0\right)x^2\right]^{1/2}} \qquad (6.4)$$

Thus, the integral yields explicitly the numerical factor by which the reciprocal of the Hubble parameter must be multiplied to give the age of the universe. This factor does not depend upon the detailed solution for R(t) but only upon the parameters Ω_0 and q_0 (or equivalently, λ_0) *evaluated at the time t_0 in question*. Values of this factor obtained by numerical integration of (6.4) are given in the Table below for a range of these parameters. Rowan-Robinson states that (6.4) ignores radiation. I think it is more accurate to say that (6.4) ignores pressure, since ρR^3 has been assumed constant in (6.2). However, this may amount to the same thing.

Omega	**Omega + lambda**			
	0.3	**0.9**	**1**	**1.02**
0.05	1.010	1.363	1.490	1.522
0.1			1.278	1.295
0.15	0.903	1.105	1.158	1.170
0.2			1.076	1.085
0.25			1.014	1.021
0.3	0.809	0.935	**0.964**	**0.970**
0.35			0.923	0.928
0.4			0.888	0.893
0.5	0.729	0.813	0.831	0.835
0.6			0.786	0.789
0.8			0.717	0.720
1	0.613	0.658	0.667	0.668

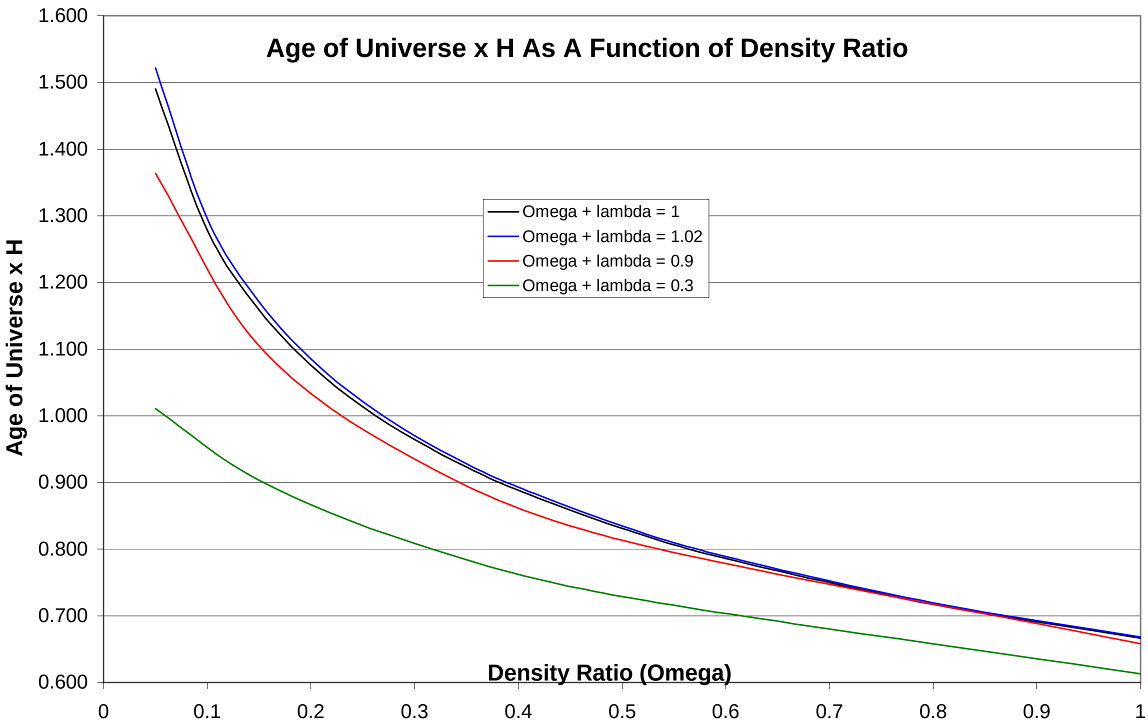
The bold blue numbers are the current best estimates. Hence, the numerical factor is very close to unity, and the age of the universe is close to the reciprocal of the Hubble parameter. In passing we note that the consensus cosmological parameters from WMAP are:- CHECK for more recent values….

$H_0 = 72 \pm 8\ \text{kms}^{-1}\text{Mpc}^{-1}$ Age of universe = 13.7 Byrs

$\Omega_{TOT} = \Omega_0 + \lambda_0 = 1 \pm 0.02$ or, $-0.02 < \dfrac{kc^2}{H_0^2 R_0^2} < 0.02$

$\Omega_0 = 0.3 \pm 0.1$ $\lambda_0 = 0.7 \pm 0.2$ $q_0 = -0.603$

Note that for matter density at the critical density ($\Omega_0 = 1$) and negligible cosmological constant fractional density ($\lambda_0 \sim 0$), we recover the result $t = 0.667\tau_0$ for a matter dominated universe, confirming that the above Table and Equ.(6.4) neglect radiation.



To add:

1) Can explicit expressions be derived for R(t) for any/all of the above FRW spaceteimes?

2) Deduce how the density parameters Ω_0, Ω_{TOT}, λ_0 evolve. Show that Ω_{TOT} diverges rapidly away from unity.

3) Repeat Sections 5 & 6 without ignoring pressure. What does the graph of Ht look like when radiation is treated properly and the abscissa spans from radiation dominance through matter dominance?

4) Explore what would happen to universes with more or less matter, or with greater or smaller G.

Annex: Further Discussion of the Solutions to the Friedmann Equation, (2.13) or (5.2), with Λ = p =0

Analytic solutions have already been given in Equations (2.15, 2.16a,b). Here we give further details and discuss some subtleties. First we note that the definitions,

$$\Omega = \frac{\rho}{\rho_{crit}}; \qquad \rho_{crit} = \frac{3H^2}{8\pi G}; \quad H = \frac{\dot{r}}{r}; \qquad \tau = \frac{1}{H} \qquad \text{(A.1)}$$

together with the Friedmann equation $\dot{r}^2 = \frac{8\pi}{3}G\rho r^2 - kc^2$ lead by simple algebraic manipulation to two expressions which are constant: one involving Ω and r, and the other involving Ω and τ. These are,

$$\frac{\Omega r}{\Omega - 1} = \frac{8\pi G\rho r^3}{3kc^2} = \frac{A^2}{kc^2} = \text{constant} \qquad \text{(A.2)}$$

$$\frac{\Omega\tau}{|\Omega - 1|^{3/2}} = \frac{8\pi G\rho r^3}{3|k|^{3/2}c^3} = \frac{A^2}{|k|^{3/2}c^3} = \text{constant} \qquad \text{(A.3)}$$

It is crucial to note that it is the reciprocal of the Hubble parameter, τ, *not* the universal time, t, which enters the latter expression. It is also crucial to note that the critical density is *not* given by either $3/8\pi Gt^2$ or by $1/6\pi Gt^2$. The first of these happens to be approximately correct in our universe at the present epoch (which has $k \approx 0$ but $\Lambda \neq 0$ and gives $H \sim 1/t$) and the second applies for $k = \Lambda = 0$ (Einstein-de-Sitter?). However, neither applies for the present case (i.e. $k \neq 0$ and $\Lambda = 0$).

These expressions are equally valid for $k > 0$ and $k < 0$, but clearly not valid for $k = 0$.

(A.2) shows that $k > 0$ requires $\Omega > 1$ (we rule out $\Omega < 0$ on physical grounds). It also shows that $k < 0$ requires $\Omega < 1$. This establishes the correspondence between open ($k < 0$) and closed ($k > 0$) universes and whether the density is less than, or greater than, the critical density ($\Omega < 1$ and $\Omega > 1$ respectively).

The exact solutions to the Friedmann equation can be written with the coefficient in three different forms, as follows,

<u>k > 0</u>

$$r = \frac{A^2}{2c^2k}(1-\cos 2\psi) = \frac{\Omega_0 r_0}{2(\Omega_0 - 1)}(1-\cos 2\psi) = \frac{ck^{1/2}\Omega_0\tau_0}{2(\Omega_0 - 1)^{3/2}}(1-\cos 2\psi) \qquad \text{(A.4a)}$$

(A.4b)

$$t = \frac{A^2}{2c^3k^{3/2}}(2\psi - \sin 2\psi) = \frac{\Omega_0 r_0}{2ck^{1/2}(\Omega_0 - 1)}(2\psi - \sin 2\psi) = \frac{\Omega_0\tau_0}{2(\Omega_0 - 1)^{3/2}}(2\psi - \sin 2\psi)$$

k < 0

$$r = \frac{A^2}{2c^2|k|}(\cosh 2\psi - 1) = \frac{\Omega_0 r_0}{2(1-\Omega_0)}(\cosh 2\psi - 1) = \frac{c|k|^{1/2}\Omega_0\tau_0}{2(1-\Omega_0)^{3/2}}(\cosh 2\psi - 1) \qquad \text{(A.5a)}$$

$$t = \frac{A^2}{2c^3|k|^{3/2}}(\sinh 2\psi - 2\psi) = \frac{\Omega_0 r_0}{2c|k|^{1/2}(1-\Omega_0)}(\sinh 2\psi - 2\psi) = \frac{\Omega_0\tau_0}{2(1-\Omega_0)^{3/2}}(\sinh 2\psi - 2\psi) \qquad \text{(A.5b)}$$

where $A^2 = \frac{8\pi}{3}G\rho_0 r_0^3 = 2GM_u$, and M_u is the mass within the radius r_0. [For the Newtonian model this is literally the mass of the universe, but great care is required in the relativistic case. When k < 0 it certainly is not the mass of the universe, since this is infinite]. The sub-script $_0$ on all quantities denotes evaluation at some arbitrary datum time ($t_0 > 0$).

We have retained the k-dependence in (A.4, A.5), rather than putting k = ±1, since the radial coordinate, r, then retains its physical meaning.

The qualitative evolution of the density parameter follows from (A.2) and (A.3). For k < 0, r increases monotonically with t. (This follows since $\dot{r}$ is easily shown to be positive). Hence, (A.2) shows that when $r \to 0$ we must have $\Omega \to 1$ from below, i.e. $\Omega = 1 - \delta$. But as $r \to \infty$ we must have $\Omega \to 0$ in order to keep (A.2) constant.

Similarly, for k < 0, although r reaches a maximum and then reduces, we find that τ becomes infinite at this point (because $\dot{r}$ is zero at the maximum of r). (A.2) shows that when $r \to 0$ we must have $\Omega \to 1$ from above, i.e. $\Omega = 1 + \delta$. But as $\tau \to \infty$ we must have $\Omega \to \infty$ in order to keep (A.3) constant.

Discussion of a Paradox

Suppose we have a universe with k < 0 and hence with Ω < 1, i.e. an open universe. Say, for sake of argument, that $\Omega = \Omega_0 = 0.5$ at some time t_0 when the size scale is r_0. Now suppose that by means of some magic we suddenly increase the density of this universe by a large factor, like x10. Thus, we have made Ω > 1 and turned the universe into a closed universe – or have we?

Actually the question is not well posed. There is insufficient information given about the nature of the change imposed to determine what sort of universe results. We are tacitly assuming that t_0 and r_0 are unaffected. The coefficient A^2 therefore increases by the same factor (call it f) by which the density is increased. But what about k? Surely we are free to declare *by fiat* that k is unchanged. But if k is unchanged, then the universe must remain open, since an open universe is defined as one for which k < 0. How can this be consistent with increasing the density? Note that we could increase the density by as much as we wish, such as by a factor of $f = 10^{100}$. This must guarantee that Ω > 1, mustn't it?

Actually, no. The resolution of the paradox is that when the density is increased, the critical density also increases. If one made the mistake of using an approximation such as $\rho_{crit} = 1/6\pi G t^2$ then one could be fooled into believing that the critical density must remain constant. But the fault lies in this approximation. The correct definitions, (A.1), together with the Friedmann equation, gives,

$$\rho_{crit} = \rho - \frac{3kc^2}{8\pi G r^2}; \text{ Hence, for } k < 0, \ \rho_{crit} = \rho + \frac{3|k|c^2}{8\pi G r^2} \qquad \text{(A.6a)}$$

Hence, for k < 0,

$$\Omega = \frac{\rho}{\rho + \frac{3|k|c^2}{8\pi G r^2}} = \frac{1}{1 + \frac{3|k|c^2}{8\pi G\rho r^2}} \qquad \text{(A.6b)}$$

From (A.6b) we see that all increasing ρ does is to drive Ω closer to unity – but it remains less than unity. The resolves the paradox for the case of fixed k.

However, we might specify some other physical condition, such as the requirement that the velocity, $\dot{r}$, remain unchanged when the density increases. In this case k must change, to k' say. The Friedmann equation before and after gives,

$$\dot{r}^2 = \frac{A^2}{r} - kc^2 = \frac{fA^2}{r} - k'c^2 \quad \text{hence } k'c^2 = kc^2 + \frac{(f-1)A^2}{r} \qquad \text{(A.7)}$$

Consequently, if we make f large enough, an initially open universe (k < 0) will indeed be changed into a closed universe (k' > 0). But there is no paradox now, because we have accepted that k has been forced to change.

What Conditions Suffice to Define a Cosmology?

We are assuming here that $\Lambda = p = 0$. Question: If we are given the density of a universe when its size scale has a specified value, does this suffice to determine if the universe is open or closed? The answer is "No". In (A.4a) or (A.5a), we know the coefficient A for the specified r, but this is not enough to find both k and ψ.

The same applies if we are given the density at a specified time. The subsequent evolution is not determined.

However, if both r and t are given when the density is specified, then this suffices to solve (A.4a,b) or (A5a,b) for k, and hence to determine the subsequent evolution. Note that if (A.4a,b) has a solution for k, then (A.5a,b) will have no possible solution, and vice-versa. There is thus a unique solution in this case. [This follows by considering the solution for k =0. If the specified r is greater than that for the k = 0 solution at the same t and for the same A, then the solution must have k > 0 and (A.4a,b) are the relevant solutions. Conversely, if the given r lies at smaller r than the k = o solution, then k < 0 and (A.5a,b) is the relevant solution].