Chapter 21 - Stellar Nucleosynthesis of Carbon and Oxygen: Derivation of Breit-Wigner Based Nuclear Reaction Rates and Introduction to the Hoyle Coincidence

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1. INTRODUCTION

In Chapter 13 we have studied in some detail how hydrogen undergoes nuclear fusion into helium-4 in the core of stars. In Chapter 17 we have described the sequence of events when the hydrogen in the core becomes exhausted. In this chapter we shall assume that conditions in the core, or in a shell around the core, have become conducive to helium fusion. This means temperatures in the region of 100 to 150 million K. The helium burning phase is much briefer than the main sequence (hydrogen burning) phase for low and medium mass stars, typically lasting for around 100 million years (Prialnik, P.152). For the numerical examples in this chapter we shall assume a density in the regions undergoing helium fusion of 3 x 10^6 kg/m^3. This was taken from Prialnik, Figure 8.6, for a star of seven solar masses. It is assumed that this is a reasonable density for a solar mass star also. Inevitably, the calculations in this Chapter are rough illustrations only.

The helium burning phase of a star's evolution is far more complicated than the hydrogen burning phase, as described in Chapter 17. Consequently it is harder to represent the nucleosynthesis during this phase in a few simple 'hand' calculations. Nevertheless, in this Chapter we shall attempt to demonstrate, albeit crudely, that published reaction rates under the assumed conditions of temperature and pressure are consistent with the description of the star's evolution given in Chapter 17. In particular, we shall be interested in the amount of carbon and oxygen produced, especially their relative abundances. We shall also be keen to establish that the nuclear power density predicted is consistent with the luminosity of the star – which during this phase is ascending the 'asymptotic' red giant branch. However, we shall not pursue nucleosynthesis beyond oxygen in this Chapter.

Of greatest interest, however, is a detailed examination of the Hoyle coincidence. This is the contention that the exact energy of a particular resonance of C12 is anthropically fined tuned to permit carbon production. Examination of this alleged coincidence will lead us to consider two things. Firstly we will need to consider the dependence of the rate of the triple alpha reaction on the energy level of this resonance. Secondly we shall then need to explore the sensitivity of this energy level to the strength of the strong nuclear force. In this Chapter we develop the theoretical framework for describing the reaction rates and their dependence on these resonance energies. A detailed examination of the implications for the Hoyle coincidence may be found in Chapter 8 of the "Critique of the Cosmic Coincidences".

2. The Reactions and The Resonances

The initial reaction in the helium fusion sequence is the so-called "triple alpha" reaction. This may be written $3\left({}^{4}_{2}\mathrm{He}\right) \rightarrow {}^{12}_{6}\mathrm{C} + \gamma$. This reaction is energetically possible since the binding energy of helium-4 is 28.2957 MeV, whereas that of carbon-12 is 92.1617 MeV, i.e. more than that of three helium-4 nuclei by 7.2747 MeV. However, the alert reader will have spotted that the reaction rate for a three-body collision will be negligibly small. Such a reaction, in effect, cannot happen

unless it proceeds via an intermediate state (so-called auto-catalysis). The obvious combination of two alpha particles is to produce ${}^{8}_{4}\mathrm{Be}$. This is indeed what happens, though beryllium-8 is highly unstable. There are three steps in the sequence that comprises the triple alpha reaction,

Reaction (10) $${}^{4}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{8}_{4}\mathrm{Be} \qquad (21.1.1)$$

Reaction (11) $${}^{4}_{2}\mathrm{He} + {}^{8}_{4}\mathrm{Be} \rightarrow {}^{12}_{6}\mathrm{C}^{*} \qquad (21.1.2)$$

$${}^{12}_{6}\mathrm{C}^{*} \rightarrow {}^{12}_{6}\mathrm{C} + \gamma \qquad (21.1.3)$$

The superscript * on the intermediate carbon-12 nucleus denotes a resonance (or excited state) of carbon-12. We will discuss this further below. For now consider the beryllium-8. Its binding energy is 56.4995 MeV. This is *less* than the binding energy of two alpha particles (which is 56.5913 MeV), by 91.8 keV. Consequently, reaction (10) is endothermic. This contrasts with the reactions in the ppI, ppII and ppIII hydrogen burning sequences, all of which are exothermic (see Chapter 13). This is the reason why reaction (10) does not occur appreciably below a temperature of about 100 million K, whereas other reactions involving two doubly charged nuclei contribute to the pp sequences at temperatures as low as 10 million K [i.e. reactions (3) and (4), which become active at ~10 MK and ~18 MK respectively, see Chapter 13]. Reaction (10) can only happen if the thermal energy of the two alpha particles exceeds 91.8 keV. Equating this energy to 1.5kT implies that most alpha particles will be sufficiently energetic to form beryllium-8 only at temperatures above ~700 MK. Hence, at 100 MK only a very small fraction of the alpha particles are sufficiently energetic – but just enough to make the reaction go. In contrast, at lower temperatures, sufficiently energetic alpha particles become so scarce that the reaction stops.

All A = 8 nuclei are unstable, including beryllium-8 which has a width of ~5eV. This corresponds to a half-life of $\tau = 1.3 \times 10^{-16}$ sec. Hence an equilibrium density of beryllium-8 will exist in which the rate of ${}^{8}_{4}\mathrm{Be}$ production is balanced by the rate of its decay. If the rate at which reaction (10) is producing ${}^{8}_{4}\mathrm{Be}$ is $R_{(10)}$ ($\mathrm{sec}^{-1}.\mathrm{cm}^{-3}$) then it is easily shown that the equilibrium density of beryllium-8 is $\tau R_{(10)} / \log_e(2) = \tau_0 R_{(10)}$ where $\tau_0 = 1.9 \times 10^{-16}$ sec. This provides a density of beryllium-8 to use to calculate the rate of the subsequent reaction (11).

Binary nuclear reaction rates, between nuclei a and b say, are conventionally expressed in the literature as the number of reactions per second per 'a' particle and per molar concentration of particle 'b'. Thus, tabulated values give r_{ab} in units of $\mathrm{sec}^{-1}.(\mathrm{mole}/\mathrm{cm}^3)^{-1}$. If the number density of particles a and b are written [a] and [b], in cm^{-3}, then the number of reactions per cm^3 per second is $r_{ab}[a][b]/A$, where A = Avogadro's number (6.03×10^{23}). Hence, $R_{(10)} = r_{(10)}[{}^{4}_{2}\mathrm{He}]^2/A$ and hence,

$$[{}^{8}_{4}\mathrm{Be}] = \tau_0 r_{(10)} [{}^{4}_{2}\mathrm{He}]^2 / A \qquad (21.1.4)$$

The rate at which carbon-12 nuclei are being produced, per second per cm^3, is,

$$r_{(11)}[{}^{4}_{2}\mathrm{He}][{}^{8}_{4}\mathrm{Be}] / A = \tau_0 r_{(10)} r_{(11)} [{}^{4}_{2}\mathrm{He}]^3 / A^2 \qquad (21.1.5)$$

in the obvious notation, $r_{(11)}$ denoting the tabulated rates of reaction (11) in $sec^{-1}.(mole/cm^3)^{-1}$. We labour this point rather so that it can be seen that the overall rate of carbon-12 production depends upon the helium-4 density cubed. Some tabulations give the carbon-12 production directly as if from a three-body reaction,

$$r_{(101)}[^4_2\mathrm{He}]^3 / A^2 \qquad (21.1.6)$$

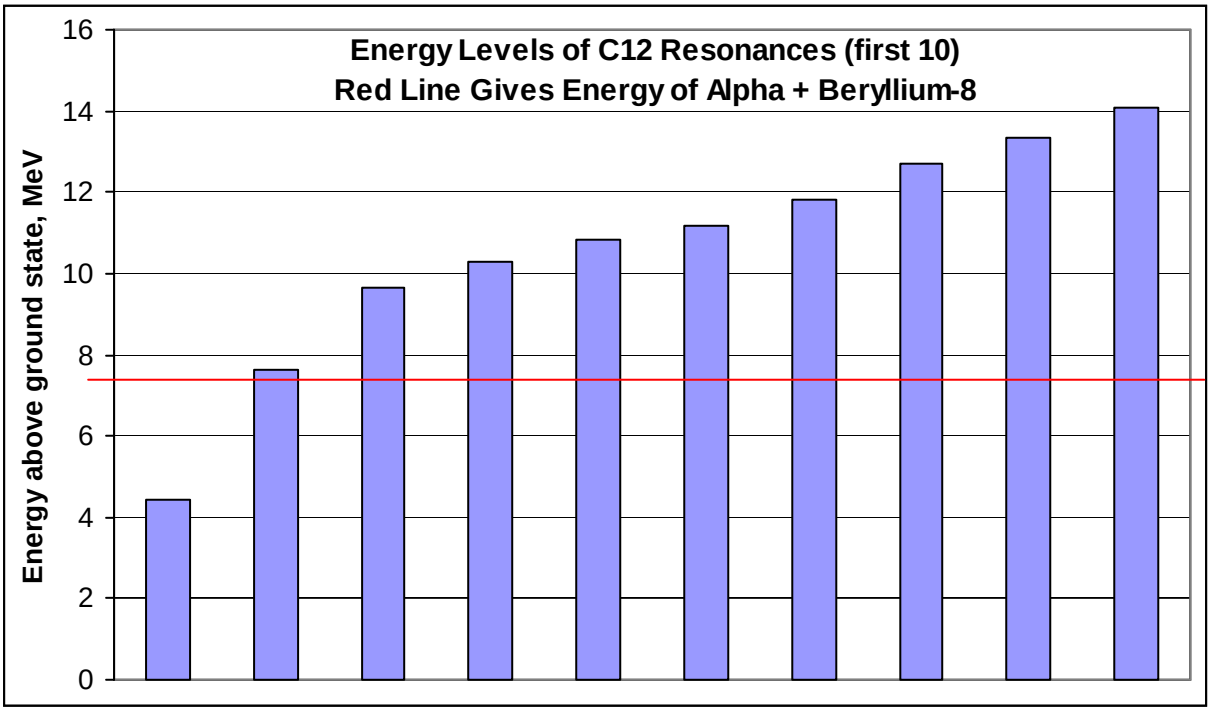
where, $r_{(101)} \equiv \tau_0 r_{(10)} r_{(11)}$. For example, CF88 give both $r_{(10)}$ and $r_{(11)}$, and also $r_{(101)}$, so this formulation can be checked. [NB: The decay of C* in (21.1.3) is so fast that it can be ignored]. Numerical values for these reaction rates will be given later. We now consider reaction (11) in more detail.

Ordinarily the fact that $^8_4\mathrm{Be}$ survives for only the order of 10^{-16} seconds would mean that there would be insufficient time for any subsequent reaction before this unstable nucleus it decayed[1]. Only an extraordinarily rapid subsequent reaction could make use of the fleeting appearance of $^8_4\mathrm{Be}$. The only reason why reaction (11) is sufficiently fast is that it occurs resonantly. That is to say, the energy of the excited state $^{12}_6\mathrm{C}^*$ coincides with that of an alpha particle plus a $^8_4\mathrm{Be}$ nucleus plus a realistically available amount of thermal energy. To put it another way, the binding energy of $^{12}_6\mathrm{C}^*$ is required to be *less* than the sum of the binding energies of $^8_4\mathrm{Be}$ and $^4_2\mathrm{He}$, but only slightly less so that the difference can be made up from the available ambient thermal energy. Reaction (11) is therefore endothermic. The binding energy of the ground state of carbon-12, $^{12}_6\mathrm{C}$, is 92.1617 MeV, whereas that of $^8_4\mathrm{Be}$ is 56.4995MeV and of $^4_2\mathrm{He}$ is 28.2957 MeV. Since the resonant capture of the alpha particle can only occur if the resonance, $^{12}_6\mathrm{C}^*$, has a slightly higher energy than the alpha-plus-beryllium (but only slightly), Hoyle was able to predict that there must be a resonance of carbon-12 just a little above 7.3666 MeV. [NB: This is the energy above the ground state of carbon-12]. The resonance was found quite quickly the same year (1954), following Hoyle's pestering of the experimentalists, at an energy of 7.6542 MeV. This leaves just 288 keV to be made up from the thermal energy.

Just how impressed should we be with Hoyle's prediction? An impression of the likelihood of a carbon-12 resonance of the required energy occurring can be gained by considering the spectrum of carbon-12 resonances, tabulated below,

[1] Recall that stellar nuclear reaction rates are often very slow, the lifetime of stars being very long. For example, the first reaction in the pp sequence takes ~10 Byrs under solar conditions. The second reaction, forming helium-3, is relatively fast, with a reaction time of just a few seconds, but this is still very slow compared with the lifetime of beryllium-8.

state of $^{12}_{6}C$	**energy above ground state, keV**
GS (0+)	0
2+	4.4389
0+	7.6542
3-	9.6415
0+	10.3
1-	10.8442
2+	11.1605
2-	11.8282
1+	12.7106
2-	13.3522
4+	14.0832



We will show in Chapter 8 of the "Critique of the Cosmic Coincidences" that if the $^{12}_{6}C^{*}$ resonance were much more than ~0.4 MeV above the alpha + beryllium threshold (the red line) then carbon production would be reduced to a very small fraction of its actual level. Only the second resonance (0^{+}) is correctly positioned in energy to result in significant carbon production. On the scale of the above histogram, the $^{12}_{6}C^{*}(0^{+})$ energy could barely be much greater without compromising carbon production. However, it cannot be much smaller without dropping below the red line, and again compromising carbon production. However likely is it that there is a carbon resonance positioned just-so? You can judge for yourself from the histogram.

We now look at the subsequent reactions following the formation of carbon. These are the capture of further alpha particles to form, firstly, oxygen-16 and then neon-20, i.e.,

Reaction (12) $${}^{4}_{2}\mathrm{He} + {}^{12}_{6}\mathrm{C} \rightarrow {}^{16}_{8}\mathrm{O}$$ (21.1.7)

Reaction (13) $${}^{4}_{2}\mathrm{He} + {}^{16}_{8}\mathrm{O} \rightarrow {}^{20}_{10}\mathrm{Ne}$$ (21.1.8)

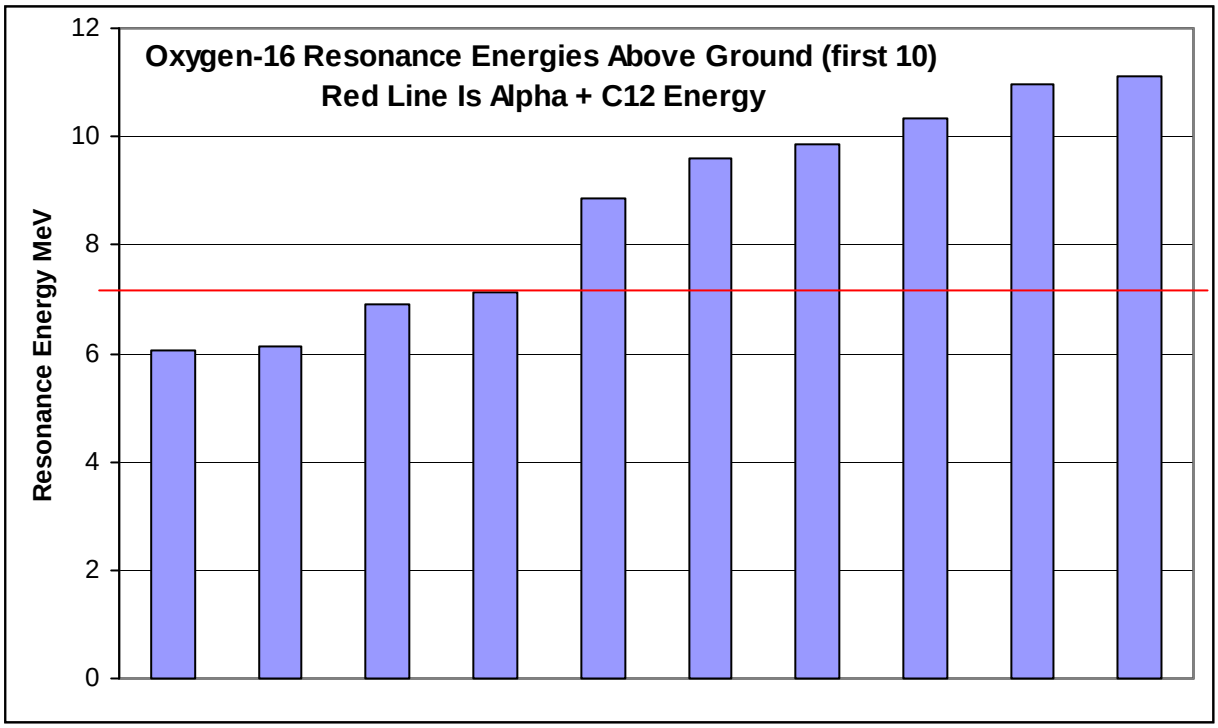
We will see below that at the temperatures and densities we use for illustration here, reaction (13) is very slow and no significant amounts of neon are formed[2].

Now if reaction (12) proceeded via a resonance, as in the case of carbon formation, it would be most unfortunate as regards the balance of elements required for conventional biochemistry. This is because reaction (12) would then be very fast and would mop up the carbon as quickly as it formed, transforming it into oxygen. However, this does not happen. There *is* a resonance of oxygen-16 which is close to the required energy – but it lies just *below* the sum of the alpha plus carbon-12 energies, and hence is non-resonant.

The binding energy of the ground state of oxygen-16 is 127.6193 MeV. This compares with the sum of the alpha and carbon-12 binding energies of 120.4574. Thus a resonant reaction would require a resonance at slightly above 7.1619 MeV with respect to the oxygen-16 ground state. The 1^- excited state of oxygen-16 has an energy of 7.1169 MeV, which is remarkably close – but is too low to produce a resonant reaction by just 43 keV. The spectrum of oxygen-16 is,

state of ${}^{16}_{8}\mathrm{O}$	**energy above ground state, keV**
GS	0
0+	6.0494
3-	6.1299
2+	6.9171
1-	7.1169
2-	8.8719
1-	9.5851
2+	9.8445
4+	10.3563
0-	10.9571
4+	11.0967

[2] This is odd because a substantial amount of neon is formed in stars. Neon is the sixth most common element in the cosmos (abundance order: H, He, O, C, N, Ne). Where does the neon come from? We shall not address this here.



In order to avoid prejudicing a good balance of carbon and oxygen production, it is necessary to avoid a resonant capture of alpha by carbon. To do this, the energy gap between the threshold energy (the red line) and the next (higher) O^{16} resonance must be as large as possible. The actual spectrum of O^{16} achieves this by having a very close miss with the 1- resonance, but which is *below* the threshold. How likely is this? Judge for yourself.

The contention of the anthropically inclined is as follows: the strong force is fine tuned to produce a resonance in carbon-12 just slightly above the sum of the energies of three alpha particles. The strong force is also tuned to provide beryllium-8 with an abnormally long lifetime (most unstable nuclei have a life of less than ~10^{-20} sec, whereas that of beryllium-8 is some 4 orders of magnitude longer). Finally, the strong force contrives to ensure that there is no oxygen-16 resonance at such an energy as to prejudice the survival of carbon by cooking it all quickly into oxygen. The fact that there is a resonance in oxygen-16 which is such a close miss in this respect is curious.

3. Published Reaction Rates and Crude Estimate of Carbon:Oxygen Ratio

An accurate estimate of the amount of carbon and oxygen formed by a star of a given mass requires a detailed stellar model. We have seen in Chapter 17 that the evolution of a star is complicated in the helium burning regime. This is so even under the simplifying assumptions of spherical symmetry and quasi-static conditions. Reality will be more complicated still. In this Section we make no pretence of accuracy. Rather we have the following more modest objectives,

(a) Are published reaction rates roughly consistent with the helium in the core being fully burnt in a time scale of about the right order?
(b) Does the resulting nuclear power agree with the luminosities suggested in Chapter 17?

(c) Most importantly, how much carbon and oxygen is produced? As a minimum we are keen to establish that substantial amounts of both carbon and oxygen are produced, as required for biochemistry.

To carry out this program we shall assume a star of solar mass. We shall assume the density of the regions undergoing helium burning (core or shell) to be 3×10^6 kg/m^3 (taken from Prialnik, Figure 8.6, for a star of seven solar masses but assumed indicative for a solar mass star). It is generally asserted that the helium burning phase lasts for around 100 million years (Prialnik, P.152), so it will be our objective to obtain consistency with this time scale. Finally, we shall assume temperatures in the helium burning regions of between 100 million K and 150 million K. This is again as quoted in many sources. Apparently this temperature is not sensitive to mass.

Reaction (10+11) $$r_{(101)}[{}^4_2\mathrm{He}]^3 / A^2 \qquad (21.1.6)$$

Reaction (12) $$ {}^4_2\mathrm{He} + {}^{12}_6\mathrm{C} \rightarrow {}^{16}_8\mathrm{O} \qquad (21.1.7)$$

Reaction (13) $$ {}^4_2\mathrm{He} + {}^{16}_8\mathrm{O} \rightarrow {}^{20}_{10}\mathrm{Ne} \qquad (21.1.8)$$

We introduce the following short-hand notation for the number density of the nuclei,

$$x = \left[{}^4_2\mathrm{He}\right], \quad y = \left[{}^{12}_6\mathrm{C}\right], \quad z = \left[{}^{16}_8\mathrm{O}\right], \quad u = \left[{}^{20}_{10}\mathrm{Ne}\right] \qquad (21.1.9)$$

Thus, (21.1.6) gives the rate of the triple alpha reaction, (10)+(11), to be,

$$\text{(10)+(11) Reaction Rate} = r_{(101)} x^3 / A^2 \qquad \mathrm{s^{-1}.cm^{-3}} \qquad (21.1.10)$$

and the rates of reactions (12) and (13) are, in similar notation,

$$\text{(12) Reaction Rate} = r_{(12)} xy / A \qquad (21.1.11)$$

$$\text{(13) Reaction Rate} = r_{(13)} xz / A \qquad (21.1.12)$$

where each $r_{(..)}$ denotes a reaction rate in $\mathrm{sec^{-1}.(mole/cm^3)^{-1}}$.

The rate of change of the number density for a given species is the difference between the rate at which such nuclei are being created and the rate at which they are being consumed. For example, carbon is created by the triple alpha reaction, (10)+(11), but consumed by reaction (12). Helium, on the other hand, is consumed by every reaction and not created (since hydrogen burning has ceased in the regions of interest here). We are therefore led to the following set of simultaneous differential equations,

$$\frac{dy}{dt} = r_{(101)} x^3 / A^2 - r_{(12)} xy / A \qquad (21.1.13)$$

$$\frac{dz}{dt} = r_{(12)} xy / A - r_{(13)} xz / A \qquad (21.1.14)$$

$$\frac{du}{dt} = r_{(13)} xz / A \qquad (21.1.15)$$

$$-\frac{dx}{dt} = 3r_{(101)} x^3 / A^2 + r_{(12)} xy / A + r_{(13)} xz / A \qquad (21.1.16)$$

The last equation follows from adding together 3 x RHS (21.1.13) plus 4 x RHS (21.1.14) plus 5 x RHS (21.1.15), because the creation of C12, O16 and Ne20 consume 3, 4 and 5 alpha particles respectively. The initial conditions are that the helium concentration starts from a known value x_0, whereas the other nuclei start from zero, $y_0 = z_0 = u_0 = 0$. For an assumed density of 3×10^6 kg/m^3, consisting of pure helium-4, we have $x_0 = 4.5 \times 10^{26}$ /cm^3.

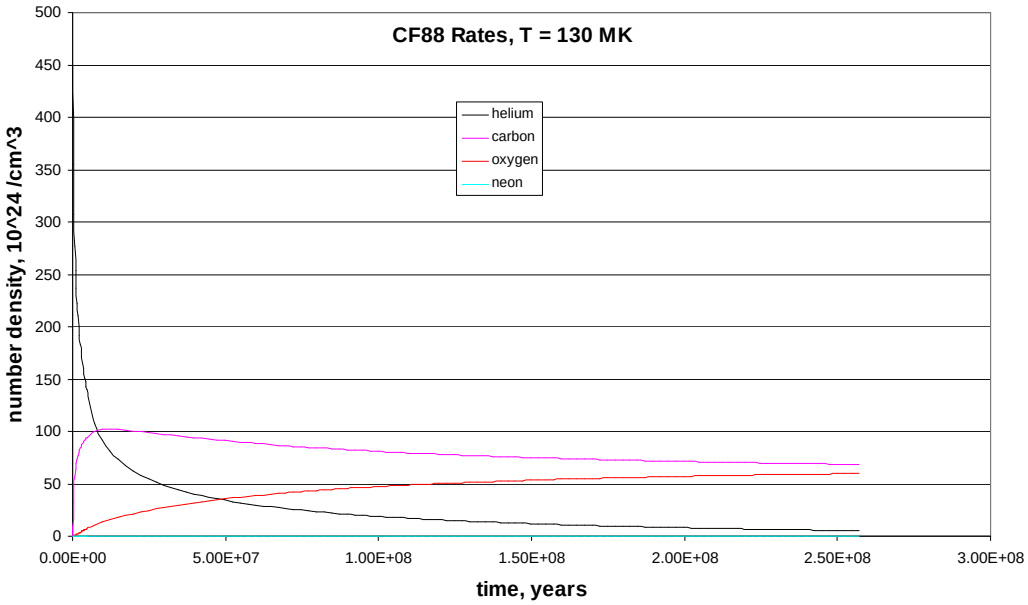
The equations (21.1.13-16) can be solved numerically by time-stepping. To do so we use the following reaction rates obtained from the CF88 (Caughlan & Fowler 1988) web site,

T (million K)	**$r_{(101)}$ $sec^{-1}.(mole/cm^3)^{-2}$**	**$r_{(12)}$ $sec^{-1}.(mole/cm^3)^{-1}$**	**$r_{(13)}$ $sec^{-1}.(mole/cm^3)^{-1}$**
100	1.36×10^{-24} 2.48×10^{-24}	9.23×10^{-21} 17.7×10^{-21}	2.74×10^{-27} 7.79×10^{-27}
110	5.62×10^{-23} 10.0×10^{-23}	6.90×10^{-20} 13.1×10^{-20}	3.74×10^{-26} 10.6×10^{-26}
120	1.21×10^{-21} 2.14×10^{-21}	4.08×10^{-19} 7.69×10^{-19}	3.78×10^{-25} 10.7×10^{-25}
130	1.61×10^{-20} 2.81×10^{-20}	1.99×10^{-18} 3.74×10^{-18}	2.99×10^{-24} 8.50×10^{-24}
140	1.44×10^{-19} 2.50×10^{-19}	8.30×10^{-18} 15.5×10^{-18}	1.93×10^{-23} 5.49×10^{-23}
150	9.55×10^{-19} 16.4×10^{-19}	3.03×10^{-17} 5.65×10^{-17}	1.05×10^{-22} 2.98×10^{-22}

The data in blue are from the NACRE site. The NACRE reaction rates are consistently a factor of 2 to 3 larger than the CF88 rates. This would lead to timescales for helium depletion which would be reduced by a similar factor. However, the relative abundances of the elements produced would be essentially unchanged, since these depend upon the relative rates only. We use only the CF88 data here. For future reference we record also the separate rates for reactions (10) and (11) separately (CF88 only) from which the rate $r_{(101)}$ was derived using $r_{(101)} \equiv \tau_0 r_{(10)} r_{(11)}$ together with $\tau_0 = 1.9 \times 10^{-16}$ sec.

T (million K)	**$r_{(11)}$ $sec^{-1}.(mole/cm^3)^{-1}$**	**$r_{(12)}$ $sec^{-1}.(mole/cm^3)^{-1}$**
100	547	1.33×10^{-11}
110	1,250	2.40×10^{-10}
120	2,460	2.63×10^{-9}
130	4,330	1.98×10^{-8}
140	6,950	1.11×10^{-7}
150	10,400	4.90×10^{-7}
200	40,000	8.27×10^{-5}

The result of numerical integration of equations (21.1.13-16) is illustrated in the graph below, which shows how the number densities of the various elements develop. The illustration is for an assumed temperature of 130 million K.



We observe the following from this graph. Firstly, the helium is close to exhaustion after a time of the order of a few hundreds of millions of years. (NB: Using the NACRE data this would therefore reduce to about 100 million years). This satisfies objective (a), above. Secondly, we see that when the helium is (nearly) exhausted, there are roughly similar amounts of carbon and oxygen produced. This satisfies objective (c), above.

It is important to emphasis how very uncertain this conclusion is, being based on a dramatically oversimplified picture. For example, considering other temperatures within the credible range changes the oxygen:carbon balance very markedly (see the Table below). At 100 MK the O:C ratio is 96%:4%, whereas at 150 MK it is 15%:85%, in contrast to the nearly 50%:50% mixture at 130 MK. However, the timescale for the reactions to be (nearly) complete vary hugely. At 100 MK we have to wait an unrealistically long 85 billion years, several times longer than the entire lifetime of the star and 2 or 3 orders of magnitude too long for the helium burning phase. At 150 MK the helium burning is essentially over in just 2.5 million years, this time two orders of magnitude too quick. Thus, we have chosen 130 MK because it

reproduces the accepted helium-burning timescale. In this sense the near equality of the oxygen and carbon abundances is confirmed.

The reaction rates become very slow, but not zero, when the helium density falls to a sufficiently low level. However, unrealistically long times would be required to reduce the helium level to virtually zero. Consequently we have truncated the integration at a time when the graphs against time appear reasonably flat. There is a degree of arbitrariness about this time, although it should correspond to the bulk of the helium having been used up. Consequently we have included the assumed 'end' time, together with the remaining % of helium, in the Table below.

Temp (MK)	**t_{end} (Myrs)**	**Reaction Rates**	**factor applied to triple alpha**	**Helium left as % of start**	**O:C Ratio**
100	85,750	CF88	1	22.6%	24
100	85,750	CF88	10	4.1%	5.1
100	42,875	CF88	100	1.1%	0.54
110	8,575	CF88	1	8.8%	6.9
110	857	CF88	10	11.6%	0.29
110	257	CF88	100	7.3%	0.051
110	857	CF88	100	3.5%	0.10
120	42,875	CF88	0.01	11.4%	750
120	8,575	CF88	0.1	7.6%	149
120	2,570	CF88	1	1.0%	2.61
120	257	CF88	10	3.8%	0.22
120	25.7	CF88	100	5.2%	0.022
130	8,575	CF88	0.01	7.0%	759
130	1,286	CF88	0.1	3.9%	63
130	257	CF88	1	1.1%	0.87
130	25.7	CF88	10	3.8%	0.094
130	2.57	CF88	100	4.6%	0.0095
140	257	CF88	0.1	0.9%	7.8
140	25.7	CF88	1	2.5%	0.40
150	257	CF88	0.01	5.1%	349
150	43	CF88	0.1	0.5%	2.6
150	2.57	CF88	1	4.7%	0.18

blue = this universe (factor = 1)

The results in the Table show the effect of varying the triple-alpha reaction rates by factors of x0.01, x0.1, x10 and x100, whilst keeping the rates of reactions (12) and (13) unchanged. Consider firstly our 'best-estimate' temperature of 130 MK.

Unsurprisingly, the effect on the oxygen:carbon balance is dramatic. Increasing the rate of the triple alpha reaction by x10 leads to a reduction in the amount of oxygen by about the same factor. This, of course, is simply because the helium becomes

exhausted after only one-tenth of the time – hence giving little time for reaction (12) to create oxygen. Increasing the triple alpha reaction rate by x100 results in virtually no oxygen being formed (O:C = 1%:99%). This departs dramatically from the observed cosmic abundances in which oxygen is about three times more abundant than carbon. Conversely, if the rate of the triple alpha reaction is reduced, the amount of carbon produced is low. A reduction by a factor of ten reduces the carbon:oxygen ratio to just 1.5%:98.5%. A reduction by a factor of 100 reduces it further to around 0.1%:99.9%.

These observations are sufficient to illustrate the sensitivity of the cosmic abundances of carbon and oxygen to the assumed reaction rates. We have, of course, been varying the reaction rates by orders of magnitude. We shall see below that this is reasonable in view of the great sensitivity of the reaction rates to temperature and resonance energies.

Nevertheless, there is an inconsistency in this illustration. We have implicitly assumed that the temperature remains at ~130 MK when we change the triple alpha reaction rate by orders of magnitude. This is unlikely since it would correspond to orders of magnitude change in the nuclear power densities, and this would cease to be consistent with the assumed temperature and prevailing heat transfer mechanisms. To a rough, but better, approximation we could assume that the temperature adjusts itself so that the nuclear power density remains the same. Thus a slower reaction rate will result in a higher temperature, and vice-versa. As a consequence, we may expect the helium burning time scale to be roughly unchanged. We may thus pick out results from the above Table which have time scales in the range $10^8 – 10^9$ years:-

Temp (MK)	t_{end} (Myrs)	Reaction Rates	factor applied to triple alpha	Helium left as % of start	O:C Ratio
110	257	CF88	100	7.3%	0.051
120	257	CF88	10	3.8%	0.22
130	257	CF88	1	1.1%	0.87
140	257	CF88	0.1	0.9%	7.8
150	257	CF88	0.01	5.1%	349

blue = this universe (factor = 1)

This is a better illustration than before, having contrived a reasonable timescale in each case. The outcome is qualitatively unchanged. For example, for a slower triple alpha reaction rate we consider higher temperatures so that the rate of C12 formation is unchanged. However, this higher temperature leads to a much faster reaction (12) so that oxygen is formed more rapidly, leading to a much larger oxygen:carbon ratio. The reverse applies for faster triple alpha reaction rates.

In conclusion, and as nearly as we can judge from such crude considerations, varying the triple alpha reaction rate whilst keeping the other reaction rates unchanged will have a marked effect on the cosmic oxygen:carbon balance. A factor of ten or one hundred reduction in the triple alpha reaction rate will reduced the carbon abundance by roughly the same factor. A factor of one hundred increase in the triple alpha reaction rate will reduced the oxygen abundance to ~5% of its value in this universe.

Luminosity

Before closing this Section we have yet to examine the luminosity. Actually we do not need to consider the reaction rates directly to do this. We need only assume a timescale of the order of 100 million years, together with an assumption on the total mass of helium fused over this time. From Chapter 17, the helium becomes exhausted in a solar mass star up to a mass coordinate of at least 0.2, and perhaps up to 0.4. Assuming 0.2 for now, the mass of helium fused is $0.2 \times 2 \times 10^{30}$ kg. Assuming the helium ends up as 50% carbon and 50% oxygen, this corresponds to a total of 3.4×10^{55} of each of these nuclei being produced. The binding energy released as heat is 7.2747 MeV per C^{12} and 14.4365 MeV per O^{16}. Hence the total heat produced is 1.2×10^{44} J. Dividing by 10^8 years = 3.15×10^{15} sec gives a mean power (luminosity) of 3.8×10^{28} W. Since solar luminosity is 4.1×10^{26} W this is 92 times solar luminosity. If we had assumed helium fusion up to a mass coordinate of 0.4, this figure would be doubled to $L/L_{solar} = 184$.

In Chapter 17 we saw that, at point F, shortly after the helium flashes, the luminosity is about $L/L_{solar} = 10^{1.7} = 50$. As the star climbs the asymptotic giant branch the luminosity rises, possibly as high as $L/L_{solar} \sim 1000$. Hence, our estimate of the *average* luminosity of $L/L_{solar} \approx 90 - 180$ appears reasonable. Note that if a star spends only a brief time at the highest luminosities, these will contribute little to the average luminosity. This would appear to be the case.

4. Theoretical Understanding of the Reaction Rates – Dependence on $E_{resonance}$

In this Section our primary aim is to determine how sensitive the triple alpha reaction rate is to the exact energy of the C^{12} 0_2^+ resonance. How much would this resonance energy have to be changed to markedly affect the reaction rate? From the preceding Section we know that a factor of 10 change in the reaction rate would be significant, and a factor of 100 change probably fatal to life in the universe. But what change in the resonance energy would cause a factor of 10 or 100 change in the reaction rate? This is deduced here from the Breit-Wigner formula for resonance cross-sections, together with the Maxwell distribution.

Since the ratio of oxygen to carbon which is formed in stars depends upon the relative rates of reactions (11) and (12), we are also interested in how sensitive reaction (12), ${}_2^4\mathrm{He} + {}_6^{12}\mathrm{C} \rightarrow {}_8^{16}\mathrm{O}$, is to small changes in the energy of the first 1^- resonance of O^{16}. Whilst this reaction is not resonant, the reaction rate may be determined by the 'tail' of the Breit-Wigner formula. We investigate if this is so below.

Before turning to the Breit-Wigner based approaches, however, we see how far we can get using the expressions derived on the basis of Schrodinger states for the formation of ${}_1^2\mathrm{D}$ and ${}_2^3\mathrm{He}$. We take the opportunity of using this approach for the relevant reactions in the pp sequences before looking also at the helium burning reactions.

4.1 Estimates for pp Reactions Based on p-D Analogy

In Appendix A3 we derived the cross-section for the reaction $\mathrm{p} + \mathrm{D} \rightarrow {}_2^3\mathrm{He} + \gamma$. If we turn off the Coulomb interaction, and modulo a possibly different numerical factor to account for spin states, the cross section is essentially,

$$\sigma_{cap} = \frac{8}{9}\pi\alpha\left(\frac{\hbar}{m_R c}\right)^2 \frac{\sqrt{E.\Delta B}}{E+\Delta B} \qquad (21.4.1.1)$$

We have written this so as to apply, by analogy, to a reaction $a + b \rightarrow c + \gamma$, where 'c' is the nucleus comprising 'a' captured on 'b'. Thus m_R is the reduced mass of a and b,

$$m_R = \frac{M_a M_b}{M_a + M_b} \qquad (21.4.1.2)$$

and ΔB is the difference between the binding energies of the final and initial states, i.e.,

$$\Delta B = B_c - (B_a + B_c) \qquad (21.4.1.3)$$

We can only apply the analogy to reactions like $a + b \rightarrow c + \gamma$ because an electro-strong interaction is implicit in the derivation of (21.4.1.1). This is because the derivation in Appendix A3 assumes Schrodinger wave-functions based on a strong nuclear potential together with an interaction between the reactants based on the electromagnetic field. The presence of α in (21.4.1.1) makes it clear that it is only applicable to a reaction resulting in photon emission.

More precisely, (21.4.1.1) is based on an assumed electric dipole interaction occurring between a free triplet P state and a bound triplet S state. For the nuclei we consider below, the initial and final states may differ from these, so our analogy is only meant to be an extremely crude guestimate. We know, however, from proton-neutron capture cross sections, that the magnetic dipole interaction can produce a far larger cross sections than the electric dipole interaction. The difference is most marked at low energies. The p-n capture cross sections can be written approximately as,

$$\sigma_{cap}^{M} = 0.8\frac{B^{3/2}}{M_p c^2 \cdot \sqrt{E}}\ (\text{fm}^2) \quad \text{and} \quad \sigma_{cap}^{E} = 2.8\frac{\sqrt{BE}}{M_p c^2}\ (\text{fm}^2) \qquad (21.4.1.4)$$

Derived from these expressions, as a crude rule of thumb, we shall estimate the magnetic dipole cross sections to be $0.28\Delta B/E$ times the electric dipole cross section given by (21.4.1.1). Furthermore, we shall replace 'E' by kT. Actually it would be better to use the Gamow peak energy in place of 'E'. This would result in estimates of the magnetic dipole cross-sections an order of magnitude or more smaller. Our estimate is again extremely crude. The n-p magnetic cross-section is based on a free singlet S state and a bound triplet S state. In the reactions below, different states may be involved. Moreover, the n-p magnetic cross-section is proportional to $(\mu_p - \mu_n)^2$. The gyromagnetic ratios for the nuclei (a and b) considered below may be very different. Since many nucleons are involved in the heavier nuclei, there is the potential for the gyromagnetic ratios to be larger. So again we emphasise that this is only a very crude guestimate.

However, the most important parts of the estimate of the reaction rates for $a + b \rightarrow c + \gamma$ are,

- The Coulomb barrier factor, and,
- The Maxwell distribution of energies.

Chapter 14 has derived the integrated effect of these two factors (the Gamow peak). Thus our estimate for the rate of reaction $a + b \rightarrow c + \gamma$ [in $s^{-1}.(mole/cm^3)^{-1}$] is,

$$R[T] = R_0 \tilde{b}^{7/3} \exp\{-f_{min}\} \quad \text{where,} \quad \tilde{b} = \frac{b}{\sqrt{kT}} \quad \text{and} \quad f_{min} = 3\tilde{b}^{2/3} \qquad (21.4.1.5)$$

(noting that we have replaced the b/2 in Chapter 14 with b here). And,

$$R_0 = \frac{8\sqrt{2}}{9}\pi\alpha Ac \cdot \left(\frac{\hbar}{m_R c}\right)^2 \frac{kT}{\sqrt{m_R c^2 \cdot \Delta B}} \qquad (21.4.1.6)$$

where A = 6.03 x 10^{29} (Avogadro's number, i.e. the number of particles per m^3 if the density is 1 mole/cm^3). In deriving (21.4.1.6) we have assumed E << B. As in Chapter 14, the Gamow peak energy, resulting in the minimum of the 'f' function, and hence the maximum in the reaction rate (i.e. the cross-section) is given by,

$$E_0 = (bkT)^{2/3} = (\pi Z_a Z_b \alpha)^{2/3} (kT)^{2/3} (2m_R c^2)^{1/3} \qquad (21.4.1.7)$$

noting that,

$$b = \frac{\pi}{2} Z_a Z_b \alpha \sqrt{2m_R c^2} \qquad (21.4.1.8)$$

The prescription (21.4.1.5) for the Gamow peak factor is consistent with the energy dependence of the electric dipole interaction and the implicitly assumed ^{3}P(free) - ^{3}S(bound) transition. Other states/interactions may have different energy dependencies, and this should really be reflected in a modified Gamow peak integral. We are deliberately avoiding this added complexity, but it may lead to serious error.

We now apply (21.4.1.5-8) to the relevant pp reactions, and associated reactions:-

$p + D \rightarrow {}^{3}_{2}He + \gamma$

We find,

$m_R = 0.667M_p = 625.5$ MeV $b = 0.406$ √MeV $\Delta B = 5.494$ MeV

$R^E_0 = 8830kT(\text{in MeV})\ s^{-1}.(mole/cm^3)^{-1}$

$R^M_0 = 0.28\frac{\Delta B}{kT}R^E_0 = 13{,}600\ s^{-1}.(mole/cm^3)^{-1}$

Reaction (2)

T (10^6 K)	$\tilde{b}$	**f_{min}**	$R^E[T]$	$R^M[T]$ $E \approx kT$	$R^M[T]$ $E \approx E_0$	**NACRE data**
14	11.7	15.43	0.0007	0.8	0.009	0.0073
30	7.97	11.97	0.018	11	0.023	0.21
50	6.17	10.10	0.11	39	0.2	1.1
100	4.37	8.01	0.8	140	0.4	5.32
300	2.52	5.55	7.7	460	0.7	13.3

The R^E estimate of rate is too small by a factor of 10. The R^M estimate of rate is too big by a factor of 100. The use of E_0 rather than kT in the R^M formula gives a better answer at lower temperatures but then becomes the poorest estimate. Overall I would tend to conclude that this is probably an electric dipole interaction (blue results), with the factor of 10 being accounted for by a difference between our estimation approach for the matrix-element-with-Coulomb interaction compared with an accurate numerical solution of the Schrodinger equation with the Coulomb potential present. A difference between these approaches of a factor of 10 was seen previously in our estimate of the pp cross section (for increased g_s).

$p + {}^{3}_{1}H \rightarrow {}^{4}_{2}He + \gamma$

We find,

$m_R = 0.75M_p = 703.7$ MeV $b = 0.430$ √MeV $\Delta B = 19.814$ MeV

$R^E_0 = 3{,}460kT(\text{in MeV})\ s^{-1}.(mole/cm^3)^{-1}$

$R^M_0 = 0.28\frac{\Delta B}{kT}R^E_0 = 19{,}200\ s^{-1}.(mole/cm^3)^{-1}$

Reaction (g)

T (10^6 K)	$\tilde{b}$	**f_{min}**	$R^E[T]$	$R^M[T]$ $E \approx kT$	$R^M[T]$ $E \approx E_0$	**CF88 data**
14	12.38	16.05	0.00016	0.73	0.086	0.046
30	8.45	12.45	0.005	11	1.7	1.1
50	6.55	10.50	0.033	42	7.6	6.0
100						
300						

In this case the magnetic dipole interaction, using the Gamow peak energy, looks to be a very good result.

$^{2}_{1}D + {}^{2}_{1}D \rightarrow {}^{4}_{2}He + \gamma$

We find,

$m_R = M_p = 938.27$ MeV $b = 0.497\ \sqrt{MeV}$ $\Delta B = 23.848$ MeV

$R^{E}_{0} = 1{,}540kT(\text{in MeV})\ s^{-1}.(mole/cm^3)^{-1}$

$R^{M}_{0} = 0.19\frac{\Delta B}{kT}R^{E}_{0} = 7{,}000\ s^{-1}.(mole/cm^3)^{-1}$

Reaction (a)

T (10^6 K)	$\tilde{b}$	**f_{min}**	$R^{E}[T]$	$R^{M}[T]$ $E \approx 1.5kT$	$R^{M}[T]$ $E \approx E_0$	**NACRE data**
10	16.91	19.76	2.5×10^{-6}	0.013		1.4×10^{-6}
14	14.29	17.67	2.0×10^{-5}	0.07		7.7×10^{-6}
30	9.76	13.70	9.1×10^{-4}	1.6		2.8×10^{-4}
50	7.56	11.60	5.2×10^{-3}	5.5		1.7×10^{-3}
100						
300						

In this case the electric dipole interaction looks to be a good result (within a factor of 3). The magnetic dipole cross-section would be expected to be zero for identical reactants (because it is proportion to the square of the difference between the magnetic moments of the two reactants).

$^{3}_{2}He + {}^{4}_{2}He \rightarrow {}^{7}_{4}Be + \gamma$

We find,

$m_R = 1.714M_p = 1608.5$ MeV $b = 2.601\ \sqrt{MeV}$ $\Delta B = 1.586$ MeV

$R^{E}_{0} = 1{,}550kT(\text{in MeV})\ s^{-1}.(mole/cm^3)^{-1}$

$R^{M}_{0} = 0.19\frac{\Delta B}{kT}R^{E}_{0} = 467\ s^{-1}.(mole/cm^3)^{-1}$

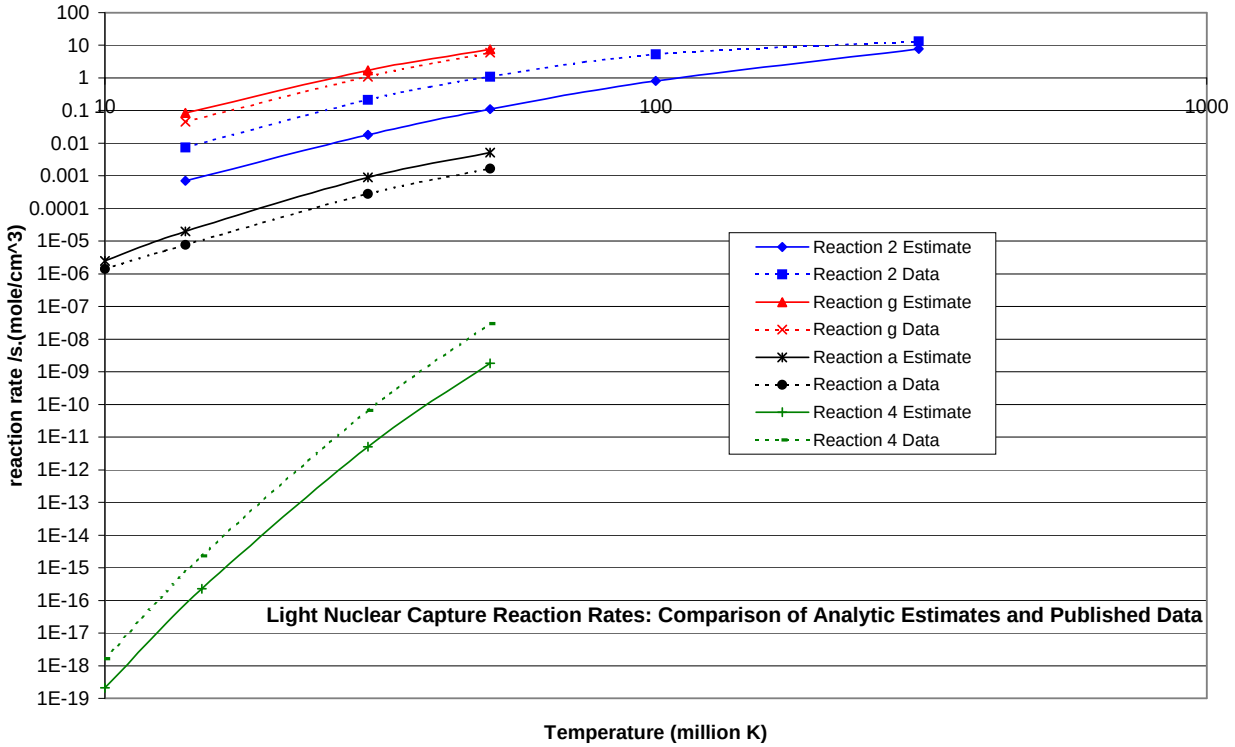
The double charges on the reactants lead to a much larger 'b' than the preceding reactions, and hence we expect far slower reaction rates, as below.

Reaction (4)

T (10^6 K)	$\tilde{b}$	**f_{min}**	$R^{E}[T]$	$R^{M}[T]$ $E \approx 1.5kT$	**NACRE data**
10	88.55	59.6	6.1×10^{-22}	2.1×10^{-19}	1.63×10^{-18}
15	72.42	52.1	1.0×10^{-18}	2.3×10^{-16}	2.31×10^{-15}
30	51.13	41.3	4.4×10^{-14}	5.1×10^{-12}	6.61×10^{-11}
50	39.61	34.9	2.6×10^{-11}	1.8×10^{-9}	3.0×10^{-8}
100					
300					

Both estimates are too low, but the magnetic dipole is far closer (about a factor of 10 too small), so this is probably a magnetic dipole interaction.

So far the examples we have considered using the "p-D" analogue have given reasonable results in comparison with the published data – within an order of magnitude or so if the better of the E or M interactions is adopted. The results so far are plotted below. The agreement looks quite impressive – on a logarithmic scale!



Before closing this Section, in passing we note that the reaction ${}^{2}_{1}\mathrm{D} + {}^{2}_{1}\mathrm{D} \rightarrow {}^{3}_{1}\mathrm{H} + \mathrm{p}$ would be expected to be fast – for two reasons. Firstly it is a purely strong reaction, i.e. no gamma is produced, so no α factor would be expected in the cross section formula. Secondly, no Coulomb barrier is penetrated. There is no particle of higher charge in the final state than at the start. Consistent with this expectation we find the rate at 15 MK is 123 $s^{-1}.(mole/cm^3)^{-1}$, far faster than any of the above reactions.

4.2 Estimates for Helium Reactions Based on p-D Analogy

$$\mathrm{p} + {}^{7}_{3}\mathrm{Li} \rightarrow {}^{8}_{4}\mathrm{Be} + \gamma\,; \qquad {}^{8}_{4}\mathrm{Be} \rightarrow {}^{4}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} + \gamma$$

We find,

$m_R = 0.875 M_p = 821.0$ MeV $b = 1.394 \sqrt{\text{MeV}}$ $\Delta B = 17.255$ MeV

$$R^{E}_{0} = 2{,}500 kT(\text{in MeV})\ s^{-1}.(mole/cm^3)^{-1}$$

$$R^{M}_{0} = 0.19 \frac{\Delta B}{kT} R^{E}_{0} = 8{,}280\ s^{-1}.(mole/cm^3)^{-1}$$

Reaction (6)

T (10^6 K)	$\tilde{b}$	**f_{min}**	$R^E[T]$	$R^M[T]$ $E \approx 1.5kT$	**NACRE data**
15	38.8	34.4	2E-11	5E-8	1.44E-5
30	27.4	27.3	2E-8	2.5E-5	0.011
50	21.2	23.0	1.4E-6	1.1E-3	0.573

This reaction has a smaller 'b' than that of reaction (4), but a larger 'b' than for the earlier reactions. We would therefore expect a much faster reaction rate than that of reaction (4) – which is true - but a slower reaction rate than the earlier reactions. It turns out that the rate of reaction (6) is anomalously fast (by at least a factor of 1000, see the above Table). Could it be that the proton capture is resonant? There is a 1^+ resonance of ^{8}Be at ~17.640 MeV. It turns out that it is not – the value of E_R = 385 keV is too large to give a significant resonant contribution to this reaction at temperatures of the order of a few tens of million K. However, we shall see below that the non-resonant Breit-Wigner cross-section will give a good estimate of the reaction rate based on the 2_3^+ resonance of ^{8}Be at 16.922 MeV, i.e. just *below* the $p + {}^7_3\mathrm{Li}$ threshold.

$${}^4_2\mathrm{He} + {}^{12}_6\mathrm{C} \rightarrow {}^{16}_8\mathrm{O} + \gamma$$

We find,

$m_R = 3M_p = 2815$ MeV $b = 10.323\ \sqrt{\text{MeV}}$ $\Delta B = 7.162$ MeV

$R_0^E = 180kT(\text{in MeV})\ s^{-1}.(\text{mole/cm}^3)^{-1}$

$$R_0^M = 0.19\frac{\Delta B}{kT}R_0^E = 245\ s^{-1}.(\text{mole/cm}^3)^{-1}$$

The multiple charges on the reactants lead to a far larger 'b' even than reactions (4) and (6). The reaction rate in the temperature range considered previously (10 – 50 million K) is therefore essentially nil. These helium-burning reactions occur at around 100 MK and above.

Reaction (12)

T (10^6 K)	$\tilde{b}$	**f_{min}**	$R^E[T]$	$R^M[T]$ $E \approx 1.5kT$	**CF88 data**
100	111.1	69.34	7E-26	1E-23	9.2E-21
120	101.4	65.24	4E-24	5E-22	4.1E-19
150	90.7	60.57	4E-22	4E-20	3.0E-17
200	78.6	55.0	1E-19	8E-18	4.7E-15

Once again we see that even the magnetic dipole interaction underestimates the reaction rate by 3 orders of magnitude. We will investigate this reaction in an alternative manner using the (non-resonant) Breit-Wigner approach below. However, in the context of this methodology we note that an explanation of the underestimate that could be offered is that we are overestimating the size of the Coulomb barrier for the increasingly large target nuclei. Implicitly, in this case, we are assuming a point

charge of 6e on the target carbon nucleus. In reality the carbon nucleus is an extended object. It might be that the alpha particle effectively remains outside the Coulomb barrier of the protons on the far side of the carbon nucleus. Thus, it only has to penetrate the Coulomb barrier of a subset of the 6 protons. This would mean that the effective value of 'b' would be smaller. Such is the sensitivity to 'b' that this could easily account for the difference. In fact, reducing the effective charge on the carbon nucleus from 6 to just 5 results in good agreement with the CF88 data.

$$ {}^{4}_{2}\mathrm{He} + {}^{8}_{4}\mathrm{Be} \rightarrow {}^{12}_{6}\mathrm{C} + \gamma $$

This is the last reaction that we will consider using the "p-D" analogy approach. We know very well that it is actually resonant, and hence we expect that the p-D method will grossly underestimate its rate. It is important to establish that this is true, however, since otherwise the importance of the 0_2^+ Hoyle resonance of C^{12} will not have been demonstrated. We find,

$m_R = 2.67M_p = 2502$ MeV $b = 6.489\ \sqrt{\text{MeV}}$ $\Delta B = 7.367$ MeV

$$R_0^E = 238kT(\text{in MeV})\ s^{-1}.(\text{mole/cm}^3)^{-1}$$

$$R_0^M = 0.19\frac{\Delta B}{kT}R_0^E = 334\ s^{-1}.(\text{mole/cm}^3)^{-1}$$

Reaction (11)

T (10^6 K)	$\tilde{b}$	**f_{min}**	$R^E[T]$	$R^M[T]$ $E \approx 1.5kT$	**CF88 data**
100	69.85	50.88		5E-16	1.33E-11
120	63.75	47.88		9E-15	2.63E-9
150	57.02	44.44		2E-13	4.90E-7
200	49.39	40.38		9E-12	8.27E-5

4.3 Breit-Wigner Resonance Cross-Section: The Triple Alpha Reactions

Firstly we develop the general treatment of a resonant reaction. A resonant reaction proceeds via an unstable intermediate state, or resonance: $a + b \rightarrow R \rightarrow c + d$. We shall assume that 'a' is the projectile particle (e.g. a proton or an alpha particle) and 'b' is a target nucleus (probably stationary in the laboratory frame in an experiment). Similarly, c is the resulting 'large' nucleus, and 'd' the relatively light particle, e.g. a photon. A reaction can only proceed via a resonance if the mass of the resonance is slightly higher than the sum of the masses of a and b. (And, obviously, it can only decay into the exit channel if it is more massive than c plus d). However, the extent of the difference, which we will call the resonance energy, $E_R = \left[M_R - (M_a + M_b)\right]c^2$, must be sufficiently small that it can be supplied as thermal energy, otherwise the resonance will not be formed. For particles a + b with combined CoM kinetic energy exactly tuned to the resonance, the cross section for its formation can be extremely large – many orders of magnitude larger than the a + b capture cross-section would be normally. Consequently, even when E_R is many times bigger than the average thermal energy (1.5kT), the resonant contribution *may* still dominate. Whether or not this is

the case depends upon the overlap between the tails of the Maxwell distribution and the resonant (Breit-Wigner) cross-section.

Warning: Confusion can occur because the resonance energy can also be defined in a different way. R will generally be an excited state of some nucleus X. Resonance energies are usually tabulated with respect to the ground state of X. Call this E'_R. The two definitions will differ by an amount equal to the difference between the binding energy of X and the sum of the binding energies of a and b. Thus we have,

$$E_R = E'_R - \left[B_X - \left(B_a + B_b\right)\right] \qquad (21.4.3.1)$$

We state without proof the Breit-Wigner equation for the cross section of the resonant reaction $a + b \rightarrow R \rightarrow c + d$. It is,

$$\sigma_{ab\rightarrow cd} = \frac{\pi}{k^2} \cdot \frac{\Gamma_{ab}\Gamma_{cd}}{\left(E_R - E\right)^2 + \left(\Gamma/2\right)^2} \qquad (21.4.3.2)$$

where k is the wavevector of a + b in the CoM system, and E their combined energy, such that $\hbar k = \sqrt{2m_R E}$, where m_R is their reduced mass. Γ is the width of the resonance, i.e. $\hbar/\Gamma$ is the half-life of the resonance. Γ_{ab} is the partial width for decay of the resonance back into the initial state, a + b. Γ_{cd} is the partial width for decay of the resonance into the exit channel, c + d. If no other decay channels occur then $\Gamma = \Gamma_{ab} + \Gamma_{cd}$, but if there are other active decay channels then $\Gamma > \Gamma_{ab} + \Gamma_{cd}$.

In what follows we shall assume a single, isolated, narrow resonance dominates. The requirement to be narrow means that $\Gamma << E_R$. It follows that the Briet-Wigner resonant cross-section is extremely sharply peaked, almost a delta-function but with a finite peak value at $E = E_R$ of,

$$\sigma^{peak}_{ab\rightarrow cd} = \frac{4\pi}{k^2} \cdot \frac{\Gamma_{ab}\Gamma_{cd}}{\Gamma^2} \qquad (21.4.3.3)$$

which is larger than the cross section at energies small compared with E_R by a factor of about $\left(E_R/\Gamma\right)^2$.

If the resonance lay at an energy typical of available thermal energies, i.e. if $E_R \sim 1.5kT$, then the cross section would be roughly as given by the peak, (21.4.3.3). We would then be "bang on the resonance". However, resonance energies with respect to the ground state (that is E'_R) typically range from a few MeV up to around 30 MeV or so (as we have seen above for C^{12} and O^{16}, though we did not list the highest levels). On the other hand, typical thermal energies are only ~0.02 MeV even at 150 million K. It is very improbable that the binding energies would contrive to almost exactly cancel with E'_R, thus ensuring that E_R, as given by Equ.(21.4.3.1), would be only 0.02 MeV. In general we will have $E_R >> kT$. Consequently only a very small proportion of the particles will have sufficient energy to create the resonance.

The probability of a and b having a combined CoM energy in the range E to E + dE is given by the Maxwell distribution,

$$P[\varepsilon]d\varepsilon = \frac{2}{\sqrt{\pi}} \cdot \sqrt{\varepsilon} \cdot e^{-\varepsilon} d\varepsilon \qquad (21.4.3.4)$$

where $\varepsilon = E / kT$.

The reaction rate for a given energy E is obtained from the cross-section, (21.4.3.2), by multiplying by the closing velocity, $v = \hbar k / m_R$, and by the number density of one of the reactants. For an assumed density of 1 mole/cm^3 the latter is A = 6.03x10^{29} m^{-3}. Thus,

$$R[E] = \frac{\pi A \hbar^2}{m_R \sqrt{2 m_R E}} \cdot \frac{\Gamma_{ab}\Gamma_{cd}}{(E_R - E)^2 + (\Gamma/2)^2} \qquad (21.4.3.5)$$

The reaction rate at temperature T is obtained by multiplying (21.4.3.5) by (21.4.3.4) and integrating over all thermal energies from zero to infinity, i.e.,

$$R[T] = \frac{\sqrt{2\pi} A \hbar^2}{m_R \sqrt{m_R kT}} \cdot \int_0^\infty \frac{\Gamma_{ab}\Gamma_{cd} e^{-\varepsilon} d\varepsilon}{(E_R - E)^2 + (\Gamma/2)^2} \qquad (21.4.3.6)$$

So far we have probably left the reader with the impression that the partial widths Γ_{ab} and Γ_{cd} are constants. Many sources are guilty of this – for reasons that will become apparent – but it is not so. It is clear that it cannot be so if for no other reason than the fact that we need to account for the Coulomb barrier. As yet, there is no Coulomb barrier in (21.4.3.6). From Chapter 14 and other places we know that the Coulomb barrier can be accounted for by factoring the cross section, or reaction rate, by a term like,

$$\text{Coulomb Barrier} = \exp\{-2b / \sqrt{E}\} \qquad (21.4.3.7)$$

noting that (21.4.1.8)gives $b = \frac{\pi}{2} Z_a Z_b \alpha \sqrt{2 m_R c^2}$, and that this b is half that defined in Chapter 14. Hence, the predominant energy dependence of the partial width Γ_{ab} will be given by (21.4.3.7). Were it not for the Breit-Wigner denominator in (21.4.3.6), the competition between this Coulomb barrier term and the Maxwell exponential would result in the integrand peaking at the Gamow peak energy, as given by (21.4.1.7).

However, if the resonance is very narrow, e.g. of the order of eV, then the Breit-Wigner term has a really extreme peak, and this acts virtually as a delta-function. The integral in (21.4.3.6) can therefore be approximated roughly as follows,

$$\int_0^\infty \frac{e^{-\varepsilon} d\varepsilon}{(E_R - E)^2 + (\Gamma/2)^2} \approx \frac{e^{-E_R/kT}\Gamma}{(\Gamma/2)^2} = \frac{4 e^{-E_R/kT}}{kT\Gamma} \qquad (21.4.3.8)$$

where we have used the fact that Γ is (literally) the width of the peak in the Breit-Wigner expression. We have not respected normalisation of our 'delta function' properly in the above. To get the numerical factor right we carry out the integral explicitly as follows,

$$\int_0^\infty \frac{e^{-\varepsilon}d\varepsilon}{(E_R - E)^2 + (\Gamma/2)^2} = \frac{1}{(kT)^2}\int_0^\infty \frac{e^{-\varepsilon}d\varepsilon}{(\tilde{E}_R - \varepsilon)^2 + (\tilde{\Gamma}/2)^2} \qquad (21.4.3.9)$$

where $\tilde{E}_R = E_R/kT$ and $\tilde{\Gamma} = \Gamma/kT$. Changing to the integration variable $\varepsilon' = \varepsilon - \tilde{E}_R$,

$$\int_0^\infty \frac{e^{-\varepsilon}d\varepsilon}{(E_R - E)^2 + (\Gamma/2)^2} = \frac{e^{-\tilde{E}_R}}{(kT)^2}\int_{-\tilde{E}_R}^\infty \frac{e^{-\varepsilon'}d\varepsilon'}{\varepsilon'^2 + (\tilde{\Gamma}/2)^2} \approx \frac{e^{-\tilde{E}_R}}{(kT)^2}\int_{-\infty}^\infty \frac{d\varepsilon'}{\varepsilon'^2 + (\tilde{\Gamma}/2)^2} \qquad (21.4.3.10)$$

The last step follows because:-
(1)We are assuming $E_R >> kT$, for the reasons discussed above, and,
(2)The sharp peak in the Breit-Wigner expression occurs at $\varepsilon' = 0$. The key assumption is that the exponential in the numerator of the integrand can therefore be ignored (i.e. taken as a constant evaluated at $\varepsilon' = 0$, where it is unity).

The second point trips off the tongue nicely, but it is actually completed unjustified for the integral as written in (21.4.3.10). The reason is that the exponential factor in the numerator actually blows up at the lower limit of the integral. Disaster! This drove me bonkers for a while.

The resolution is that this is where (21.4.3.7) comes in. Recall that we should really have left an energy dependent partial width within the integrand. Equ.(21.4.3.7) gives the energy dependence of this partial width. It has the effect of dragging the integrand down to zero at low energies, i.e. near the bottom limit of the integral. This saves us from this disaster and means that (21.4.3.10) is valid after all.

The integral in (21.4.3.10) is just $2\pi/\tilde{\Gamma}$. Hence we get,

$$\int_0^\infty \frac{e^{-\varepsilon}d\varepsilon}{(E_R - E)^2 + (\Gamma/2)^2} \approx \frac{2\pi e^{-\tilde{E}_R}}{(kT)^2\tilde{\Gamma}} = \frac{2\pi\exp\{-E_R/kT\}}{kT\Gamma} \qquad (21.4.3.11)$$

i.e. a factor of 2π in place of the factor of 4 in the rough estimate of (21.4.3.8). Substituting (21.4.3.11) into (21.4.3.6) gives the reaction rate at temperature T for a resonant reaction,

$$R[T] = A\left(\frac{2\pi[\hbar c]^2}{m_R c^2 kT}\right)^{3/2}\frac{\Gamma_{ab}\Gamma_{cd}}{\hbar\Gamma}\exp(-E_R/kT) \qquad (21.4.3.12)$$

Application to $^4_2\mathrm{He} + {}^8_4\mathrm{Be} \rightarrow {}^{12}_6\mathrm{C} + \gamma$ **:-**

The 'ab' partial width of the C^{12} 0^+_2 resonance is to re-emit the alpha particle. It so happens that this is overwhelming more probable than the resonance decaying to a stable C^{12} nucleus. Hence, we have $\Gamma \approx \Gamma_{ab} = \Gamma_\alpha$, and consequently dependence on this partial width cancels out of (21.4.3.12). [For interest, it's value is 8.5 eV so the half-life of the resonance is roughly 10^{-16} sec.]. For completeness we write this case explicitly,

$$R[T] = A\left(\frac{2\pi[\hbar c]^2}{m_R c^2 kT}\right)^{3/2} \frac{\Gamma_\gamma}{\hbar} \exp(-E_R / kT) \qquad (21.4.3.13)$$

The partial width that matters then is the minority one that leads to stable carbon. This is $\Gamma_{cd} = \Gamma_\gamma = 0.0037\mathrm{eV}$. As noted previously, E_R = 0.288 MeV, and the reduced mass is $2.67M_p$ = 2502 MeV. We therefore deduce the following reaction rates,

Reaction (11)

T (10^6 K)	R[T] **Resonant**	**CF88 data**
100	1.44E-11	1.33E-11
120	2.8E-9	2.63E-9
150	5.2E-7	4.90E-7
200	8.6E-5	8.27E-5

The agreement with the CF88 data is excellent – so much so that it is probable that CF88 used the same formula and virtually the same input data.

Application to $^4_2\mathrm{He} + {}^4_2\mathrm{He} \rightarrow {}^8_4\mathrm{Be} \rightarrow {}^4_2\mathrm{He} + {}^4_2\mathrm{He}$

In this case the beryllium-8 nucleus is itself the resonance, since even the ground state is so unstable. The two partial widths are the same, of course, since the final state is the same as the initial state, and both equal Γ = 5eV. The reduced mass is $2M_p$ and the 'resonance' energy is E_R = 91.8 keV above the 2α threshold. Equ.(21.4.3.12) results in the following reaction rates (after also multiplying by 2 to account for the identical reactants):-

Reaction (10)

T (10^6 K)	R[T] **Resonant**	**CF88 data**
100	406	547
120	1,826	2,460
150	7,690	10,400
200	29,330	40,000

Not a bad estimate given the sensitivity to temperature. The CF88 data are ~30% larger at all temperatures.

4.4 Application of Breit-Wigner to the Non-Resonant ${}^{4}_{2}\mathrm{He} + {}^{12}_{6}\mathrm{C} \rightarrow {}^{16}_{8}\mathrm{O} + \gamma$

In deriving (21.4.3.12) we have assumed that the Breit-Wigner peak dominates over the Gamow peak of the integrand. This will be the case so long as the resonance is sufficiently narrow and so long as the resonance is not too distant from the a + b threshold, i.e. so long as E_R/kT is not too large. However, for a sufficiently distant resonance, the Gamow peak may dominate the integral. We must now evaluate the integral using the Coulomb barrier explicitly, i.e.,

$$R[T] = \frac{\sqrt{2\pi}A\hbar^2\Gamma_\gamma}{m_R\sqrt{m_R kT}} \cdot \int_0^\infty \frac{\Gamma_\alpha(E_0)\exp\left\{\frac{2b}{\sqrt{E_0}}\right\}\exp\left\{-\varepsilon - \frac{2b}{\sqrt{E}}\right\}d\varepsilon}{(E_R - E)^2 + (\Gamma/2)^2} \qquad (21.4.3.14)$$

Where we have approximated the energy variation of the partial width for re-emission of an alpha particle simply by the Coulomb barrier of (21.4.3.7), that is we have put,

$$\Gamma_\alpha(E) \approx \Gamma_\alpha(E_0)\exp\left\{\frac{2b}{\sqrt{E_0}}\right\}\exp\left\{-\frac{2b}{\sqrt{E}}\right\} \qquad (21.4.3.15)$$

where $\Gamma(E_0)$ denotes the partial width *at* the Gamow peak. The integral in (21.4.3.14) is evaluated assuming the Gamow peak dominates as done previously (e.g. in Chapter 14), thus,

$$\begin{aligned}
\int_0^\infty \frac{\exp\left\{-\varepsilon - \frac{2b}{\sqrt{E}}\right\}d\varepsilon}{(E_R - E)^2 + (\Gamma/2)^2} &\approx \frac{e^{-f_{min}}}{(E_R - E_0)^2}\int_0^\infty \exp\left\{-0.5f_0''(\varepsilon - \varepsilon_0)^2\right\}d\varepsilon \\
&\approx \frac{e^{-f_{min}}}{(E_R - E_0)^2}\int_{-\varepsilon_0}^\infty \exp\left\{-0.5f_0''y^2\right\}dy \\
&= \frac{e^{-f_{min}}}{(E_R - E_0)^2}\cdot\sqrt{\frac{2\pi}{f_0''}} \\
&= \frac{e^{-f_{min}}}{(E_R - E_0)^2}\cdot\sqrt{\frac{4\pi}{3}}\cdot\tilde{b}^{1/3}
\end{aligned} \qquad (21.4.3.16)$$

where, $f_0'' = \frac{3}{2\varepsilon_0}$ and $\varepsilon_0 = \frac{E_0}{kT} = \tilde{b}^{2/3}$ because $E_0 = (bkT)^{2/3}$ and $\tilde{b} = \frac{b}{\sqrt{kT}}$ and $f_{min} = 3\tilde{b}^{2/3}$ and $b = \frac{\pi}{2}Z_a Z_b \alpha\sqrt{2m_R c^2}$. Taking into account the positive exponent of $2b/\sqrt{E_0}$ in (21.4.3.14), the overall exponent is found to be,

$$\exp\left\{\frac{2b}{\sqrt{E_0}}-f_{min}\right\}=\exp\left\{\frac{2b}{(bkT)^{1/3}}-3\tilde{b}^{2/3}\right\}=\exp\left\{2\tilde{b}^{2/3}-3\tilde{b}^{2/3}\right\}=\exp\{-\varepsilon_0\}\quad(21.4.3.17)$$

Hence (21.4.3.14) reduces to,

$$R[T]=\frac{A(Z_aZ_b)^{1/3}}{\sqrt{3}}\left(\frac{2\pi}{m_Rc^2\sqrt{kT}}\right)^{4/3}\frac{(\hbar c)^2\Gamma_\alpha^{E_0}\Gamma_\gamma}{(E_R-E_0)^2}\cdot c\cdot\exp\left\{-\frac{E_0}{kT}\right\}\qquad(21.4.3.18)$$

where the superscript on the alpha partial width is a reminder that we must evaluate it at the Gamow peak energy.

Note three major differences with respect to the resonant reaction rate, (21.4.3.13),

- Most importantly, the exponential factor depends upon the Gamow peak energy E_0 rather than the Breit-Wigner resonance peak at E_R.
- Secondly, the denominator of the Briet-Wigner expression in (21.4.3.18) is evaluated at E_0, so the total width Γ has been ignored in comparison with the far bigger energy scale (E_0 - E_R). Thus – ignoring the effect of the exponential term - the non-resonant reaction rate would be smaller than the resonant reaction rate by a factor of $[\Gamma/2(E_R-E_0)]^2$. If E_0 is sufficiently smaller than E_R, however, the exponential term will make the non-resonant rate faster overall.
- The non-resonant reaction rate, (21.4.3.18), depends upon *both* partial widths, unlike the resonant rate in the case when one partial width is dominant, i.e. (21.4.3.13).

Which of (21.4.3.13) or (21.4.3.18) applies will generally be whichever gives the larger reaction rate.

The non-resonant reaction rate, (21.4.3.18), has been derived for the case when there is a resonance at a positive energy gap, E_R, above the a + b threshold, but the Gamow peak occurs at a lower energy $E_0 < E_R$. Another way in which the resonance peak leading to (21.4.3.12 or 13) may not be appropriate is if the resonance lies *below* the a + b threshold. In this case, E_R is negative. Since we integrate over positive thermal energies only, clearly we never pass through the resonant (Breit-Wigner) peak. The Gamow peak will then always be dominant (excepting possibly the case where $-E_R$ is comparable with the width Γ, which we will not consider). Equ.(21.4.3.18) is also applicable in this case, and may be re-written for $E_R < 0$ as,

$$R[T]=\frac{A(Z_aZ_b)^{1/3}}{\sqrt{3}}\left(\frac{2\pi}{m_Rc^2\sqrt{kT}}\right)^{4/3}\frac{(\hbar c)^2\Gamma_\alpha^{E_0}\Gamma_\gamma}{(|E_R|+E_0)^2}\cdot c\cdot\exp\left\{-\frac{E_0}{kT}\right\}\qquad(21.4.3.19)$$

Hoyle gives the prescription for finding the alpha partial width at the Gamow peak energy as follows,

$$\Gamma_\alpha^{E_0} = \frac{G_s}{\sqrt{E_0}} \cdot \exp\left\{ 4\left[\frac{2\alpha m_R c^2 Z_a Z_b R}{\hbar c} \right]^{1/2} - \frac{2b}{\sqrt{E_0}} \right\} \qquad (21.4.3.20)$$

where, $R = 1.6\left(A_a + A_b\right)^{1/3}$ fm. (21.4.3.21)

and $G_s/\sqrt{E_0}$ is the alpha partial width "without a Coulomb barrier". The combination of the two exponentials in (21.4.3.19) and (21.4.3.20) are, by definition, maximised at the Gamow peak energy, and so the two expressions may be combined as,

(21.4.3.22)

$$R[T] = \frac{A\left(Z_a Z_b\right)^{1/3}}{\sqrt{3}} \left(\frac{2\pi}{m_R c^2 \sqrt{kT}} \right)^{4/3} \frac{(\hbar c)^2 \Gamma_\gamma c}{\left(|E_R| + E_0\right)^2} \cdot \frac{G_s}{\sqrt{E_0}} \cdot \exp\left\{ 4\left[\frac{2\alpha m_R c^2 Z_a Z_b R}{\hbar c} \right]^{1/2} \right\} \exp\left(-f_{min}\right)$$

where, as before, $f_{min} = 3\tilde{b}^{2/3}$ and $\tilde{b} = b/\sqrt{kT}$.

Application to $^4_2\mathrm{He} + {}^{12}_6\mathrm{C} \rightarrow {}^{16}_8\mathrm{O}$

Hoyle (1954) suggests Γ_γ = 1 eV and $G_s/\sqrt{E_0}$ = 0.1 MeV. From modern tables, the half-life of the 1^- resonance at 7116.85keV of O^{16} is 8.3 fs, i.e. a width of 0.0794 eV, which can be interpreted as the gamma width. This is about an order of magnitude smaller than Hoyle's suggestion. We can keep the product of the two partial widths essentially the same as Hoyle's by choosing the alpha partial width "without barrier" as $G_s/\sqrt{E_0}$ = 1 MeV. How can this be justified? Where is it tabulated? What the heck does the alpha partial width mean in any case, since it is energetically impossible for the 7116.85keV resonance to decay into an alpha plus C^{12}??

The other key data for use in (21.4.3.22) are easily derived as, b = 10.323 $\sqrt{\text{MeV}}$, $m_R = 3M_p$ = 2815 MeV, $E_R = E'_R - \Delta B$ = 7.1169 - 7.1619 = -0.045 MeV. Note that E_R is negative, i.e. the reaction is *not* resonant. We find the reaction rates to be as follows,

Reaction (12)

T (10^6 K)	R[T] Equ.(21.4.3.22)	**CF88 data**	**NACRE data**
100	2.1E-20	0.92E-21	1.77E-20
120	9.1E-19	4.1E-19	7.7E-19
150	6.6E-17	3.0E-17	5.7E-17
200	1.0E-14	0.47E-14	0.89E-14

The above are not really predictions because we have used an arbitary value for $G_s/\sqrt{E_0}$ namely 1 MeV. However, we see that this is a good value (essentially in agreement with Hoyle as regards the size of the product of the two partial widths) because the NACRE reaction rates are reproduced very well.

4.5 Application of Breit-Wigner to Reaction (6)

$$p + {}^7_3\mathrm{Li} \rightarrow {}^8_4\mathrm{Be}[2^+_3] \rightarrow {}^8_4\mathrm{Be}[GS] + \gamma \rightarrow {}^4_2\mathrm{He} + {}^4_2\mathrm{He} + \gamma$$

A quick check shows that the 1^+ resonance of $^8_4$Be at 17.640 does not lead to a significant contribution to the reaction rate for temperatures in the region of 10 to 50 million K. For example, at 10 MK we find $\tilde{E}_R \approx 445$, so that the exponential term in (21.4.3.12) is $e^{-445} \sim 10^{-193}$.

However, the 2_3^+ resonance of $^8_4$Be at 16.922 MeV, which is just below the p + Li^7 threshold, does provide a good estimate of the reaction rate via the non-resonant Breit-Wigner based Equ.(21.4.3.22). The decay of the 2_3^+ resonance to the ground state of $^8_4$Be has a partial width of 74 keV. The partial width to the original p + Li^7 is assumed to be given by the same prescription as for the alpha-channel in Section 4.4, i.e. a partial width "without barrier" of $G_s/\sqrt{E_0}$ = 1 MeV, together with the temperature dependence given by (21.4.3.20), which leads to (21.4.3.22).

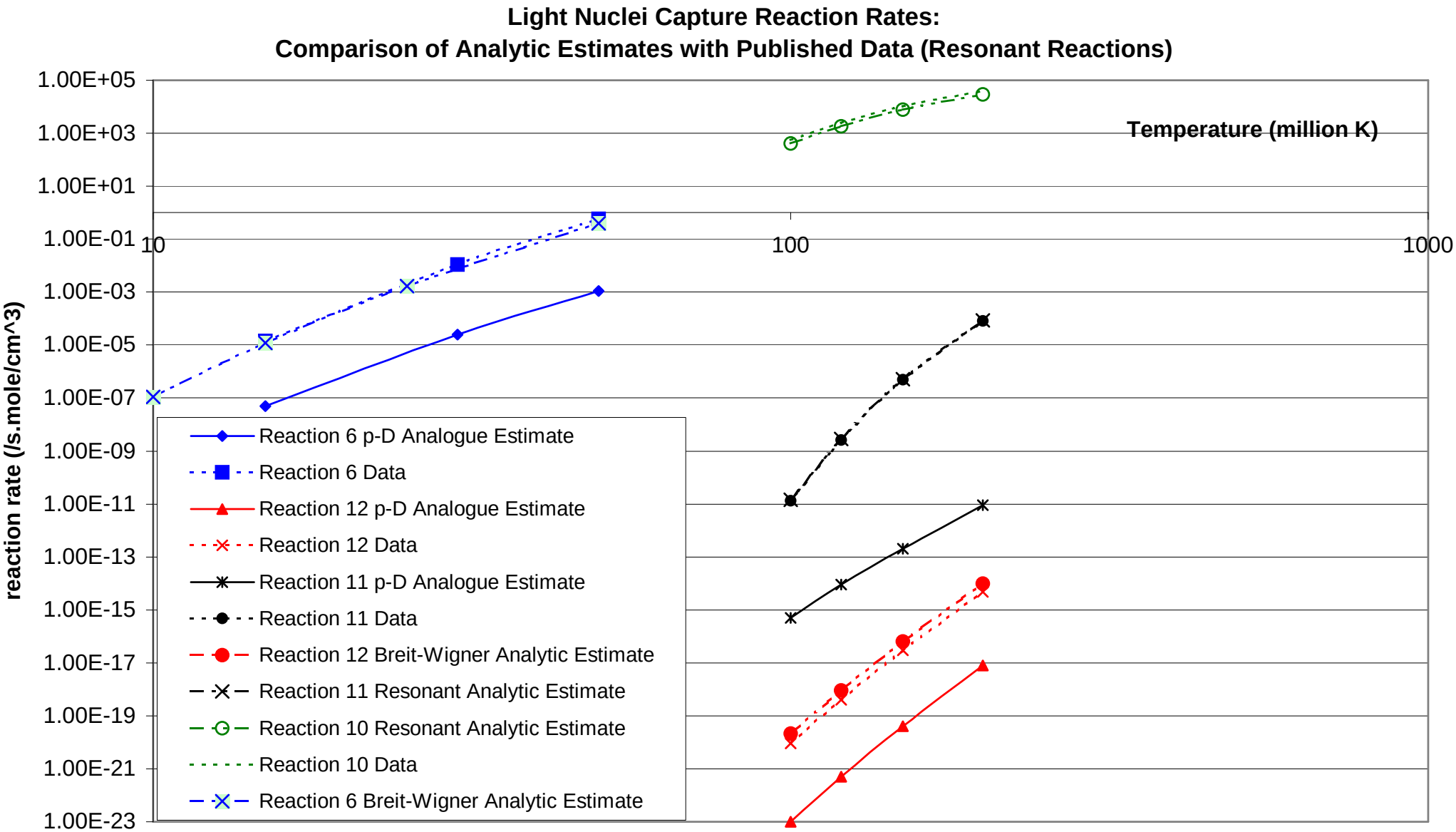
The binding energy difference between the ground state of $^8_4$Be and p + Li^7 is 17.255 MeV. Hence the 2_3^+ resonance of $^8_4$Be at 16.922 MeV has E_R = -0.333 MeV. We find the reduced mass to be 821 MeV and b = 1.394 MeV. Equ.(21.4.3.22) then gives the following reaction rates,

Reaction (6) [Non-Resonant]

T (10^6 K)	R[T] Equ.(21.4.3.22)	**NACRE data**
10	1.1E-7	1.3E-7
15	1.2E-5	1.44E-5
25	1.7E-3	2.25E-3
50	0.40	0.565

Hence, the non-resonant Breit-Wigner based reaction rate from Equ.(21.4.3.22) is a very good representation of the published data.

The reaction rates estimated based on Breit-Wigner, whether resonant (21.4.3.12/13) or non-resonant (21.4.3.22), are compared with data in the following graph. Note that the agreement is sufficiently good that on this log scale the results virtually over-plot. The continuous curves are the very poor results obtained from the p-D analogy in Section 4.2.


Light Nuclei Capture Reaction Rates:
Comparison of Analytic Estimates with Published Data (Resonant Reactions)
reaction rate (/s.mole/cm^3)
1.00E+05
1.00E+03
1.00E+01
1.00E-01
1.00E-03
1.00E-05
1.00E-07
1.00E-09
1.00E-11
1.00E-13
1.00E-15
1.00E-17
1.00E-19
1.00E-21
1.00E-23
10
100
1000
Temperature (million K)
Reaction 6 p-D Analogue Estimate
Reaction 6 Data
Reaction 12 p-D Analogue Estimate
Reaction 12 Data
Reaction 11 p-D Analogue Estimate
Reaction 11 Data
Reaction 12 Breit-Wigner Analytic Estimate
Reaction 11 Resonant Analytic Estimate
Reaction 10 Resonant Analytic Estimate
Reaction 10 Data
Reaction 6 Breit-Wigner Analytic Estimate