Chapter 20: The Entropy Of The Universe

Last Update: 6 November 2005

1. Introduction

According to the Big Bang theory, the universe is supposed to have started[1] in a highly dense and extremely hot state consisting of radiation and particles in random motion, devoid of structure. And yet here we are, 13.7 billion years later. We inhabit a planet which is replete with natural order, not least its flora and fauna, including ourselves. It is not immediately obvious how this apparent transition from disorder to order can be reconciled with the second law of thermodynamics, namely that the entropy of the universe should never decrease. Nor is the spontaneous production of order a purely terrestrial phenomenon. The creation of individual energy production centres, stars, out of uniform matter is also the emergence, it would seem, of greater order. More convincing still are the galaxies, whose inner workings as the nurseries and domiciles of stars is sufficiently complex to resemble living organisms[2]. Of course, it is clear what is responsible for the formation of galaxies, stars and planets: it is gravity. Could it be, therefore, that the process of gravitational collapse is not subject to the second law of thermodynamics after all, leading instead to states of reduced entropy?

We will not attempt to examine the entropy book-keeping involved in the formation of galaxies, still less in the emergence of biological life. Probably too little is known of both these subjects at present for such a thing to be attempted. However, the mystery of apparently reducing entropy can be resolved convincingly by an illustration. We consider a cloud of gas collapsing under its own gravity. This may be considered a very simple surrogate for star formation, though we restrict ourselves to the assumption that the temperature and density of the cloud remain uniform as it collapses. Nor do we impose constraints that must apply in reality, namely hydrostatic equilibrium – so we are not concerned with the pressure of the gas.

This problem is treated in Appendix B1, with the added condition that the gas remain gravitationally bound as it collapses – a reasonable requirement for a model of star formation. Appendix B1 shows that this later requirement implies that the gas cloud must loose energy as it collapses, presumably in the form of radiation through some cooling mechanism. This is only to be expected since "collapsing" implies moving to a more compact state, and hence with increasingly large, negative, gravitational potential energy. Because the gas is loosing energy, i.e. $dQ_{gas} < 0$, it follows that the entropy of the gas is also reducing since,

$$dS_{gas} = \frac{dQ_{gas}}{T} < 0 \qquad (1)$$

This strictly only applies for reversible changes. So, it appears that gravity is indeed the philosophers' stone and provides us with the mechanism for reducing the entropy of matter. This is not entirely whimsical. In fact, it is true. This is indeed the mechanism which has lead to the formation of structured galaxies, stars and planets

[1] "Started" here means at time 0.01 seconds, or 1 second, or some such time – but after any inflationary phase or at times sufficiently early that things may have been otherwise.
[2] This is an analogy, not some cosmic Gaia hypothesis.

out of gases and dust clouds of randomly moving particles. And the structured nature of these collapsed agglomerations of matter is indeed the consequence of the gas reducing its entropy. Strangely, the standard texts seem not to emphasise this important truth.

However, the second law is not violated. Appendix B1 shows that whatever carries off the heat lost from the gas (presumably radiation) must have an increased entropy, i.e.,

$$dS_{radiation} = \frac{dQ_{radiation}}{T} \geq -\frac{dQ_{gas}}{T} > 0 \qquad (2)$$

Thus, the overall entropy of the universe is not reduced, the radiation simply carries away the entropy lost by the gas. Note that this is essentially a reversible process. We can imagine an external source of radiation causing the gas to re-expand, evaporating the star. Hence, the equality in (2) is likely to apply, so that the gravitational collapse occurs at roughly constant overall entropy.

Having disposed of this most important issue, there nevertheless remains some interesting questions about entropy in the universe. Amongst these are,

- What is the total entropy of the (observable) universe?
- Where is most of the entropy of the universe located?
- How much does dark matter contribute to the entropy?
- How much does dark energy contribute to the entropy?
- What happens to the entropy as the universe expands?
- How much has the entropy of the universe increased, if at all?
- What processes lead to the entropy increase?
- What is the typical entropy of a star?
- What is the entropy of a black hole?

But before getting to these problems we first ask a directly related question: what has given rise to the greater number of photons – the Big Bang or all the stars in the history of the universe?

2. Where Have Most Photons Come From?

The current temperature of the cosmic microwave background (CMB) is 2.728^oK. The photon density is thus $0.2436\left(\frac{kT}{\hbar c}\right)^3$ = 4.1×10^8 m^{-3}. The age of the universe is 13.7 Byrs, so, assuming a flat universe with $\Lambda = 0$ for illustration, the volume of the observable universe is $\frac{4\pi}{3}(3ct)^3$ = 2.5×10^{80} m^3. So, the total number of CMB photons in the observable universe is 10^{89}.

Recalling from Chapter 4 that the photon:baryon ratio is 1.9×10^9, the number of baryons in the observable universe is thus $\sim 5 \times 10^{79}$, which gives the total mass of ordinary matter in the observable universe to be $\sim 9 \times 10^{52}$ kg, or about 4.4×10^{22} solar masses.

To estimate the number of photons produced by stars, we shall assume all the stars to be of solar mass, and hence that there are 4.4 x 10^{22} of them in the observable universe. Moreover, we shall assume that all these stars have been burning with the current brightness of the Sun for the whole age of the universe (13.7 Byrs). This is not unreasonable since the lifetime of a solar mass star is indeed comparable with the age of the universe (perhaps closer to 10 Byrs).

The Sun's photospheric temperature is ~5890°K, so the photon flux from its surface is, from Stefan's law for a black body, $J_\gamma^N = \rho_\gamma^N c/4$ where $\rho_\gamma^N = 0.2436\left(\frac{kT}{\hbar c}\right)^3 = 4.1$ x 10^{18} m^{-3}, hence $J_\gamma^N = 3.1$ x 10^{26} $m^{-2}s^{-1}$. The radius of the Sun is 7 x 10^8 m, so its surface area is 6.15 x 10^{18} m^2. The total emission of photons from the Sun is thus 1.9 x 10^{45} per second. Over the life of the universe (13.7 Byrs = 4.3 x 10^{17} sec) such a star will produce ~8 x 10^{62} photons. Hence, the whole complement of 4.4 x 10^{22} stars in the observable universe will have produced 3.6 x 10^{85} photons.

Thus, large though the number of photons produced by the stars is, the number of CMB photons is 2,750 times larger still.

One may argue that the assumption that all the stars are of stellar mass might seriously skew the result. To examine the sensitivity of the result to this assumption we now consider instead that all the stars are of 10 solar masses. Now from Chapter 18 we know that the total luminosity of a main sequence star increases as $M^{3.4}$. The luminosity is proportional to surface area times T^4, where T is the surface temperature, and the surface area varies as $M^{1.6}$. Hence, the surface temperature increases as $M^{0.45}$. The total number of photons emitted per second is proportional to surface area times T^3 and thus increases as $M^{2.95}$. Hence, each 10 solar mass star emits virtually 1000 times as many photons per second as a solar mass star. Now such stars will only burn for about 40 million years at most, i.e. only ~0.3% of the age of the universe. However, for the sake of our estimate, we can assume that, immediately one such star burns out, another of 10 solar masses is created to take its place. Since there would be only 1/10 as many 10 solar mass stars as solar mass stars for the same total mass, and each produces 1000 as many photons, the overall production of photons would be 100 times greater, i.e. ~3.6 x 10^{87} photons. This is likely to be an upper bound estimate since only a fraction of stars are so massive (though stars in the early universe were probably more massive). However, even stars of 10 solar masses would produce less photons that the CMB, by a factor of ~27.

Could it be that the photon production by supernovae overturns this argument? No! Supernovae can easily be seen to increase the stellar output of photons only negligibly. If there is one supernova per galaxy every X years, which shines with the luminosity of the whole galaxy for, say, Y weeks, then the fractional increase in the average photon output of the galaxy is Y/52*X. Now X ~ 100 years and Y is a few weeks, so supernovae might cause a fraction of a percent increase in the number of stellar photons, no more.

Most likely our estimate of the number of stellar photons based on the assumption of solar mass stars is reasonably close to the truth. So the number of photons produced by the Big Bang outnumbers those produced by stars by the order of a factor of 1000 or so. Thus, about 99.9% of the photons in the universe originated from the Big Bang, and only about 0.1% from stars.

3. What Is The Total Entropy Of The Observable Universe?

The question is slightly ambiguous. Do we mean "the entropy of observable things" or "the entropy of all things within the observable boundary"? The difference is whether dark matter and dark energy are included. In principle we wish to include them, but whether we are able to estimate their entropy is another question. We consider each major constituent in turn:-

3.1 The Entropy of the Photons and Neutrinos

The entropy of N black body photons is simply,

$$S_\gamma = 3.6Nk_B \quad \text{(see Appendix A0)} \tag{3.1}$$

Since we have already calculated the number of photons in the observable universe, we have

$$S_\gamma / k_B = 3.6 \times 10^{89} \tag{3.2}$$

We note that this originates virtually entirely from the Big Bang. The number of Big Bang photons is constant[3], so the entropy of the photon content of the universe is virtually constant. Stars will have increased the photon entropy only by about 0.1% over the age of the universe.

The entropy of N black body fermions is,

$$S_\nu = 4.2Nk_B \qquad \text{(see Appendix A0)} \tag{3.3}$$

The number density of black body fermions is $0.0914N_sN_a\left(\frac{kT}{\hbar c}\right)^3$. The fermions in question are the neutrinos, of which there are three different types, and the number of different spin/antiparticle states is $N_sN_a = 2$ (probably). Hence, the number density of all neutrinos is $0.5481\left(\frac{kT}{\hbar c}\right)^3$. However, we must remember than the cosmic neutrino background temperature is not that of the CMB (i.e. not 2.728°K) but only 1.947°K (see Chapter 5). Thus, the total neutrino density in the present universe is expected to be 3.36×10^8 m^{-3} (though of course it has never been observed). Hence the total number of neutrinos in the observable universe is 8.4×10^{88} and their entropy is,

$$S_\nu / k_B = 3.5 \times 10^{89} \tag{3.4}$$

almost identical to the entropy of the photons.

Include a comment on the number of neutrinos produced by stars. Presumably this must be smaller than the number of photons produced by stars? (Check by looking at

---

[3] Constant, that is, once electron-positron annihilation has finished.

the reactions). If so, what percentage is the neutrino entropy increased by due to the stars?

Note that the entropy of "radiation", in units of k_B, is essentially just the number of quanta, times a factor of order 4.

3.2 The Entropy of Ordinary Matter

In this section we estimate the entropy of the ordinary matter. For simplicity we consider this to consist of only protons and electrons, i.e. we ignore the neutrons and nuclei. In the present state of the universe, this ordinary matter occurs primarily in two forms: stars and gas clouds. We consider both, and, in the case of gas clouds, a range of radically different conditions and cloud sizes.

However, prior to galaxy and star formation, matter was not so concentrated. Consequently we also estimate the entropy of matter prior to, and just after, recombination.

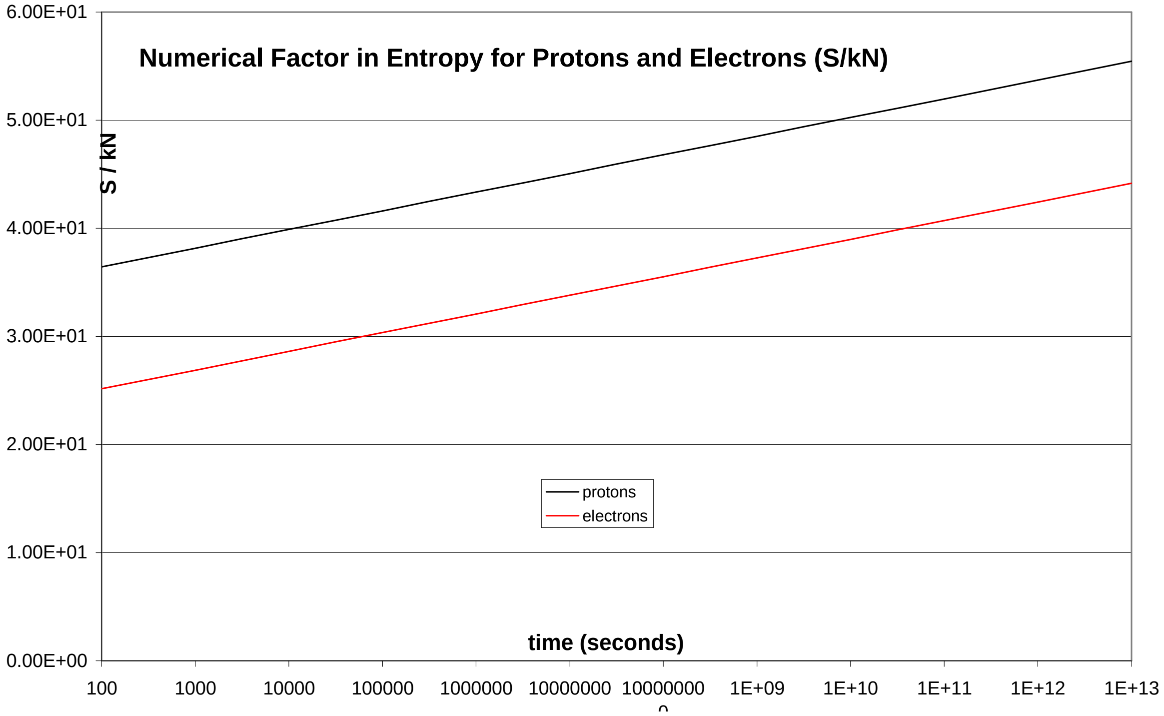
Of course, since the photon:baryon ratio is 1.9 x 10^9, and since, as we have seen, the entropy ($/k_B$) is just the number of particles times a small numerical factor, we can anticipate that the entropy of matter will be negligible compared with that of radiation. Nevertheless we shall evaluate it for interest's sake.

3.2.1 Entropy of Ordinary Matter Up To Recombination

In the crude approximation that all baryons are protons, the protons and electron density is $\rho_p^N = \rho_e^N = \rho_\gamma^N / 1.9\text{x}10^9$, where $\rho_\gamma^N = 0.2436\left(\frac{kT}{\hbar c}\right)^3$. The entropy of an ideal gas is given by the Sackur-Tetrode equation, e.g. for the protons,

$$\frac{S_p}{k_B} = N_p \left\{ \frac{5}{2} + \log\left[\left(\frac{M_p k_B T}{2\pi\hbar^2} \right)^{3/2} \frac{1}{\rho_p^N} \right] \right\} \qquad (3.5)$$

and similarly for the electrons. The {…} in (3.5) takes the place of the numerical factors of 3.6 or 4.2 in (3.1) or (3.3) for photons and neutrinos. Because of the logarithm in the {…} we can anticipate that this factor will not differ too much for matter particles. It can be calculated explicitly as a function of time using $T = 1.02\text{x}10^{10} / \sqrt{t}$. The calculation applies after electron-positron annihilation is complete. The numerical factor is shown below:-



time (sec)	Sp/kN	Se/kN	(Sp+Se)/kN	(Sp+Se)/k
100	3.64E+01	2.52E+01	6.16E+01	3.24E+81
1000	3.82E+01	2.69E+01	6.50E+01	3.42E+81
10000	3.99E+01	2.86E+01	6.85E+01	3.60E+81
1.00E+05	4.16E+01	3.03E+01	7.20E+01	3.78E+81
1.00E+06	4.33E+01	3.21E+01	7.54E+01	3.97E+81
1.00E+07	4.51E+01	3.38E+01	7.89E+01	4.15E+81
1.00E+08	4.68E+01	3.55E+01	8.23E+01	4.33E+81
1.00E+09	4.85E+01	3.72E+01	8.58E+01	4.51E+81
1.00E+10	5.02E+01	3.90E+01	8.92E+01	4.69E+81
1.00E+11	5.20E+01	4.07E+01	9.27E+01	4.87E+81
1.00E+12	5.37E+01	4.24E+01	9.61E+01	5.06E+81
1.00E+13	5.54E+01	4.42E+01	9.96E+01	5.24E+81

The final column, above, gives the total entropy of all the ordinary matter (protons and electrons) in the observable universe, based on $N = N_\gamma / 1.9 \times 10^9 = 10^{89} / 1.9 \times 10^9 = 5 \times 10^{79}$. Note that by "observable universe" we actually mean that part of the universe that will be observable by the present epoch, although, of course, only a small fraction of it would have been observable at the times in question, above.

As expected, the entropy of ordinary matter is 8 orders of magnitude less than that of the radiation (photons and neutrinos).

The entropy of $5.24 \times 10^{81} k_B$ applies just before recombination. Just after recombination the entropy is approximately equal to the proton entropy previously, since hydrogen atoms have essentially the same mass (and the electrons no longer have any degree of freedom). Thus, the entropy of matter reduces to 2.9×10^{81} k_B. This is based on the assumption that the temperature does not change during recombination. The reduction in the entropy of matter will be at least balanced by the additional entropy carried away by the emitted photons. This additional photon entropy, ~2.3×10^{81} k_B, is a negligible percentage increase. It is interesting that, even at this primitive stage of the universe, matter has contrived to reduce its share of the universe's entropy.

3.2.2 Entropy of Ordinary Matter Now

3.2.2.1 Entropy of Stars

We shall again use the simplification that the matter in question is just protons and electrons, i.e. ignoring nuclei. We shall use the approximation that all the stars are of solar mass. Finally, in this Section, we shall assume that all ordinary matter is in the form of stars, i.e. ignoring the matter in gas clouds.

Now the temperature and density vary markedly through the depth of a star. Consequently we carry out a very crude "integration", actually just three non-zero regions, to estimate the total entropy. Within each region the Sackur-Tetrode expression, (3.5), is used for the entropy. The temperature and density for each region is estimated from the solar model of Chapter 13, with the central values assumed to be 13.7×10^6 K and 90,000 kg/m^3 and a radius of R = 7×10^8m.

r/R	$\langle T \rangle / T_c$	$\langle \rho \rangle / \rho_c$	ΔV	ΔM	$N_p = N_e$	$\log X_p$	$\log X_e$	S / k_B
0 – 18%	0.85	0.60	9.1×10^{24}	4.9×10^{29}	2.9×10^{56}	12.4	1.1	0.54×10^{58}
18 – 37%	0.5	0.22	6.4×10^{25}	1.26×10^{30}	7.6×10^{56}	12.6	1.3	1.44×10^{58}
37 – 74%	0.14	0.0055	5.1×10^{26}	2.5×10^{29}	1.5×10^{56}	14.4	3.1	0.34×10^{58}
74 – 100%	0.015	~0	8.5×10^{26}	~0	-	-	-	~0
TOTAL	-	-		2×10^{30}		-	-	2.3×10^{58}

where ΔV and ΔM are the volume and mass in the spherical annulus indicated by the first column, and N_p, N_e are the (equal) number of protons and electrons. X_p and X_e are the expressions which appear inside the logarithm in the Sackur-Tetrode equation, (3.5). Hence, the total entropy given in the last column is just $(5 + \log X_p + \log X_e)N$. We see that 92% or so of the entropy is due to the protons, by virtue of their greater mass.

But, if all (ordinary) matter is in the form of stars, there are 4.4×10^{22} stars. Hence the total entropy of ordinary matter in the present observable universe is 10^{81}. This is about 34% of the entropy of all ordinary matter just after recombination. Hence, consistent with our analysis of the collapsing gas cloud (Appendix B1), the entropy of ordinary matter has indeed reduced during the evolution of the universe. From Appendix B1 we can associate this with the reduction in entropy due to the gravitational collapse which forms the stars, the balance of the entropy being emitted as radiation.

What about the entropy of the inventory of photons that a star contains at any instant? This is easily evaluated from the black body formula by integrating over the temperature distribution within the star. For a solar mass star we estimate the entropy of the photons it contains to be of the order of 2.8 x 10^{54}. This is four orders of magnitude smaller than the entropy of the matter content, and hence is negligible.

3.2.2.2 Entropy Of Gas Clouds

Need typical sizes, temperatures and densities of gas clouds – then just use Sackur-Tetrode as before. Expect a greater entropy than for stars (assuming all mass in the form of gas clouds) – but less than that of matter just after recombination. That's quite a tight range to hit!

3.3 Loss of Matter Entropy and Gain of Photon Entropy

From Section 3.2.1 we have seen that the entropy of ordinary matter just after recombination was 2.9 x 10^{81} (all entropies will be in units of k_B unless otherwise stated). From Section 3.2.2 we have seen that the minimum entropy of ordinary matter now (i.e. if it were all in the form of stars) is 10^{81}. Thus, the ordinary matter in the observable universe has reduced its entropy at most by 1.9 x 10^{81}.

In contrast, from Section 3.1, Equ.(3.2), we know that the photons in the observable universe have increased their entropy by roughly 3.6 x 10^{86}, since Section 2 has shown that stars will have increased the number of photons by around 0.1%. Consequently, the increase in entropy of the photons is about 2 x 10^{5} times larger than the reduction in the entropy of ordinary matter. The second law of thermodynamics is therefore respected, i.e. the total entropy has increased. However, we note that the process of gravitational collapse is a negligible contributor to the increase in photon entropy. This is to be expected since the overwhelming majority of photons are produced due to the heat generated by nuclear reactions – not by the heat of gravitational collapse. The increased entropy of the universe, modest though it is in percentage terms, occurs due to the irreversible nature of the nuclear reactions in stars.

3.4 Entropy of Dark Matter

Dark matter is estimated to account for about 5 times more mass in the universe than ordinary matter.

If dark matter is composed of neutrinos, then this entropy has already been calculated in Section 3.1 (all three types). If not, then its entropy will depend sensitively on its nature.

If dark matter is composed of particles whose mass is not too much less than that of the electron, then their entropy will be negligible compared with that of the CMB radiation. (The point being that the greater their mass, the smaller their number – and it is their number which principally determines their entropy). A special case of this is the possibility that dark matter consists of "Jupiters" or brown dwarfs. Another is that dark matter is WIMPs.

On the other hand, if the dark matter particles are very light (say 100eV or less) then its entropy becomes significant. There are 10^{89} CMB photons. The total mass of dark matter is about 5 times the mass of ordinary matter and hence is ~4.5 x 10^{53} kg. The entropy of dark matter is comparable with that of the photons if there is the same number of them. Hence, this requires the dark matter particles to be of mass ~4.5 x

$10^{53} / 10^{89}$ kg = 4.5 x 10^{-36} kg = 5 x 10^{-6} m_e = 2.5eV. Thus, if the dark matter were composed of particles very much lighter than 2.5eV then their entropy would dominate the universe. Unfortunately we don't know.

3.5 Entropy of Dark Energy

Haven't a clue! All we know about dark energy is that it accounts for 71% of the universal density, and it has negative pressure. I don't know how the negative pressure affects entropy. Can it be treated as particles – I think not. So how is its entropy to be determined – even in principle?

3.6 The Entropy of Black Holes

The entropy to be associated with a black hole is controversial. Rigorous theorems in general relativity, due to Hawking, prove that, when two black holes merge, the surface area of the resulting (larger) black hole is greater than, or equal to, the sum of the surface areas of the original pair of black holes. This suggests an analogy with entropy which Bekenstein was the first to take seriously. However, the idea that the entropy of a black hole could be identified with a constant times its surface area became credible only when Hawking discovered that a black hole is not black. A black hole emits radiation as a black body at a well defined temperature. This temperature is,

$$kT = \frac{\hbar c}{8\pi m_G} = \frac{\hbar c^3}{8\pi GM} \qquad (3.6)$$

where, m_G is the geometrical mass,

$$m_G = \frac{GM}{c^2} \qquad (3.7)$$

and the radius of the black hole is, $R_{bh} = 2m_G$ (3.8)

By 'radius' and 'area' of a black hole we are referring to the Schwartzschild radius, which is the event horizon for a non-spinning black hole. In this Section we shall assume a non-rotating black hole. Once we have an expression for temperature, the entropy follows from the definition,

$$dS = \frac{dQ}{T} \qquad (3.9)$$

where we identify the 'heat' increment with the energy equivalent of the in-falling mass[4], $dQ = c^2 dM$. Thus, using (3.6) in (3.9) and integrating gives,

$$\frac{S}{k} = \int c^2 dM \cdot \frac{8\pi GM}{\hbar c^3} = \frac{4\pi GM^2}{\hbar c} = \frac{A}{(2L_{Planck})^2} \qquad (3.10)$$

[4] This is, of course, a most peculiar identification since mass is hardly regarded normally as being heat. This is symptomatic of the difficulties associated with black hole thermodynamics, which we will make no attempt to address.

where A is the black hole surface area, $4\pi R_{bh}^2$, and the Planck length is given by,

$$L_{Planck} = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.61 \times 10^{-35}\,\mathrm{m} \qquad (3.11)$$

Thus, a solar mass black hole would have a radius of 3 km and an entropy of 1.06 x 10^{77} (in units of k). Since the entropy of the Sun is ~2.3 x 10^{58} (see Section 3.2.2.1), the entropy of a black hole is a factor of ~5 x 10^{18} greater than that of the star from which it formed.

The entropy of a gas is essentially just the number of particles it contains, times a numerical factor of modest size. Thus, for a solar mass star, we have roughly,

$$S_{star} \approx 10(N_p + N_e)k \approx 20\frac{M}{M_p}k \qquad (3.12)$$

Since the stars entropy is proportional to its mass, but the black hole's entropy varies as the mass-squared, (3.10), we can find a mass at which the two are equal. Actually we can ask three interesting questions,

1) For what mass is a black hole's entropy equal to the original entropy of the matter of which it is composed? [M such that $S_{bh} = S_{matter}$?]

2) For what mass is a black hole's size equal to that of a fundamental particle (say, the Compton wavelength of a proton)? [M such that $2m_G = \hbar / M_p c$?]

3) For what mass would a black hole formed in the Big Bang have just evaporated by now? [M such that $t_{bh} = t_{universe}$?].

Curiously, the answer to all three questions is virtually the same. In the first two cases, this can be shown algebraically. In the last case, the age of the universe in the current epoch happens to be tuned to make it true. This observation becomes slightly less mysterious when we recall that the current age of the universe is tuned (anthropomorphically) to be at least of order of the lifetime of solar mass stars. We now demonstrate these coincidences:-

(1) M such that $S_{bh} = S_{matter}$?

Equating (3.10) and (3.12) gives,

$$M \approx \left(\frac{20}{4\pi}\right)\left(\frac{\hbar c}{G M_p}\right) = \left(\frac{20}{4\pi}\right)\frac{M_{Planck}^2}{M_p} \qquad (3.13)$$

where the Planck mass is,

$$M_{Planck} = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.18 \times 10^{-8} \text{kg} \qquad (3.14)$$

The value of (3.13) is 4.5 x 10^{11} kg.

(2) M such that $R_{bh} = 2m_G = \hbar / M_p c$ **?**

Using (3.7) gives,

$$M \approx \left(\frac{1}{2}\right)\left(\frac{\hbar c}{GM_p}\right) = \left(\frac{1}{2}\right)\frac{M_{Planck}^2}{M_p} \qquad (3.15)$$

Thus, (3.15) and (3.13) are essentially the same, to within a factor of 10/π. Even this would become closer to unity if the electrons and protons were combined into neutrons, i.e. (3.13) would be halved. (3.15) gives a result M = 1.43 x 10^{11}kg.

NB: It is better to consider the Compton wavelength as a measure of the proton mass than its "size". The Compton wavelength of the proton is 0.21fm, whereas the range of the strong nuclear force (which is, arguably, a better measure of its size) is nearly an order of magnitude larger at ~1.4fm. The latter is better regarded as the (inverse) mass of the pion.

(3) M such that $t_{bh} = t_{universe}$ **?**

To calculate the lifetime of a black hole we use the Stefan radiation formula applied its temperature as given by (3.6), together with the surface area derived from (3.8). Thus,

(3.16)

$$\dot{M} = -\frac{Power}{c^2} = -\frac{\sigma T^4 A}{c^2} = -\frac{1}{c^2} \cdot \frac{\pi^2 k^4}{60 c^2 \hbar^3} \cdot \left(\frac{\hbar c^3}{8\pi kGM}\right)^4 4\pi \left(\frac{2GM}{c^2}\right)^2 = \frac{1}{15360\pi} \cdot \frac{\hbar c^4}{G^2 M^2}$$

Integrating gives,

$$t_{bh} = 5120\pi \frac{G^2 M^3}{\hbar c^4} \qquad (3.17)$$

or,

$$M = \left[\frac{\hbar c^4}{5120\pi G^2} t_{bh}\right]^{1/3} \qquad (3.18)$$

Thus, the black hole mass that would just evaporate by now (13.7 Byrs = 4.32 x 10^{17} s) is 1.73 x 10^{11} kg. This is very close to the result from (3.15).

Note that our estimate for the lifetime of the black hole, (3.17), is not really correct as it ignores all radiation other than electromagnetic radiation. Neutrinos will also be emitted, as will a spectrum of all particles – though those with higher masses will contribute little except very near the end "explosion". Consequently, the lifetime will be rather less than given by (3.17), by roughly a factor of 2? 3? for the size of black hole in question here (check – Page has calculated it). However, (3.18) is not very sensitive to the lifetime.

3.6.1 What is the Entropy of the Radiation Emitted during the Evaporation of a Black Hole? Does it Exceed the Initial Entropy of the Black Hole?

We have seen that the formation of a black hole of mass greater than about 4.5 x 10^{11} kg respects the second law of thermodynamics in that the entropy of the black hole exceeds the entropy of the material from which it formed. Is the second law also respected by the evaporation of the black hole? In other words, does the total entropy of the emitted radiation exceed that of the black hole?

How could this possibly be true in reverse as well? The reason is that the evaporation of the black hole produces principally photons and neutrinos, of zero rest mass or very small rest mass. Thus, for every nucleon that goes into forming the black hole initially, a very large number of photons and neutrinos eventually emerges. That this is true for a solar mass black hole can be seen simply. (3.6) gives the initial temperature of a solar mass black hole to be a mere kT ~ 10^{-30} J ~ 10^{-11} eV, T ~ 10^{-7} K. Thus, the energy of the typical emitted photon is a factor of around 10^{20} times less than the rest mass of a nucleon. As the black hole evaporates, it gets hotter, so this factor will decrease. Nevertheless, we therefore expect the emitted radiation to have a total entropy up to a factor 10^{20} greater than that of the original matter from which the black hole formed. We have seen above that indeed the black hole entropy is about a factor of 10^{18} greater than that of the original matter.

To check this in detail we note that the number of photons emitted per second is,

$$\dot{N}_\gamma = J_\gamma^N A, \quad J_\gamma^N = \frac{c}{4}\rho_\gamma^N, \quad \rho_\gamma^N = 0.2436\left(\frac{kT}{\hbar c}\right)^3 \tag{3.19}$$

and A is the surface area of the black hole given by the radius in (3.7, 3.8). We can substitute for the mass in terms of the temperature from (3.6). To integrate over time we can change variables to, say, temperature, using (3.6) and (3.16), to give,

$$\dot{T} = \frac{1}{15360\pi}\frac{\hbar c^4}{G^2}\left(\frac{8\pi kG}{\hbar c^3}\right)^3 T^4 \tag{3.20}$$

which results in,

(3.21)

$$N_\gamma = \int_{T_0}^{\infty}\frac{c}{4}0.2436\left(\frac{kT}{\hbar c}\right)^3 4\pi.\frac{4G^2}{c^4}\left(\frac{\hbar c^3}{8\pi kG}\right)^2\frac{1}{T^2}dT\frac{15360\pi G^2}{\hbar c^4}\left(\frac{\hbar c^3}{8\pi kG}\right)^3\frac{1}{T^4}$$
$$= 9.305\frac{G}{\hbar c}\left(\frac{\hbar c^3}{8\pi kG}\right)^2\int_{T_0}^{\infty}\frac{dT}{T^3} = 4.652\frac{G}{\hbar c}M_0^2$$

where the subscript $_0$ represents the start of life values, and note that the upper limit of the temperature integral is infinite since the temperature is divergent as the mass reduces to zero at the end of life. (3.21) is converted to entropy merely by multiplying by 3.6 (see 3.1), hence,

$$\frac{S^{radiation}_{bh\ evaporation}}{k} = 16.75\frac{GM_0^2}{\hbar c} = 16.75\left(\frac{M_0}{M_{Planck}}\right)^2 \qquad (3.22)$$

Thus, for a solar mass black hole the emitted entropy as it evaporates is 1.41 x 10^{77} (in units of k). This is essentially the same as the entropy of the black hole (see after 3.11) except that (3.22) suggests that the emitted radiation has entropy a factor of 4/3 larger. This factor may be traced to the distinction between the photon entropy in differential form $dS = dU / T$ and in integrated form,

$$S = \frac{4}{3}U$$

(see Appendix A0). But which is actually correct? In any case, the second law of thermodynamics is preserved by the evaporation process.

We have avoided the question of whether information is preserved in the process of forming and then evaporating a black hole. However, we note that since there is vastly more entropy after than before, there is scope for information to be preserved. At one time it was common to believe that information could not be preserved, on the grounds that the thermal nature of the Hawking radiation would not provide a vehicle for the conveyance of information. However, Hawking has recently changed his mind on this matter, and many others have long believed that information would be preserved. The issue hinges on whether the black hole can be regarded as a quantum mechanical pure state, and upon the unitarity of the S-matrix. These matters are too hard for me.

3.6.2 What Is The Total Black Hole Entropy In The Universe?

To answer this question we need to know how many black holes there are, and what is their distribution of masses. Unfortunately this is not known with any precision. The consensus amongst astronomers (if there is one) is that,

(a) There are probably millions of single stellar mass black holes per galaxy – or at least it seems that way in the Milky Way, and we assume that this is a typical spiral galaxy;

(b) There is probably a super-huge black hole at the centre of most galaxies, whose mass is believed to lie in the range one million to one billion solar masses.

Consider the effect that just one black hole of solar mass would have on the galactic entropy. A typical galaxy contains 10^{11} stars. Thus, each star contributes a fraction 10^{-11} of the galactic entropy, assuming all the stars are just that – stars not black holes. We have seen above that a solar mass black hole has ~5 x 10^{18} times the entropy of the Sun. We'll call it 10^{18} for the sake of argument. Now this means that a single solar mass black hole in a galaxy otherwise devoid of black holes will dominate the entire galaxy's entropy, which will be ~10^7 times larger as a consequence. Now recall that the CMB photons have an entropy about 10^8 times larger than that of ordinary matter. Consequently if all galaxies contained just one solar mass black hole, or a handful of solar mass black holes and no more, then the CMB would still dominate the entropy of the universe (setting aside dark matter and dark energy).

But each galaxy probably contains the order of a million such black holes. Consequently the entropy of each galaxy will be $\sim 10^{13}$ times larger than we originally estimated (based on the assumption of no black holes). Thus, even if we ignore the supermassive black holes at the centres of galaxies, the entropy of 'matter' is increased by a factor of 10^{13} by the stellar mass black holes. This alone causes the entropy of 'matter' (including black holes) to exceed that of the CMB by a factor of about 10^{5}.

What about the supermassive black holes? If these are of one million solar masses, their entropy will exceed that of the matter from which they are composed by a factor of $\sim 10^{24}$. Their entropy as stars, before collapsing to form a black hole, would have been a fraction 10^{-5} of the galaxy's entropy. Thus, the galaxy's entropy is increased by the supermassive black hole by a factor of 10^{19}. If all galaxies contain just one such supermassive black hole, the entropy of the 'matter' (including black holes) in the universe is also increased by a factor of 10^{19}. This compares with the factor of increase of 10^{13} due to a million individual solar mass stars forming black holes. (The difference is due to the entropy of a black hole being proportional to the square of its mass). The latter is negligible compared to the former. Thus, the entropy of 'matter' (including black holes) may exceed that of the CMB by a factor of 10^{11}.

If the supermassive black holes are of one billion solar masses, their entropy will exceed that of the matter from which they are composed by a factor of $\sim 10^{27}$. Their entropy as stars, before collapsing to form a black hole, would have been a fraction 10^{-2} of the galaxy's entropy. Thus, the galaxy's entropy is increased by the supermassive black hole by a factor of 10^{25}. If all galaxies contain just one such supermassive black hole, the entropy of the 'matter' (including black holes) in the universe is also increased by a factor of 10^{25}. This compares with the factor of increase of 10^{13} due to a million individual solar mass stars forming black holes. (The difference is due to the entropy of a black hole being proportional to the square of its mass). The latter is negligible compared to the former. Thus, the entropy of 'matter' (including black holes) may exceed that of the CMB by a factor of 10^{17}.

Thus, including the entropy of all these black holes, the entropy of the universe (excluding dark matter and dark energy) may have increased by a massive factor of 10^{17} since black holes first started to form. This is in stark contrast to the picture ignoring black holes. In this case the entropy of matter is negligible, and the entropy of the CMB photons has increased due to stellar photon production by a mere fraction ~0.001.