Chapter 19 – Solar Opacity and Solar Models

Last Update: 24/9/06

1. INTRODUCTION

In Chapter 18 we have presented a solar model without imposing the constraint of heat transport. How do we know that the rate of heat transport required by this model, i.e. in order to balance the power produced, is consistent with the available heat transport processes? We don't! We only have faith that it does so because we tuned our model using results available from the literature for more complete models which *do* include the heat transport constraint. In this brief Chapter we examine whether this faith is justified. This is by way of a cheap alternative to a full model solution. One advantage of this, rather piecemeal, manner of building up our solar model – which we started way back in Chapter 11 – is that it exposes clearly the role played by the various different physical constraints.

We will compare the heat flux required by our model, i.e. to balance the rate of nuclear heat production, with that which would actually occur under the local conditions of density, temperature and temperature gradient predicted by the model. We shall consider heat transport by radiation only. In Chapter 18 we saw that the heat flowing through a sphere of radius r is given in terms of the temperature gradient and the opacity, κ, of the stellar medium by,

$$Q_{rad} = A\frac{r^2T^3}{\rho\kappa}\cdot\frac{dT}{dr} \text{ (Watts)} \quad \text{where, } A = \frac{64\pi}{3}\sigma_{SB} = 3.80 \times 10^{-6} \text{ W/m}^2\text{K}^4 \qquad (1)$$

This should equal the luminosity, L, of the material within the sphere of radius r, calculated by integrating the nuclear power density up to radius r. Starting from the density and temperature profile derived in Chapter 18, we may therefore calculate the opacity, κ, required at each point in order that $Q_{rad} = L$ everywhere. By comparing the required opacity with the opacity data from published sources we can examine whether our faith in the model of Chapter 18 is justified, i.e. whether the model is consistent in terms of heat transport.

2. REQUIRED AND ACTUAL OPACITIES

We first observe that whilst there appear to be substantial variations in the nuclear reaction rates, and hence power densities, in the literature, the model is overall consistent with the observed surface temperature of ~5900 K, and hence the luminosities calculated in the model must be quite accurate.

Key field quantities against radial position from the model are given below, together with the opacity required to balance the luminosity from Equ.(1). [NB: Equ.1 gives the opacities in m^2/kg which should be multiplied by 10 to get the values in the Table, which are in cm^2/g].

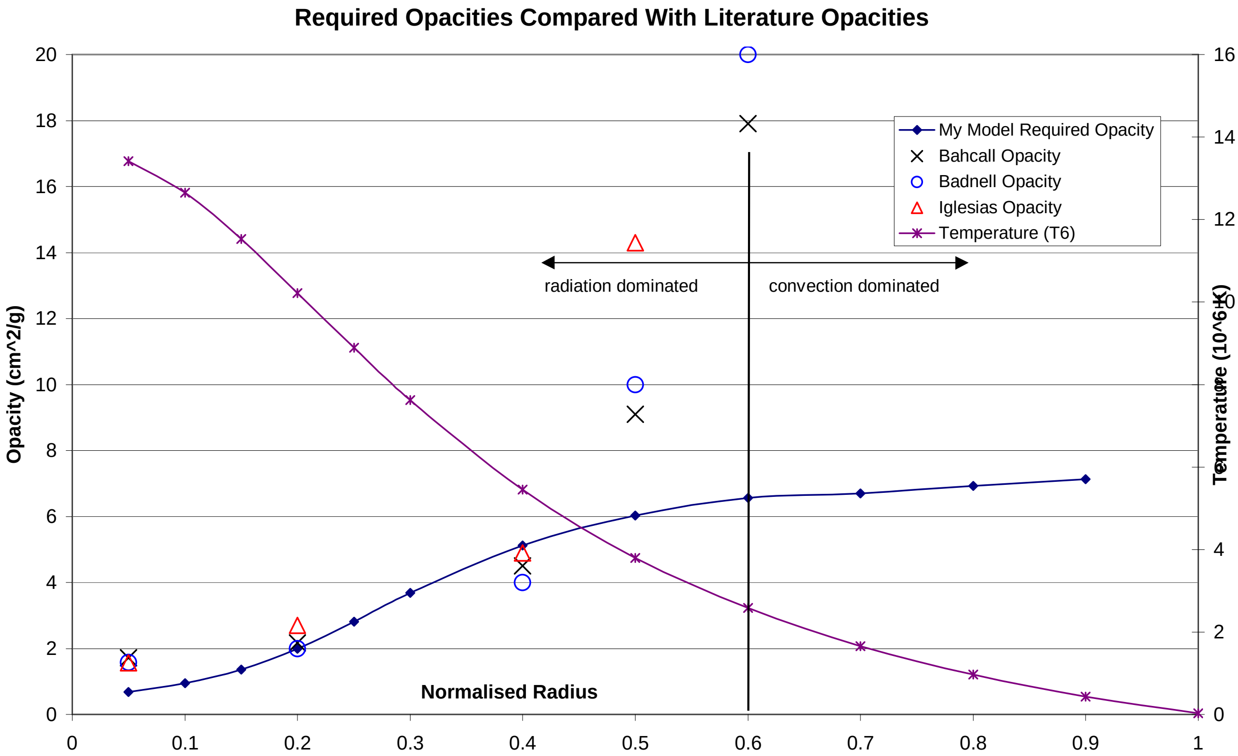
r/R*	**m/M***	**ρ (kg/m³)**	**T (10^6 K)**	**L@ (10^{26} W)**	**κ# (cm²/g)**	**κ(1) (cm²/g)**	**κ(2) (cm²/g)**	**κ(3) (cm²/g)**
0.05	0.007	84,600	13.42	0.274	0.68	1.72	1.58	1.56
0.1	0.05	70,700	12.65	1.50	0.95			
0.15	0.143	53,300	11.53	3.08	1.36			
0.2	0.274	37,100	10.22	4.08	2.0	2.17	2.0	2.7
0.25	0.420	24,300	8.89	4.50	2.82			
0.3	0.559	15,200	7.62	4.63	3.69			
0.4	0.777	5,500	5.45	4.67	5.12	4.51	4.0	4.9
0.5	0.905	1,850	3.80	4.67	6.03	9.1	10.0	14.3
0.6	0.968	570	2.58	4.67	6.56	17.9	20.0	23.0
0.7	*0.991*	*152*	*1.66*	*4.67*	*6.70*	*35*	*35*	*52.0*
0.8	*1*	*30*	*0.97*	*4.67*	*6.93*	*85*	*50*	
0.9	*1*	*2.4*	*0.43*	*4.67*	*7.13*	*273*	*140*	
1.0	*1*	*0.00088*	*0.031*	*4.67*	*780.0*	*1.E5*	*1500*	

*R = 6.96 x 10^8 m, M = 2.2 x 108 kg; @from integrated nuclear power; #from model via Equ.(1)

Figures in italics denote the region where convection dominates.

The opacities from the literature in the above Table were obtained from:-
(1) Bahcall & Pinsonneault (2001)
(2) Opacity Project [Badnell et al]
(3) Iglesias & Rogers [quoted in Prialnik]

The comparison between my model's required opacities and the opacities from the literature is displayed in the graph below:-



Thus, the opacities from the literature are in reasonable agreement with each other. They also compare well with the opacity required in my model, at least below r/R ~ 0.45. At larger radii the true radiative opacity increases steeply. However, beyond r/R ~ 0.6, i.e. for temperatures below about 2.5 million K, the heat transport is actually convection dominated. It is no surprise, then, that the effective opacity required by my model is far smaller than the true radiative opacity at these larger radii. The model with an artificially low opacity is merely mimicking the faster heat transport occurring under convective conditions.

In conclusion, within the region where radiation does dominate, the accepted opacities are broadly compatible with my model. Note that this encompasses the whole of the region where nuclear reactions are taking place (r/R < 0.25, for temperatures above ~9 million K).

3. WHY IS THE HIGH TEMPERATURE OPACITY AS GIVEN ABOVE?

From the above graph and Table we see that the opacity within the region where the nuclear reactions are occurring (i.e. above ~9 million K, and densities of 24,000 – 90,000 kg/m^3) is in the range 1.6 to 3.2 cm^2/g. These are significantly larger than the Thompson opacity due to energy conserving free electron scattering, which is only 0.32 cm^2/g. This is probably due to a combination of at least three factors:-

The first is that, even for hydrogen opacity alone, the contribution of the bound-free and free-free transitions has not reduced to negligible proportions at (say) central solar densities (90,000 kg/m^3) and a temperature of 10 million K. Hence, the lower bound Thompson opacity has not quite been achieved at 10^7 K. This is illustrated below, where the graph is based on the hydrogen opacity formulation of Chapter 16, Equ.(7.1), plus the assumption that the free-free transitions contribute the same opacity as the bound-free transitions.

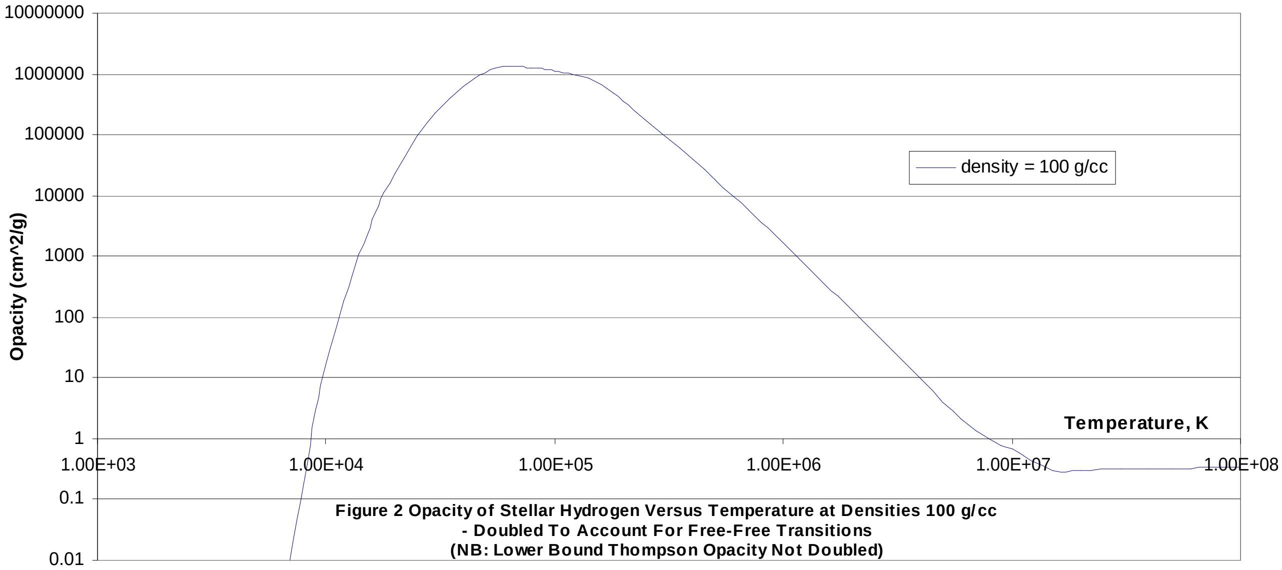


Figure 2 Opacity of Stellar Hydrogen Versus Temperature at Densities 100 g/cc - Doubled To Account For Free-Free Transitions (NB: Lower Bound Thompson Opacity Not Doubled)

However, for a calculation as in Chapter 16, the opacity is very close to the Thompson lower bound – particularly at the solar central temperature of 13.7 million K.

The second factor is the contribution to the opacity due to 'metals'. The Sun contains about 1% by mass of oxygen plus carbon, and around another 0.3% or so of higher mass nuclei (making about ~1.3% in all which is neither H nor He). These 'metals' may contribute a non-negligible amount to the opacity in the important region, i.e. 2 – 14 million K. I'm not sure about this, but in view of the result derived below for the Rosseland mean opacity it would appear that the metals give the dominant contribution to the opacity in this key range. If they do, then it is unfortunate since this implies that a quantification of the mechanism of 'metal' opacity is required to properly understand solar-mass stars' heat transfer and hence structure. It also implies that early stars, with negligible metal content, will behave significantly differently.

The third factor is the recognition of a shortcoming in the method of Chapter 16 for estimating the effective opacity of hydrogen. Chapter 16, Equ.(7.1) actually gives an expression for the opacity which would apply if all the photons had the same energy E_γ. In the numerical evaluations of Chapter 16 we have assumed that this photon energy is the thermal average energy, i.e. $E_\gamma = 2.7kT$. This is assumed in deriving the above graph. However, at any given temperature, photons of all energies can occur, in accord with the black body spectrum. Thus, the effective opacity at a given temperature should actually be a weighted average over the whole of the above graph, where the weighting is determined by the black body spectrum.

There is an 'obvious' way of performing this weighted average which is wrong. It seems reasonable to suppose that the relevant probability distribution is just the normalised number density. Thus, the probability that a photon picked at random has an energy in the range E_γ to $E_\gamma + dE_\gamma$ is,

$$dP[x] = \frac{0.416x^2 dx}{e^x - 1} \quad \text{where,} \quad x = \frac{E_\gamma}{kT} \tag{2}$$

It may be supposed, therefore, that the required averaged opacity would be given by,

$$\langle \kappa \rangle = \int_{x=0}^{x=\infty} \kappa(x) dP[x] \tag{3}$$

where $\kappa(x)$ is the opacity resulting from Chapter 16, Equ.(7.1) for the photon energy indicated by x. Now we are interested in the factor by which $\langle \kappa \rangle$ exceeds the opacity based on the average photon energy, i.e. the factor $\langle \kappa \rangle / \kappa(\langle E_\gamma \rangle)$. Using Equ.(3) gives very large factors. For example, at 10 million K the factor is about 1700. This would result in a mean opacity, $\langle \kappa \rangle$, which is far too big. The reason for it being so large can be seen from the above graph. The contribution to the opacity from lower energy photons can be several orders of magnitude larger than that at the mean thermal energy when the temperature is high.

However, Equ.(3) is simply <u>not</u> the correct weighted average. The correct weighted average is that for which the heat transfer equation, (1), is true. It is apparent from Equ.(1) that this will involve the average of the *reciprocal* of the opacity, $1/\kappa$, i.e. essentially the average of the radiative conductance of the medium. The larger opacities at lower energies will thus contribute little to the average of the reciprocal. The derivation of the correct form of weighted average is given in standard texts, see for example Stahler & Palla Appendix G, Equ.(G.6) of which gives the average as,

$$\frac{1}{\langle \kappa \rangle} = \int_0^\infty \frac{dP}{\kappa} \quad \text{where,} \quad X \cdot dP[E_\gamma] = \frac{d\xi(T, E_\gamma)}{dT} dE_\gamma \tag{4}$$

(where the photon energy, E_γ, has been used rather than its frequency). Thus, not only is it the reciprocal of the opacity which is averaged, but the correct probability density differs from the obvious one of Equ.(2), being given in terms of the temperature density of the black body energy spectrum. [NB: $\xi(T, E_\gamma) dE_\gamma$ is the black body photon energy density at temperature

T due to photons with energies in the range E_γ to $E_\gamma + dE_\gamma$]. The correctly weighted average opacity, as given by (4), is known as the Rosseland mean opacity.

The normalisation factor X in (4) is the temperature derivative of the total black body energy density, and hence is given by,

$$X = \frac{16\sigma_{SB}}{c} T^3 = \frac{4\pi^2}{15} k \left(\frac{kT}{\hbar c} \right)^3 \tag{5}$$

The black body energy spectrum is,

$$\xi(T, E_\gamma) = \frac{1}{\pi^2 (\hbar c)^3} \cdot \frac{E_\gamma^3}{e^{E_\gamma / kT} - 1} \tag{6}$$

and hence the required temperature derivative is,

$$\frac{d\xi(T, E_\gamma)}{dT} = \frac{1}{\pi^2 (\hbar c)^3 kT^2} \cdot \frac{E_\gamma^4 e^{E_\gamma / kT}}{\left(e^{E_\gamma / kT} - 1\right)^2} \tag{7}$$

Now, for photon energies large compared with the hydrogen ionisation potential, B = 13.6eV, Chapter 16, Equs.(5.11, 7.1) show that we can write the bound-free hydrogen opacity for a photon energy E_γ as,

$$\kappa(E_\gamma) = \frac{A}{E_\gamma^{7/2}} \tag{8}$$

in accord with Kramer's Law. Substituting (5), (7) and (8) into (4) gives,

$$\frac{1}{\langle \kappa \rangle} = \frac{15}{4\pi^4 (kT)^5} \int_0^\infty \frac{E_\gamma^{7/2}}{A} \cdot \frac{E_\gamma^4 e^{E_\gamma / kT}}{\left(e^{E_\gamma / kT} - 1\right)^2} dE_\gamma = \frac{15 (kT)^{7/2}}{4\pi^4 A} \Im \tag{9}$$

where, $$\Im = \int_0^\infty \frac{x^{15/2} e^x}{\left(e^x - 1\right)^2} dx = 14116.18 \tag{10}$$

(where the value of $\Im$ has been obtained by numerical integration). Thus the factor by which the Rosseland mean opacity, as given by (9), exceeds the opacity based on the mean photon energy can be found by normalising by,

$$\kappa(\langle E_\gamma \rangle) = \frac{A}{\langle E_\gamma \rangle^{7/2}} = \frac{A}{(2.7kT)^{7/2}} \tag{11}$$

giving, $$\langle \kappa \rangle = \frac{4\pi^4 (2.7)^{7/2}}{15\Im} \kappa(\langle E_\gamma \rangle) = 0.0595 \kappa(\langle E_\gamma \rangle) \tag{12}$$

Thus, we see that the Rosseland mean opacity is actually *less* than the opacity evaluated at the mean energy, namely only about 6% of it. Hence this correction, rather than helping explain the required opacity, actually makes things worse – i.e. the Rosseland mean opacity for hydrogen is negligible at 10 million K.

We conclude that it must be the ‘metal’ opacity which dominates at around 10 million K. I need to get better data on the metal contributions to check this important conclusion. It implies that the Sun cannot be understood properly based on hydrogen opacity alone. It also implies that early stars, with negligible ‘metal’ content, would behave significantly differently. They would be less opaque than the Sun and hence would have reduced temperature gradients for a given luminosity. I think this means that a star of a given mass would be less luminous and have lower temperatures.

NB: it is reasonable that the mean opacity derived from averaging the reciprocal will be less than the opacity at the mean energy. Take as a simple example three values: 10, 1, and 0.1 occurring at a low, mean and high energy respectively. Their simple average is 3.7, i.e. the large value at the low energy pulls the mean value up above the value at the mean energy. However, their reciprocals are 0.1, 1 and 10, so the mean of the reciprocals is also 3.7 and hence the reciprocal of the mean reciprocal (the Rosseland mean) is only 0.27. In this case the low value at the high energy pulls the ‘Rosseland mean’ down below the value at the mean energy.