Chapter 18 – Stellar Structure 2: Numerical Solutions, Heat Production, Heat Transport, and Dependence on Mass

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1. Introduction

In Chapter 11 we derived the equation for hydrostatic equilibrium in several different forms:-

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \tag{1a}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}.\frac{dP}{dr}\right) = -4\pi r^2 G\rho \tag{1b}$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \tag{1c}$$

$$Gm(r)^2 = -8\pi\int_o^r r'^4 dP \tag{1d}$$

where P, ρ and m all vary with radius (r), and m is the mass within radius r. None of these equations involve just one dependent variable, and hence do not suffice to find a solution. We require further physical input in the form of another equation.

We noted in Chapter 11 that, in certain circumstances, the pressure and density are related by a polytropic (power-law) expression,

$$P = K\rho^{\gamma} \tag{2}$$

If a polytropic P-ρ relation is assumed, the hydrostatic equilibrium equation becomes,

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{\gamma K r^2}{\rho^{2-\gamma}}.\frac{d\rho}{dr}\right) = -4\pi\rho G \tag{3}$$

This is now an equation with just one dependent variable (i.e. density, when written in this form) and hence can be solved subject to suitable boundary conditions.

In Chapter 11 we discussed a simple analytical approximation for the pressure, density and temperature distributions, accurate only near the centre of stars (the Clayton model). This did not make use of the polytropic equation [Equs.2 or 3], but only of hydrostatic equilibrium [Equs.1]. Thus, whilst hydrostatic equilibrium is respected by the Clayton model, Equs.[2,3] are not. Hence the Clayton model is not really a correct solution, and this shortcoming becomes apparent at larger radii. In particular, the Clayton model is completely wrong as regards the temperature predicted near the surface of the star.

In this Chapter we shall derive numerical solutions of Equ.(3), assuming values for the polytropic parameters K and γ. Since Equ.(3) is a second order differential equation, it requires two boundary conditions to uniquely specify a solution. One of these is $d\rho / dr|_{r=0} = 0$, i.e. that the density is maximum at the centre. The other is conveniently taken for the purposes of numerical integration as the value of the density at the centre, ρ_c - although at this point we do not know what this central density is, for, say, a star of a given mass.

Actually, we have no right to expect the resulting distributions of pressure, density and temperature to be correct. This is because, thus far, we have completely ignored the issue of nuclear heat production and the transport of heat outwards from the centre. The star's power density will be sensitive to the temperature and density, whilst the heat flux will be related to the temperature gradient. To maintain a steady temperature it is necessary for the rate of heat production to be balanced by the (net) rate of heat transport away from its point of production. Since both the power density and the heat flux will be determined by the density and temperature distributions, it would be rather remarkable if the solution to the polytropic-hydrostatic equation, (3), just happened to respect local thermal equilibrium (LTE) without this physical constraint having being built into the equations.

There are, however, three degrees of freedom available, namely the values of K, γ and ρ_c. One of these degrees of freedom is removed by requiring a star of a given mass, M. This leaves two degrees of freedom which can be tuned to best respect LTE. We illustrate this with a solar model which appears to give a reasonably good representation of all physical quantities. In particular, the surface temperature is predicted quite well from the nuclear energy production.

Nevertheless, it is still unclear if any solution exists [i.e. to Equ.(3)] if LTE is also required. Strict LTE is a constraint at all radii, and hence cannot be achieved with a finite number of constraints (i.e. the equation set is over-constrained). The solution to this paradox is that polytropic behaviour is not strictly valid and has been introduced only as a simplification. We therefore present in Section 3.4 the true equation set in which equations representing LTE replace the polytropic expression, (2). We do not present detailed solutions but instead we derive how the solar model can be scaled to apply, approximately, for stars of different mass. Amongst other interesting scaling behaviours, we find that the luminosity of a star is sensitive to its mass ($L \propto M^{\alpha_L}$) and that the value of the exponent, α_L, is ~3.

2. A Polytropic-Hydrostatic Solar Model

2.1 Values of the Parameters K, γ and ρ_c

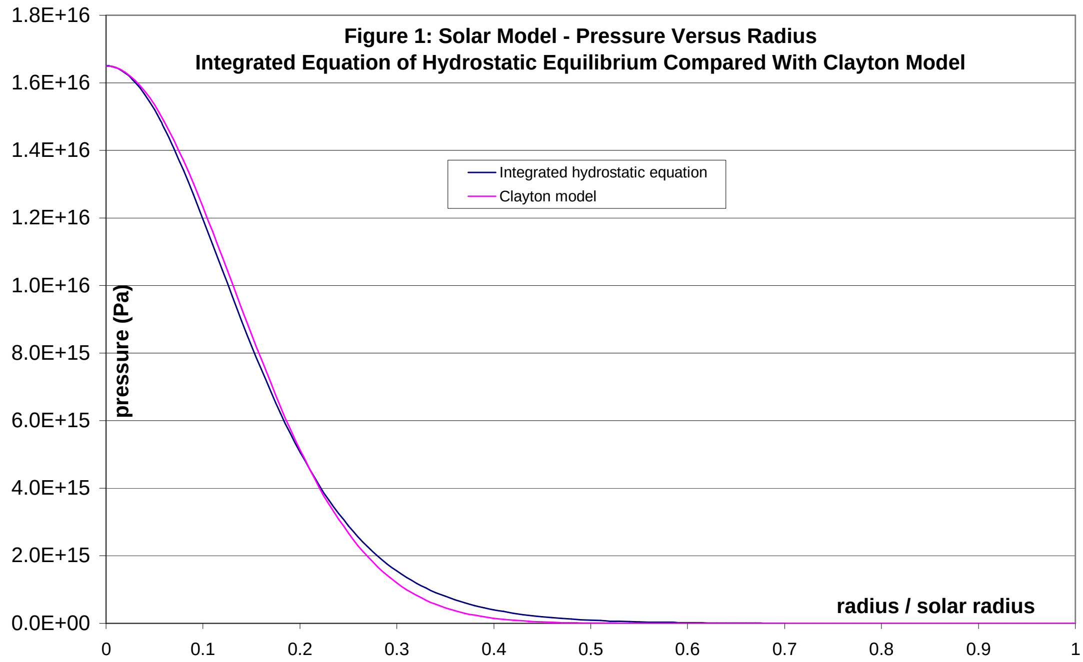
In Chapter 11 we have seen that the Clayton model is consistent with a polytropic equation at the centre of the star given by $\gamma = 4/3$ and $K = 0.444GM^{2/3}$. For a solar mass star ($M = 2 \times 10^{30}$ kg) this gives $K = 4.70 \times 10^9$ (MKSA units). Hence, from (2), the central pressure corresponding to an assumed central density of 90,000 kg/m^3 is 1.90×10^{16} Pa, and the central temperature is 15.8×10^6 deg.K (for an average particle mass of 1.0321×10^{-27} kg). These are slightly too high for a solar model (more rigorous models apparently giving around 1.65×10^{16} Pa and 13.7×10^6 deg.K).

More seriously, perhaps, using $\gamma = 4/3$ and $K = 4.70 \times 10^9$ (MKSA units) and integrating Equ.(3) numerically, from a central density of 90,000 kg/m^3, gives an inconsistent total mass of ~2.7 x 10^{30} kg (obtained by integrating the density), i.e. 35% larger than we initially assumed to estimate K. The resulting radius of the star is also too large to represent the Sun, by about 5%. It might be imagined that these shortcomings could be fixed by reducing the assumed central density a little. However, rather surprisingly, it turns out that changing the assumed central density alone does not change the resulting total mass. Instead, the radius of the star reduces (proportionally as $1/\rho_c^{1/3}$) so that the total mass is invariant.

To change the total mass it is necessary to change the polytropic parameter, K. Thus, to fine-tune the model for the sun we have used $\gamma = 1.33$ and $K = 4.25 \times 10^9$ (MKSA units). These reproduce the target pressure and temperature at the centre (1.65 x 10^{16} Pa and 13.7 x 10^6 deg.K) for a central density of 90,000 kg/m^3. Integration of Equ.(3) results in the correct radius for the Sun (7.0 x 10^8 m), and an integrated mass which is within 10% of the correct value (2.2 x 10^{30} kg). This appears to be the best we can do with our highly simplified model. [Note that using $\gamma = 1.33$ rather than $\gamma = 4/3$ makes a significant difference – about 4% on central temperature, but far larger differences to the surface temperature, perhaps 16%-65%, due to the extreme sensitivity of nuclear reaction rates to the temperature. More importantly, note that using $\gamma = 1.33$ is a cheat, since this is actually an unphysical value. For reasons we shall not go into here, values of γ less than 4/3 imply an unstable star].

Unfortunately it is not clear how to generalise the above parameters for stars of different mass. From the Clayton solution one possibility might be to assume $\gamma = 1.33$ and $K = 4.25 \times 10^9 (M / M_{Sun})^{\gamma/2}$ (MKSA units), but this is no more than a guess. Moreover, we have as yet given no method for finding the central density. It is worth asking yourself whether intuition suggests the central density would be larger or smaller for higher mass stars.

2.2 Model Results for the Sun Compared With Clayton Model



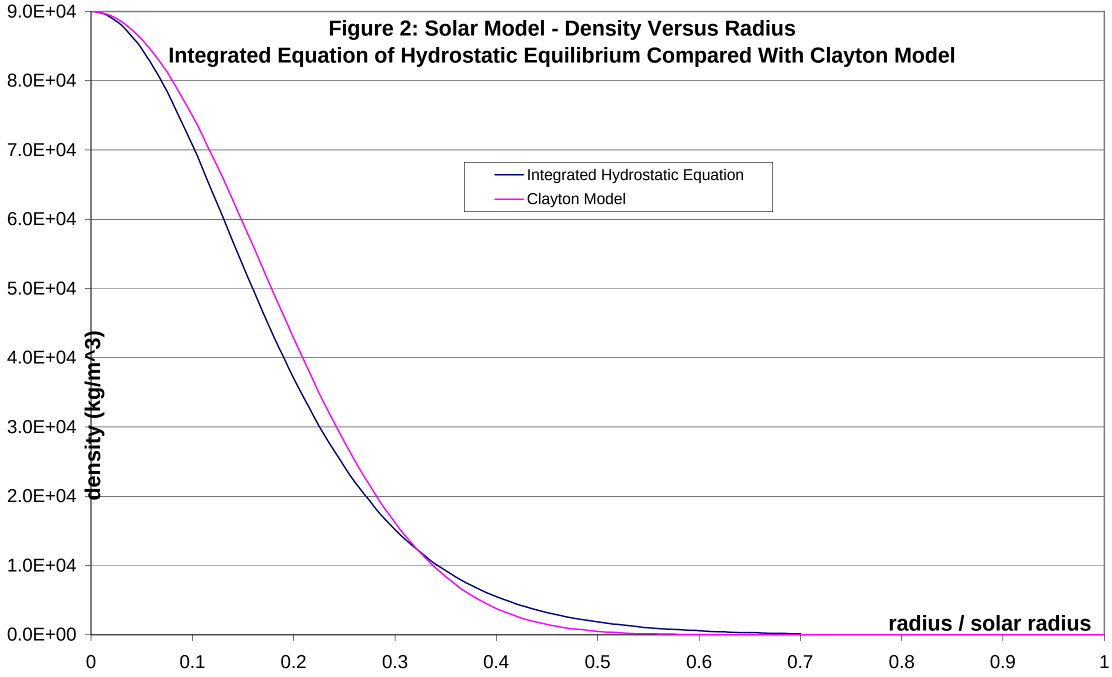


Figure 2: Solar Model - Density Versus Radius
Integrated Equation of Hydrostatic Equilibrium Compared With Clayton Model
Integrated Hydrostatic Equation
Clayton Model
density (kg/m^3)
radius / solar radius
9.0E+04
8.0E+04
7.0E+04
6.0E+04
5.0E+04
4.0E+04
3.0E+04
2.0E+04
1.0E+04
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0.1
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0.9
1

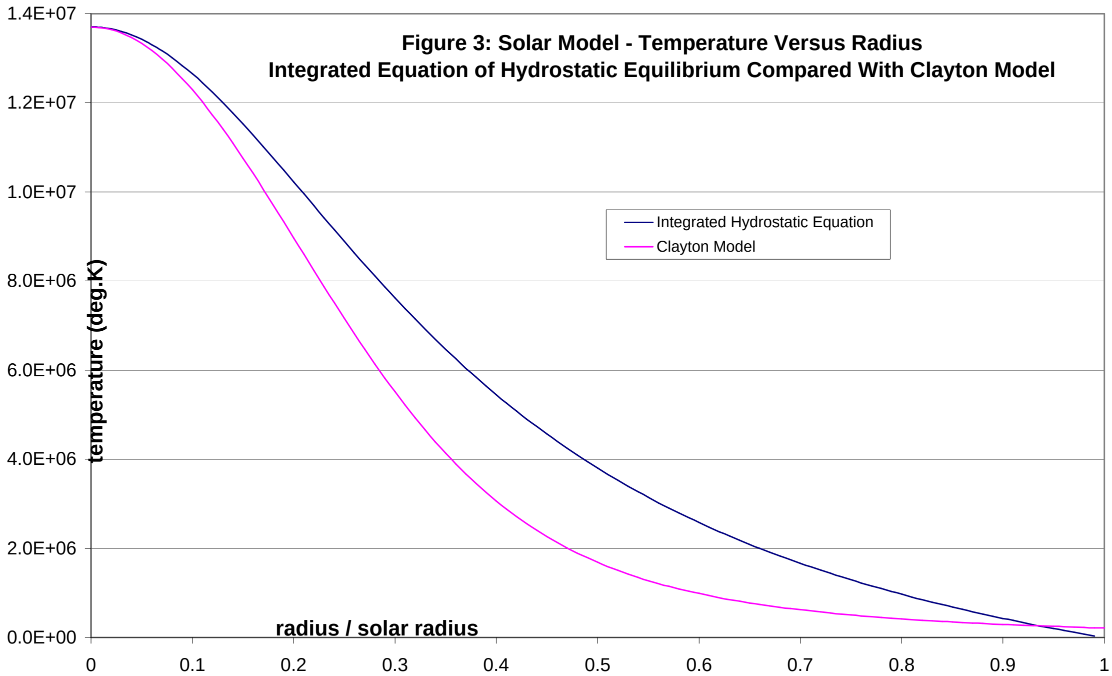
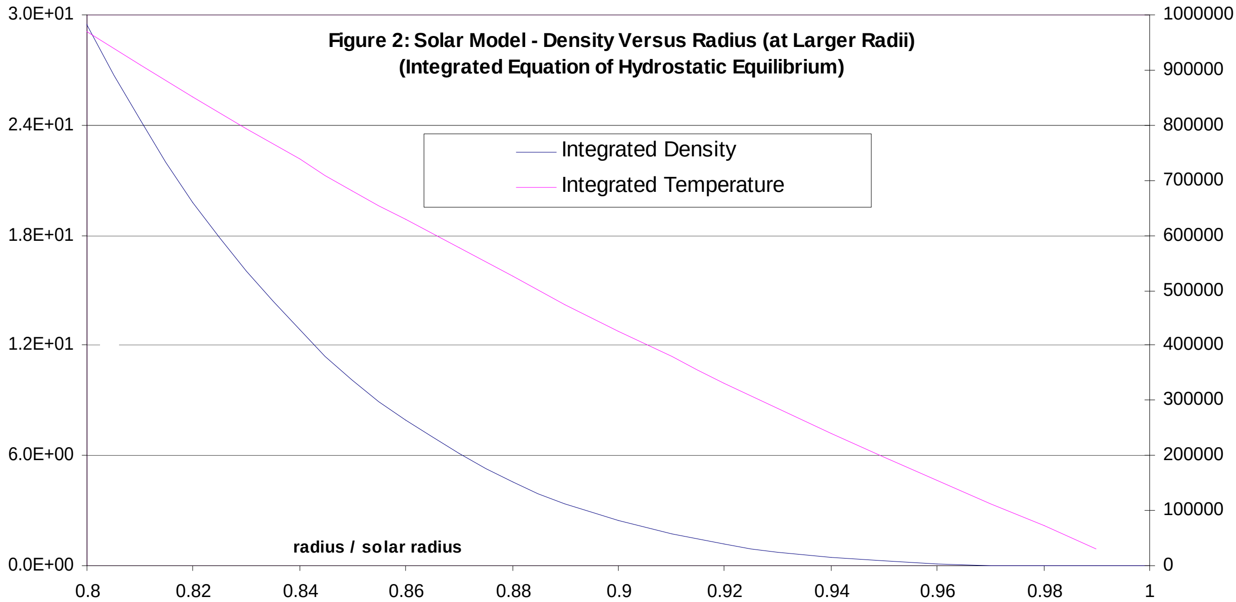


Figure 3: Solar Model - Temperature Versus Radius
Integrated Equation of Hydrostatic Equilibrium Compared With Clayton Model
Integrated Hydrostatic Equation
Clayton Model
temperature (deg.K)
radius / solar radius
1.4E+07
1.2E+07
1.0E+07
8.0E+06
6.0E+06
4.0E+06
2.0E+06
0.0E+00
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Thus, we see that the Clayton model results for pressure and density are very close to the numerical integration of the polytropic-hydrostatic differential equation, (3) – at least for radii less than about half the solar radius. The temperature profile, however, is significantly different. This is important as regards predicting the surface temperature and heat transfer rates.

Note that, unlike the Clayton model, the pressure, density and temperature all fall to zero at a well defined radius – so the numerical model gives a clear prediction for the stellar size. This is illustrated by plotting the fields at larger radii on an expanded scale, below:-



r (m)	density, kg/m^3	pressure, Pa	Temperature, deg.K	r/Rsun
6.68E+08	0.113035	234000000	154672	0.96
6.75E+08	0.043227	65153969	112629	0.97
6.82E+08	0.010882	10404263	71444	0.98
6.89E+08	0.000875	363860	31090	0.99
6.96E+08	< 0	< 0	< 0	1

This leaves us with the problem of determining what we mean by the temperature of the "surface" of a star, and what its value is in this case.

2.3 Nuclear Heat Production and Surface Temperature

Once the temperature and density are known at all radii, evaluating the total power production follows from knowledge of the nuclear reaction rates. As regards "hydrogen burning", i.e. when the consumed fuel is the inventory of protons, there are two families of reaction paths of importance: the pp sequences and the CN cycle. For solar mass stars, the pp sequences dominate heat production (see Section 3.1)

The pp sequences have been discussed in Chapter 13. There are three distinct sequences of reactions, denoted ppI, ppII and ppIII. However, the net heat produced by ppI and ppII are almost the same, and these two sequences account for 99.98% of the reactions. Thus, a weighted average of 26.13 MeV of heat is produced per helium-4 nucleus created (or, equivalently, an average heat production of 6.53 MeV per proton consumed). All three pp sequences share the common initial reaction $p + p \rightarrow D + e^{+} + \nu$, and since this is the slowest, and hence rate-determining, step, the power is simply the rate of this reaction times 6.53 MeV times 2 for the two protons consumed. The rate of this reaction has been taken from Hoffman et al, the relevant numerical data being:-

T (^oK)	**E = 1.5kT (MeV)**	**Reaction Rate s^{-1} (mole/cm^3)$^{-1}$**
10 x 10^6	0.00129	7.21 x 10^{-21}
30 x 10^6	0.00388	4.59 x 10^{-19}
50 x 10^6	0.00647	1.9 x 10^{-18}

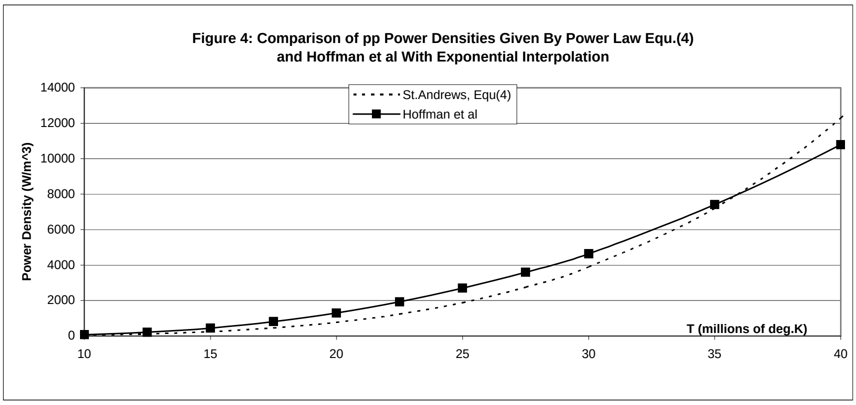
To interpolate at different temperatures we have fitted to both a power-law and an exponential form, i.e. $\text{Rate} \propto T^{\alpha}$ or $\text{Rate} \propto \exp(-b/\sqrt{E})$. The values of the parameters fitted to the Hoffman data are,

T (^oK)	**α**	**b √MeV**
10 x 10^6 to 30 x 10^6	3.78	0.352
30 x 10^6 to 50 x 10^6	2.78	0.393

A power-law fit which is sometimes used is,

$$\text{Power density} = 120\left(\frac{X_h}{0.5}\right)^2\left(\frac{\rho}{10^5 kg/m^3}\right)^2\left(\frac{T}{15x10^6 K}\right)^4 \text{ W/m}^3 \qquad (4)$$

Thus, we see that the Hoffman data gives a comparable power law exponent (3.78) for the temperature range 10 to 30 million ^oK. (Hoffman implies a less steep dependence on temperature at temperatures above 30 million degrees, but this is less important since the CN sequences dominate above 20 million K in any case). The two sources compare as follows: for an assumed density of 90,000 kg/m^3 and a star comprising 75% hydrogen, the predicted power densities are:-



Using Hoffman et al with the density and temperature distributions derived above from the numerically integrated polytropic-hydrostatic model gives a total energy production of 4.21 x 10^{26} W (obtained by numerical integration over the radius). This assumes nuclear reaction rates are negligible below 10 million degrees K, as illustrated in the above graph. The surface temperature of the Sun can be calculated as the temperature required for the Sun's luminosity to equate with this rate of production of heat, i.e. by setting $4\pi R^2 \sigma_{SB} T_S^4$ equal to 4.21 x 10^{26} W. The radius is 7 x 10^8 m and the Stefan-Boltzman constant is 5.67 x 10^{-8} $Wm^{-2}K^{-4}$, so we find T_S = 5915 deg.K. This should align with the Sun's apparent temperature as measured bolometrically, and is indeed in good agreement (the latter is perhaps closer to 5885 deg.K).

2.3 Heat Transport – Consistency of the Model?

So far we have said nothing whatsoever about the transport of heat radially through the star. For our model to be physically self-consistent the rate at which the heat is transported outwards through any sphere of radius r must equal the rate at which heat is being produced within that same sphere. Were this not the case, the temperature distribution could not be static. We expect the heat flux to be proportional to the local temperature gradient. We can therefore define an effective thermal "conductivity", K_{eff}, in the usual way, i.e.,

$$Q = K_{eff} \frac{dT}{dr} \tag{5}$$

where Q is the heat flux (i.e. heat through unit area per unit time), and we have assumed the temperature gradient to be radial (spherically symmetric star). Of course, the actual mechanism of heat transfer is unlikely to be conduction. In very dense stars, e.g. white dwarfs, this might be the case. For main sequence stars, the heat transfer is often dominated by radiation in the region where nuclear reactions occur. There are generally other regions, however, where convection dominates. The radiative or convective behaviour of stars is very complex. The regions which are dominated by one mechanism or the other vary in size and position as the star ages. For now we consider radiation only. The effective "conductivity", K_{eff}, is therefore determined by the extent to which the stellar medium absorbs the radiation.

We have seen in Chapter 12 that the stellar medium is opaque. Were it not, gamma rays from the centre at tens of millions of K would escape into space. The more opaque the stellar medium is to radiation, the lower the effective conductivity when radiation dominates. The opacity, κ, of the stellar medium is defined, roughly speaking, to be a property of the constituent particles (in their prevailing state of ionisation and/or excitation), rather than a measure of how many such particles there are. Hence, we expect the degree of absorption of radiation to also be proportional to the density (i.e. double the number of particles, double the absorption for a given particle opacity). Clearly, the absorption will also be proportional to the thickness of stellar material traversed, for sufficiently short distances. Putting these together, the decrease in the radiant energy density over a short length Δr can be written,

(roughly) $$\Delta I = -\kappa\rho I \Delta r \quad \text{or} \quad \frac{dI}{dr} = -\kappa\rho I \tag{6}$$

from which it follows by integration that, if the opacity and density are constant over a region, then the radiant energy density will reduce exponentially with Δr in the usual way. More generally, the radiant energy density at point 2 in terms of that at point 1 is,

$$I_2 = I_1 \exp\{-\int_{r_1}^{r_2} \kappa\rho dr\} \qquad (7)$$

The exponent $\int_{r_1}^{r_2} \kappa\rho dr$ is known as the optical depth (dimensionless). At any point, the mean free path of a photon is $1/\kappa\rho$, so the optical depth is the thickness of medium traversed in units of the mean free path. Thus, values much bigger than unity imply most of the radiation will be absorbed, whilst values much smaller than unity mean that little radiation will be absorbed. [Note that "optical" depth is really a misnomer since the preceding discussion applies whatever the frequency of the radiation in question]. Finally we note that opacity, κ, has units of $g^{-1}.cm^2$ or $kg^{-1}.m^2$.

However, we note that Equ.(6) is not really correct. This is because it is only that part of the radiation which is travelling radially which suffers net absorption. If we consider radiation in the circumferential direction, it will be absorbed just as much (per unit length). However, radiation is also re-emitted. The net effect is that there is no circumferential temperature gradient because loss of radiation circumferentially is balanced by the gain of radiation from circumferentially displaced regions. Thus, the correct form of Equ.(6) must involve directed quantities. Crudely speaking we may associate 1/3 of the energy density with each Cartesian direction. Hence, if we now consider just the radially oriented radiation, we replace I on the LHS of (6) with I/3 (the same factor results from considering the LHS to be the radiation pressure). For consistency, on the RHS, I is replaced by the heat flux in the radial direction (divided by c), the heat flux (Q) being just the luminosity (L) divided by $4\pi r^2$. Thus (6) becomes,

$$\frac{1}{3}\frac{dI}{dr} = -\frac{\kappa\rho}{c}\cdot\frac{L}{4\pi r^2} \qquad (6b)$$

Assuming that the departure from thermal equilibrium is slight, we can approximate I by the black body energy density, i.e. $I = \frac{4}{c}\sigma_{SB}T^4$ (see Appendix A0). The radial variation of I thus follows from that of the temperature field, i.e.,

$$\frac{dI}{dr} = \frac{dI}{dT}\frac{dT}{dr} = \frac{16}{c}\sigma_{SB}T^3\cdot\frac{dT}{dr} \qquad (8)$$

Hence we get,

$$\frac{dT}{dr} = -\frac{3\kappa\rho}{16\sigma_{SB}T^3}\cdot\left(\frac{L}{4\pi r^2}\right) = -\frac{3\kappa\rho}{16\sigma_{SB}T^3}\cdot Q \qquad (9)$$

from which we see that the effective "conductivity" is,

$$K_{eff} = \frac{16\sigma_{SB}T^3}{3\kappa\rho} = \frac{16\lambda\sigma_{SB}T^3}{3} \qquad (10)$$

where λ is the photon mean free path at the location in question.

This is all very well, but we can make no progress unless the opacity (or, equivalently, λ or K_{eff}) is known. This is not an easy problem in general (see Section 3.3 or Chapters 16 or 19 for further discussion). In order to examine the self-consistency of our solar model we will limit ourselves to just a quick reasonableness check.

To this end we consider only the radius at which the temperature falls to 10 million K, this being a temperature below which the nuclear reactions are becoming insignificant. From the numerically integrated polytropic-hydrostatic equation we find that this occurs at a radius $r/R_{sun} = 0.21$ ($r = 1.46 \times 10^8$ m) at which point the temperature gradient is 0.0385 deg.K/m. The integrated energy production within the sphere of radius 1.46×10^8 m (luminosity) was found to be 4.2×10^{26} W. To get the same total heat transport through the sphere of radius r for the above temperature gradient requires an effective "conductivity" of 4.08×10^{10} W/mK. This is achieved for an opacity of 0.217 $kg^{-1}m^2$ (= 2.17 $g^{-1}cm^2$), using the density at this radius which is 34,200 kg/m^3. Is this opacity correct, for the prevailing local pressure and temperature? Well, it appears to be about right - see Section 3.3.

3. Models For Stars of Other Masses

3.1 The CN Sequence

For solar mass main sequence stars, for which the temperatures do not exceed ~14 million K, the conversion of protons to helium-4 is dominated by the pp sequences. The CN sequence becomes dominant above a temperature of about 19 million K (see below). We have already seen that the rate of the pp sequence is sensitive to temperature, but the CN sequence is dramatically more sensitive. Thus, for stars a little more massive than the Sun, it is necessary to include the power production by the CN sequence.

The rate controlling (slowest) step in the CN sequence is ${}^{14}_{7}N + p \rightarrow {}^{15}_{8}O + \gamma$. Data on the rate of this reaction has been obtained from the NACRE web site:-

T (^{o}K)	**Reaction Rate s^{-1} (mole/cm^3)$^{-1}$**	**α***	**b#**
8×10^6	1.03×10^{-24}	24.2	
9×10^6	1.78×10^{-23}	23.3	
10×10^6	2.08×10^{-22}	22.5	
11×10^6	1.77×10^{-21}	21.8	
12×10^6	1.18×10^{-20}	21.2	1.675
13×10^6	6.44×10^{-20}	20.6	
14×10^6	2.97×10^{-19}	20.1	
15×10^6	1.19×10^{-18}	19.7	
16×10^6	4.24×10^{-18}	19.1	1.773
18×10^6	4.00×10^{-17}	18.3	
20×10^6	2.76×10^{-16}	17.3	1.879
25×10^6	1.31×10^{-14}	16.1	1.908
30×10^6	2.48×10^{-13}	14.9	1.991
40×10^6	1.78×10^{-11}	13.6	2.053
50×10^6	3.68×10^{-10}	12.6	
60×10^6	3.68×10^{-9}	-	

**Exponent in the power law fit: Rate $\propto T^{\alpha}$. The value given relates to the temperature interval ending at the next temperature listed.*

#Parameter in the exponential fit: Rate $\propto \exp\{-b/\sqrt{E}\}$, where E =1.5kT

An expression for the power density produced by the CN sequence which is sometimes employed is ,

$$\text{Power density} = 2.0\left(\frac{X_h}{0.5}\right)\left(\frac{X_{N14}}{0.006}\right)\left(\frac{\rho}{10^5\,kg/m^3}\right)^2\left(\frac{T}{15x10^6\,K}\right)^{16}\ \text{W/m}^3 \qquad (11)$$

Note that the power from the CN cycle depends linearly on the nitrogen-14 density, because this controls the rate of the rate determining reaction step. A first generation star will not initially contain any nitrogen-14, and hence the CN sequence will not initially be relevant.

We note that, in the temperature range of greatest interest, perhaps 19 to 25 million K, the NACRE solution implies a power law exponent of about 16-18, in agreement with Equ.(11) (although the exponent could probably be "anywhere between 13 to 18"). The origin of this extreme sensitivity to temperature is discussed in an Annex to this Chapter, below.

Rather disconcertingly, Hoffman et al, whilst giving the same reaction rate as NACRE at 50 million K, give a rate a factor of 19 times faster than NACRE at 30 million K. The Hoffman rate looks odd since it corresponds to a power law exponent of only about 8.5. For this reason we prefer the NACRE solution.

We shall compare the power law power density from Equ.(11) with that based on the NACRE reaction rate assuming a density of 90,000 kg/m^3, and a composition by <u>mass</u> fraction of 75% protons and 0.6% N14. The number density of protons is thus 5.4×10^{25} cm^{-3}, which equals 89.5 moles per cm^3. Since N14 is ~14 times as massive as a proton, the number density of N14 is (0.6 / 14*75) 5.4 x 10^{25} cm^{-3} = 3.1 x 10^{22} cm^{-3} = 3.1 x 10^{28} m^{-3}. Now each $^{14}_{7}N + p \rightarrow ^{15}_{8}O + \gamma$ reaction is part of a cycle which converts four protons to a helium-4 nucleus. Hence the associated heat energy released for each such reaction is, overall, 26.2 MeV = 4.18 x 10^{-12} J. Thus, the reaction rate in $s^{-1}(mole/cm^3)^{-1}$ is converted to a power density by multiplying by the factor,

Power Density =
Rate in $s^{-1}(mole/cm^3)^{-1}$ x 89.5 moles per cm^3 x 3.1 x 10^{28} m^{-3} x 4.18 x 10^{-12} J
= Rate in $s^{-1}(mole/cm^3)^{-1}$ x 1.16 x 10^{19} (W/m^3)

Thus the predicted power densities compared are:-

T (K)	Equ.(11) W/m^3	NACRE W/m^3
10	0.0041	0.0024
11	0.019	0.02
12	0.076	0.136
14	0.9	3.4
15	2.7	13.7
16	7.6	49
18	50	461
20	269	3,180
25	9,570	151,000
30	177,000	2,860,000

Figure 5: Comparison of CN Power Densities Given By Power Law Equ.(11) and NACRE data (from Internet)

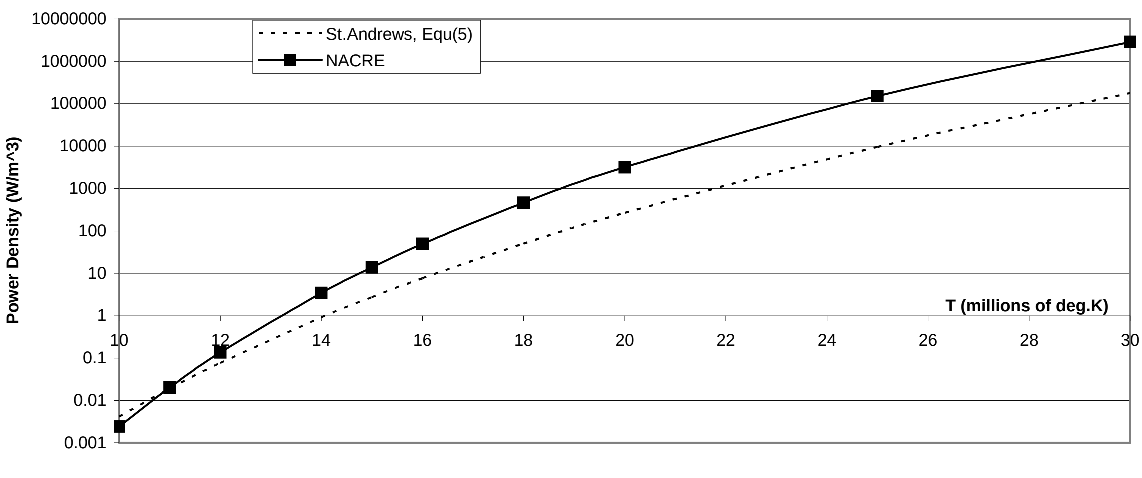
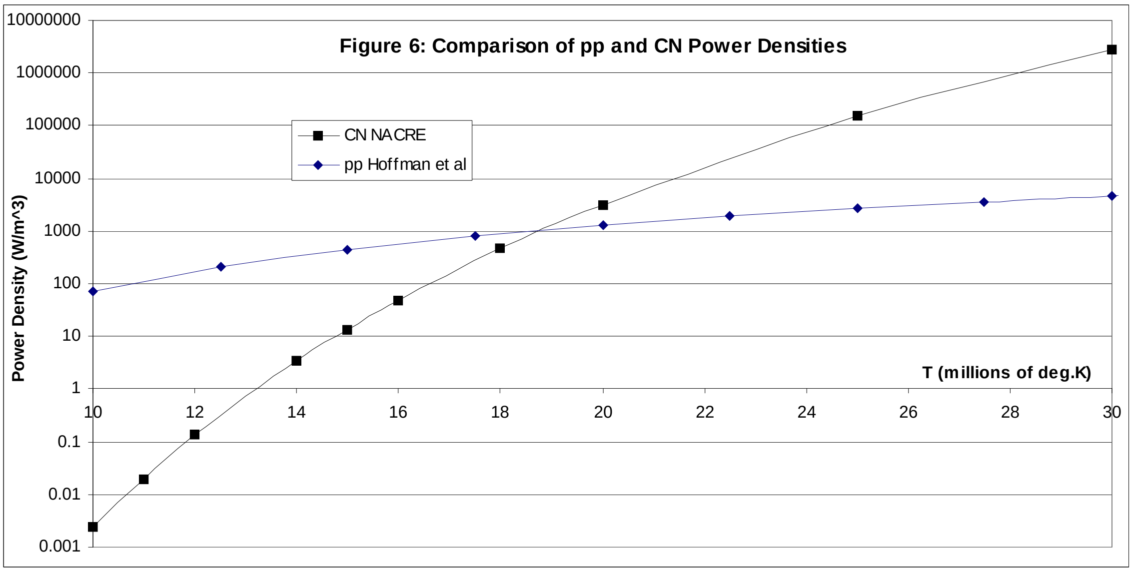


Fig.5 shows that there is a substantial difference between the NACRE and Equ.(11) power densities for the CN sequence, by up to an order of magnitude at ~30 million K, although this is equivalent to only a few (million) degrees shift in temperature. Fig.6 shows that the CN power density increases by 9 orders of magnitude between 10 million K and 30 million K, whereas the pp rate increases by less than 2 orders of magnitude – hence their very different power law exponents. Fig.6 also shows that the power densities from the pp and CN sequences are about equal at around 19 million K. A similar result is obtained by equating the expressions, Equs.(4) and (11), which gives equality of power density at $60^{1/12}$ times 15 million K, i.e. at about 21 million K.



3.2 The Lifetime Paradox

In passing we note that John Bahcall gives a characteristic time for the $^{14}_{7}N + p \rightarrow ^{15}_{8}O + \gamma$ reaction as 3 x 10^8 years at central solar conditions. This is in contrast to his quoted characteristic time for the initial pp reaction ($p + p \rightarrow D$), i.e. 1.4 x 10^{10} years. The argument is made that the latter is what controls the lifetime of solar mass stars, i.e. of the order of 10^{10} years. The paradox here is that Bahcall's reaction time for the CN sequence would naively seem to imply that a solar mass star would have a lifetime of only 300 million years. Taking the $^{14}_{7}N + p \rightarrow ^{15}_{8}O + \gamma$ reaction rate to be 1.19 x 10^{-18} (i.e. at 15 million K), the proton density is 89.5 mole/cm^3 so that the reaction rate per N14 nucleus is 1.07 x 10^{-16} s^{-1}. The reciprocal of this is the typical time between reactions for a given N14 nucleus, and is indeed 3 x 10^8 years, which confirms Bahcall's result. But we know from the preceding results that at solar temperatures (taken here as ~15 million K) the nuclear reactions are dominated by the pp sequence, so how can we apparently conclude a lifetime some two orders of magnitude shorter based on a reaction sequence (CN) which is hardly significant at all?

The resolution of the paradox is that the typical reaction time per N14 nucleus is not the relevant timescale as regards estimating the star's life. In the CN sequence, the N14 acts as a catalyst. It is not consumed overall, but rather is created and consumed in equal measure. To determine the star's lifetime we must consider the rate at which the fuel is being consumed. The fuel is the inventory of protons. Recall that both the pp and CN sequences have, as their net effect, the conversion of four protons to helium-4. We therefore need to consider the typical reaction time per proton. This is less than that per N14 nucleus simply because the ratio of their numerical preponderance is (0.6/75*14) = 0.00057. Thus, the reaction time per proton is around 3 x 10^8 years / 0.00057 = 5 x 10^{11} years. This is now far greater than the pp based lifetime (of ~10^{10} years), as it should be since the CN reactions are minor at 15 million K.

The utility of this observation is that it gives us a crude means of estimating the order of magnitude of a star's life if we know its central temperature (and if we can assume a density of 90,000 kg/m^3 and an N14 mass fraction of 0.6% - both of which will be wrong in general, i.e. for stars not of solar mass and composition). Thus, assuming CN dominance, at 20 million K we estimate 2 billion years, whereas at 25 and 30 million K we estimate 50 million years and 2-3 million years respectively. We will see in Section 3.5 that these are not bad estimates.

3.3 Formulation of Heat Transport: Radiation versus Convection

Radiation versus Convection….Equs for each; Convective stability – where is the convection boundary?

The photosphere – why it's tricky to model

More on opacity. **<<< This Section needs adding >>>**

3.4 Equation Set With Radiative Heat Transport

In the Introduction we alluded to the shortcoming of the polytropic assumption, Equ.(2), namely that there was no degree of freedom left to impose the constraints of heat transport. In this Section we present a set of equations based on the assumption of radiation dominated heat transport, but drop the polytropic assumption. Polytropic behaviour may or may not arise as an approximate property of the solutions of such equation sets. As regards radiation dominance of the heat transfer, this is expected to be a good approximation within the region of nuclear heat production for stars of solar mass and above.

The equations are conveniently expressed in terms of the mass, m, within radius r as the independent variable. They are:-

1) The definition of m: $$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \qquad (12a)$$

2) Local Thermal Equilibrium: $$\frac{dL}{dm} = \frac{\varepsilon}{\rho} = \varepsilon_m \qquad (12b)$$

3) Hydrostatic Equilibrium: $$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \qquad (12c)$$

4) Radiative Heat Transport: $$\frac{dT}{dm} = -\left(\frac{3}{256\pi^2 \sigma_{SB}}\right)\frac{\kappa L}{r^4 T^3} \qquad (12d)$$

5) The Gas Law (Equation of State): $$P = A\rho^{\chi_\rho} T^{\chi_T} \qquad (12e)$$

where L is the luminosity at radius r; ε is the nuclear heat production per unit volume at radius r; κ is the opacity of the stellar medium at radius r; and A is some constant. For perfect gas behaviour we have $\chi_\rho = \chi_T = 1$. For later convenience we have also introduced the nuclear heating rate per unit mass, i.e. $\varepsilon_m = \varepsilon / \rho$.

To solve equations (12a-e) we assume that the opacity, κ, and the power density, ε, can be found at any radius r in terms of the local temperature and density, T and ρ, and the local composition of the stellar medium. In other words, we assume we can calculate numerically the value of the functions,

$$\varepsilon[T, \rho, \text{composition}] \quad \text{and} \qquad \kappa[\mathrm{T}, \rho, \text{composition}] \tag{13}$$

In this case we see that the five equations (12a-e) involve five dependent variables: r, L, T, P and ρ, (where the independent variable is m) and hence form a soluble system.

3.5 Scaling With Mass – The Homologous Model

Explicit solution of equations (12a-e) must be done numerically due to their non-linearity and the fact that the opacity and power density functions, (13), are available only numerically. This, together with augmentations to account for convective heat transport, forms the basis of comprehensive stellar models. We shall not attempt this here. Instead we shall use a much simpler, approximate, approach which suffices to establish how the main properties of a star (size, temperature, luminosity, density and lifetime) depend upon its mass. This approach has two great advantages,

- It is not necessary to solve any differential equations at all;
- Numerical data for the opacity and nuclear reaction rates are not required – only their dependence upon temperature and density.

Thus we shall assume that (13) are power-laws,

$$\varepsilon_m = \varepsilon_0 \rho^\lambda \mathrm{T}^\nu \quad \text{and} \qquad \kappa = \kappa_0 \rho^n \mathrm{T}^{-s} \tag{14}$$

[Noting that it is the power density per unit mass, not per unit volume, which we have defined in (14)]. The coefficients κ_0 and ε_0 must depend upon composition. However, we shall consider stars only of the same composition as the sun, and hence κ_0 and ε_0 are constants for this family of stars. We are interested in how the properties of these stars depend on their mass.

We note that the complete set of equations, i.e. Equs.(12a-e) plus (14), involve only power laws. Since we are considering a one-parameter family of solutions, parameterised by the stellar mass, M, this suggests that the solutions may be self-similar. In other words, if for the sun of mass M_0 the independent variable is called m_0 and the solutions are written as,

$$r = f_r(m_0), \quad L = f_L(m_0), \quad P = f_P(m_0), \quad T = f_T(m_0), \quad \rho = f_\rho(m_0) \tag{15}$$

then, under a scaling of the independent variable, m, as follows,

$$m = \frac{M}{M_0} \cdot m_0 \tag{16}$$

the new solution, for a star of mass M, will scale as powers of the mass ratio, i.e.,

$$r(m) = \tilde{f}_r(m) = \tilde{f}_r\left(\frac{M}{M_0} m_0\right) = \left(\frac{M}{M_0}\right)^{\alpha_r} f_r(m_0) \tag{17a}$$

$$L(m) = \tilde{f}_L(m) = \tilde{f}_L\left(\frac{M}{M_0} m_0\right) = \left(\frac{M}{M_0}\right)^{\alpha_L} f_L(m_0) \tag{17b}$$

$$P(m) = \tilde{f}_P(m) = \tilde{f}_P\left(\frac{M}{M_0} m_0\right) = \left(\frac{M}{M_0}\right)^{\alpha_P} f_P(m_0) \tag{17c}$$

$$T(m) = \tilde{f}_T(m) = \tilde{f}_T\left(\frac{M}{M_0} m_0\right) = \left(\frac{M}{M_0}\right)^{\alpha_T} f_T(m_0) \tag{17d}$$

$$\rho(m) = \tilde{f}_\rho(m) = \tilde{f}_\rho\left(\frac{M}{M_0} m_0\right) = \left(\frac{M}{M_0}\right)^{\alpha_\rho} f_\rho(m_0) \tag{17e}$$

In other words, the solutions for all stellar masses are essentially "the same", i.e. graphs of the dependent variables r, L, P, T, ρ for different masses would overplot as long as the independent variable were scaled as in (16), and the dependent variables were scaled as in (17a-e). This conclusion is a consequence of the power law assumptions in (14) and is known as the "homologous model". It remains to find the values of the exponents α_r, α_L, α_P, α_T, α_ρ from the equations (12a-e) plus (14).

Firstly we notice that the derivatives can be written,

$$\frac{dr}{dm} = \frac{dm_0}{dm} \cdot \frac{dr}{dm_0} = \left(\frac{M_0}{M}\right) \cdot \left(\frac{M}{M_0}\right)^{\alpha_r} f_r'(m_0) = \left(\frac{M}{M_0}\right)^{\alpha_r - 1} f_r'(m_0) \tag{18a}$$

where the dash denotes the derivative function, and similarly,

$$\frac{dL}{dm} = \left(\frac{M}{M_0}\right)^{\alpha_L - 1} f_L'(m_0) \tag{18b}$$

$$\frac{dP}{dm} = \left(\frac{M}{M_0}\right)^{\alpha_P - 1} f_P'(m_0) \tag{18c}$$

$$\frac{dT}{dr} = \left(\frac{M}{M_0}\right)^{\alpha_T - 1} f_T'(m_0) \tag{18d}$$

Hence, taking the log of the LHS of (12a) gives,

$$\log\left(\frac{dr}{dm}\right) = (\alpha_r - 1)\log\frac{M}{M_0} + \log f_r'(m_0) \tag{19a}$$

whereas the log of the RHS of (12a) is, substituting (17a) and (17e),

$$-2\alpha_r \log\frac{M}{M_0} - \alpha_\rho \log\frac{M}{M_0} + \log\left(\frac{1}{4\pi f_r^2(m_0) f_\rho(m_0)}\right) \tag{19b}$$

Equating (19a) and (19b) gives,

$$[3\alpha_r + \alpha_\rho - 1]\log\frac{M}{M_0} = \log\left(\frac{1}{4\pi f_r^2(m_0) f_\rho(m_0) f_r'(m_0)}\right) = 0 \tag{19c}$$

The expression in the log on the right is unity, by virtue of Equ.(12a) when the stellar mass is M_0. Since log(1) = 0 it follows that the LHS is always zero (<u>not</u> just for mass M_0), and hence we require,

$$3\alpha_r + \alpha_\rho - 1 = 0 \tag{19d}$$

Each of Equs.(12b,c,d) give similar algebraic constraints on the α exponents, obtained, we can now see, simply by collecting together the powers of M/M_0. Thus, Equ.(12b) gives,

$$(\alpha_L - 1) - \lambda\alpha_\rho - \nu\alpha_T = 0 \tag{20}$$

[recalling that the exponents λ and ν are defined for the power density per unit mass, (14)]. Next, Equ.(12c) gives in the same way,

$$(\alpha_P - 1) - 1 + 4\alpha_r = 0 \tag{21a}$$

but from the gas law, Equ.(12e), together with (17c,d,e) we have,

$$\alpha_P = \chi_\rho \alpha_\rho + \chi_T \alpha_T \tag{21b}$$

and (21a) + (21b) give,

$$4\alpha_r + \chi_\rho \alpha_\rho + \chi_T \alpha_T - 2 = 0 \tag{21c}$$

Finally, Equ.(12d) gives in the same way,

$$(\alpha_T - 1) - \alpha_L - n\alpha_\rho + s\alpha_T + 3\alpha_T + 4\alpha_r = -\alpha_L - n\alpha_\rho + (4+s)\alpha_T + 4\alpha_r - 1 = 0 \tag{22}$$

Equs.(19d, 20, 21c, 22) can be written as a single matrix equation,

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 4 & \chi_\rho & 0 & \chi_T \\ 0 & \lambda & -1 & \nu \\ 4 & -n & -1 & 4+s \end{pmatrix} \begin{pmatrix} \alpha_r \\ \alpha_\rho \\ \alpha_L \\ \alpha_T \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} \tag{23}$$

The determinant of the matrix is,

$$D = (3\chi_\rho - 4)(\nu - s - 4) - \chi_T(3\lambda + 3n + 4) \tag{24}$$

In terms of which the solutions to the simultaneous equations (23) are,

$$\alpha_r = \frac{1}{3D}[D - 2\xi] \qquad \text{where, } \xi = \chi_T + \nu - s - 4 \tag{25a}$$

$$\alpha_\rho = \frac{2}{D}\xi \tag{25b}$$

$$\alpha_L = 1 + \frac{1}{D}[2\lambda\xi - 2\nu\eta] \qquad \text{where, } \eta = \chi_\rho + \lambda + n \tag{25c}$$

$$\alpha_T = -\frac{2}{D}\eta \tag{25d}$$

We shall now insert specific numbers into Equs.(25) to see how the properties of stars vary with their mass. We shall assume the following:-

(a) The stellar medium behaves as a perfect gas, and hence in Equ.(12e) we shall use $\chi_\rho = 1, \ \chi_T = 1$.

(b) Since we have used equations relating to the region of nuclear reactions (i.e. radiation dominance) it is consistent to use the opacity in this region. Since the temperatures in this region are above ~10^7 K we may approximate the opacity to be a constant, independent of both density and temperature, thus in Equ.(14) we have $n = s = 0$.

(c) In Equ.(14) $\lambda = 1$. This is accurate, rather than an approximation. Note that the power density per unit volume is proportional to density squared (i.e. the reaction rate per particle is proportional to the density, and the number of particles per unit volume is also proportional to the density). However, it is the power per unit mass which enters our equations, and this is the power density per unit volume divided by the density, and hence is proportional to the density.

Thus, only the exponent of temperature, ν, in the power mass density remains. Substitution of the above values into (24) gives $D = -(3 + \nu)$ and (25c) then gives

$$\alpha_L = 3 \tag{26}$$

irrespective of ν, whose dependence cancels from this exponent. Thus we find that the luminosity varies as mass cubed, i.e.,

$$\frac{L}{L_0} = \left(\frac{M}{M_0}\right)^3 \tag{27}$$

irrespective of the nuclear reaction rates. The lifetime of a star can be deduced from (27) provided that we assume that the proportion of fuel consumed at the end of life is the same for all masses. In this case, the lifetime is simply proportional to the amount of fuel, i.e. the mass, divided by its rate of burning, the latter being proportional to the luminosity. Hence, lifetime $\propto M / L$, or,

$$\frac{\text{Lifetime}}{\text{Stellar Lifetime}} = \left(\frac{M}{M_0}\right)^{-2} \tag{28}$$

Notice the minus sign! Contrary to naïve expectation, a more massive star lives for a much shorter time. How do these results compare with more detailed models or with observations? Well, α_L is probably more like 3.4 to 3.6, so our simple estimate is not too bad. For future use, however, note that (27) and (28) are, more accurately,

$$\frac{L}{L_0} \approx \left(\frac{M}{M_0}\right)^{3.4 \text{ to } 3.6} \quad \text{and} \quad \frac{\text{Lifetime}}{\text{Stellar Lifetime}} \approx \left(\frac{M}{M_0}\right)^{-2.4 \text{ to } -2.6} \tag{29}$$

However, returning to our simple model, what about its predictions for the other properties? These do depend upon the temperature sensitivity of the nuclear reactions (i.e. upon ν). For stars less massive than the sun, where the pp sequences dominate, $\nu \approx 4$ (see Section 2.3). For stars more massive than the sun, where the CN sequences dominate, $\nu \approx 15 - 19$ (see Section 3.1). Equs.(24, 25) give:-

ν	α_L	α_r	α_ρ	α_T	α_P	α_T(surface)
15	3	$\frac{7}{9}$	$-\frac{4}{3}$	$\frac{2}{9}$	$-\frac{10}{9}$	$\frac{13}{36}$
19	3	$\frac{9}{11}$	$-\frac{16}{11}$	$\frac{2}{11}$	$-\frac{14}{11}$	$\frac{15}{44}$
4	3	$\frac{3}{7}$	$-\frac{2}{7}$	$\frac{4}{7}$	$\frac{2}{7}$	$\frac{15}{28}$

Hence, the mass dependencies for stars of greater mass than M_0 are,

$$\frac{R}{R_0} \approx \left(\frac{M}{M_0}\right)^{0.78 \text{ to } 0.82} \quad \frac{T}{T_0} \approx \left(\frac{M}{M_0}\right)^{0.18 \text{ to } 0.22} \tag{30}$$

$$\frac{\rho}{\rho_0} \approx \left(\frac{M}{M_0}\right)^{-1.33 \text{ to } -1.45} \quad \frac{P}{P_0} \approx \left(\frac{M}{M_0}\right)^{-1.11 \text{ to } -1.27} \tag{31}$$

Whereas, for stars of lower mass than M_0, the mass dependencies are,

$$\frac{R}{R_0} \approx \left(\frac{M}{M_0}\right)^{0.43} \qquad \frac{T}{T_0} \approx \left(\frac{M}{M_0}\right)^{0.57} \tag{32}$$

$$\frac{\rho}{\rho_0} \approx \left(\frac{M}{M_0}\right)^{-0.29} \qquad \frac{P}{P_0} \approx \left(\frac{M}{M_0}\right)^{0.29} \tag{33}$$

These two sets of results apply for CN dominance and pp dominance respectively. The boundary between the two is at a temperature of ~19 million K. Since the solar central temperature is ~13.7 million K, it follows from Equ.(32) that the transition occurs for stars of about 1.8 solar masses. Thus M_0 is ~$1.8M_{solar}$.

We note some remarkable, and some counterintuitive, features of these results:-

- Bigger stars (i.e. with larger radius) have larger mass. That's not surprising. However…

- The density reduces with increasing mass. This is because the radius grows faster than $M^{1/3}$. The density and radius variations are consistent in the sense that the average density is proportional to M/R^3, i.e. we expect $\alpha_\rho = 1 - 3\alpha_r$, which is true.

- For stars of greater mass than M_0, (31) implies that the pressure is reduced compared with that of the mass M_0 reference star. Curiously, though, (33) implies that stars with mass less than M_0 also have lower pressure! This is because α_P changes sign between CN and pp dominance. This suggests that stars of mass M_0 have the greatest achievable pressures of any stars! (That is, any stars on the main sequence and with this composition. This excludes neutrons stars, red giants, white dwarfs, etc, which are not covered by our analysis).

- The temperatures given by (30, (32) apply only within the nuclear burning region (essentially where temperatures are above about 10 million K, or at least within the millions of K). In particular they do not apply to the surface temperature. The surface temperature is best found from the luminosity which is proportional to R^2T^4. Hence the surface temperature exponent is $\alpha_T^{surface} = (\alpha_L - 2\alpha_2)/4$, giving,

$$\frac{T^{surface}}{T_0^{surface}} = \left(\frac{M}{M_0}\right)^{0.34\ \text{to}\ 0.36} \text{ for } M > M_0; \qquad \frac{T^{surface}}{T_0^{surface}} = \left(\frac{M}{M_0}\right)^{0.54} \text{ for } M < M_0 \tag{34}$$

  If the observed luminosity relation, Equ.(29), is used for α_L instead of our model prediction (i.e. α_L ~ 3.4 instead of 3.0) then the exponents in the above are increased by about 0.1. Hence, crudely, the surface temperature increases as the square-root of the mass.

3.6 The Main Sequence

The Hertzsprung-Russell diagram plots the absolute luminosity of stars against their colour, both on logarithmic scales. A surrogate for the colour is the surface temperature. If the homologous model of Section 3.5 is correct, we expect main sequence stars to lie on a straight line on the HR diagram (i.e. power law relationships become straight lines when plotted on logarithmic scales). Specifically, the homologous model gives,

$$L \propto T^{\beta} \qquad \text{where, } \beta = \frac{\alpha_L}{\alpha_T^{surface}}$$

and the numerical values for this β exponent are,

ν	α_L	**α_T(surface)**	**β**
15	3	13/36	8.31
19	3	15/44	8.80
4	3	15/28	5.60

The HR diagram in Rowan-Robinson, Figure 2.2, has a main sequence slope which can be derived from the following two points:-

1) (The Sun): Surface temperature 5885°K; Luminosity taken as unity.
2) A star of luminosity 10^4 times that of the sun; Surface temperature 22,000°K.

Hence, $\beta = (4-0)/(\log_{10} 22000 - \log_{10} 5885) = 7.0$. This compares reasonably well with the values in the above Table.

4 Summary

In Chapter 11 we derived an analytic approximation for the pressure, density and temperature variation sufficiently near the centre of a solar mass star, the Clayton model. In this Chapter we initially consider an alternative approximate model based on the assumption of a polytropic equation of state, i.e. that pressure is proportional to a certain fixed power of density, P = Kρ^γ. Together with hydrostatic equilibrium this leads to a second order, non-linear differential equation for the density.

Integrating this hydrostatic-polytropic equation provides a fairly good representation of the Sun. This model contains four arbitrary constants: the two coefficients in the polytropic equation, plus two integration constants. One constant (K) is fixed by specifying the mass of the star, whilst the requirement that the density is maximum at the centre determines one of the integration constants. The other two constants, taken to be γ and the density at the centre of the star, are not determined by the model and are adjusted to give a reasonable fit to the known results for the Sun. Specifically we find that a central density of 90,000 kg/m^3 together with γ = 1.33 gives a good representation of the Sun. These reproduce:

- the correct central pressure (1.65 x 10^16 Pa);
- the correct central temperature (13.7 million.K);
- the correct radius (7.0 x 10^8 m);
- a mass which is within 10% of the correct value (2.2 x 10^30 kg, cf. 2.0 x 10^30 kg);

An advantage of the polytropic model over the Clayton model is that it leads to a clearly defined surface where the pressure, density and temperature drop to zero. Using the known nuclear reaction rates for a given temperature and density, the total luminosity of the star can be found from the above model results. We estimate a surface temperature of 5915 deg.K (cf. the accepted value of ~5885 K) for a luminosity of 4.21 x 10^{26} W.

An expression is derived for the radiative heat flux in terms of the temperature, the density, the temperature gradient and the opacity.

Simple power-law expressions in temperature and density are given for the power density resulting from the pp and CN reaction sequences. They are compared with power densities obtained from published reaction rates. The CN power density is proportional to the density of nitrogen-14, and hence the CN sequence does not apply early in the life of first generation stars. The pp power density is roughly proportional to T^4 whereas the CN power density is roughly proportional to T^16. Consequently, the CN sequence dominates at higher temperatures. We find that the pp sequence dominates below 19 million K, and the CN sequence dominates above this temperature.

Whilst the hydrostatic-polytropic model is quite successful in terms of the above results, it was necessary to be provided with two parameters from a more detailed model. Consequently, the hydrostatic-polytropic model is not fundamental. The ingredient which is missing from the models considered so far is the requirement that the rates of heat transport balance against the rate of heat production at every point (without which we would not have a quasi-static solution). The correct, complete, set of fundamental equations is presented here for the case of radiation dominated heat transport. The polytropic equation is not required, but rather the pressure and the density will both follow from solving the complete set of equations.

We do not attempt to solve the complete equation set here. This would have to be by numerical integration. Instead we note that very informative scaling relations may be obtained in the case that the opacity and the power density can both be approximated by power laws in density and temperature. In this case it is shown that the solution is essentially 'the same' for stars of different masses, when suitably scaled. The scaling required is that the dependent variables pressure, density, temperature and luminosity are scaled by (M/M0)^x and the distributions are plotted against the fractional mass parameter (m/M). The stellar equation set is used to determine the powers x for each field quantity for this so-called '*homologous model*'.

Assuming the (constant) Thompson opacity and perfect gas behaviour we deduce that luminosity is proportional to the cube of the star's mass, and that the stellar lifetime is *inversely* proportional to the square of the mass. These results obtain independently of the nuclear reaction rates. Thus, contrary to naïve expectation, more massive stars burn out far more quickly than lighter stars. More detailed models suggest an even greater sensitivity to mass, the exponents being closer to 3.5 and -2.5 respectively.

Two sets of scaling behaviours for the pressure, density, central temperature and stellar radius are presented: one for CN dominance and one for pp dominance. The latter applies for stars of mass below M0 ~ 1.8 solar masses, and the former for heavier stars. Some results are unsurprising, e.g. that the radius of the star and its central temperature increase monotonically with increasing mass. However, we find, contrary to naïve expectation, that the density *reduces* monotonically with increasing mass. More massive stars are disproportionately larger and hence have lower densities. More surprisingly still, we conclude that stars with masses either less than or greater than M0 have lower pressures! Thus, of all main sequence (hydrogen burning) stars, those of 1.8 solar masses have the largest pressure.

The homologous model result for luminosity implies a surface temperature proportional to mass to a power ~0.34 for more massive stars, or to power ~0.64 for less massive stars.

Hence, more massive stars are hotter – but note that this statement relates solely to main sequence stars. [Red giants might be very massive, but with low surface temperature]. Having deduced both the luminosity and the surface temperature we can finally derive an expected slope for the Hertzsprung-Russell diagram, which is a logarithmic plot of luminosity versus surface temperature (colour). Our estimates derived from the homologous model, are 5.6 to 8.8, which compare reasonably well with the observed value of ~7.

Finally, in an Annex to this Chapter, we derive from first principles the temperature dependence of the rate-controlling CN reaction. The purpose is to explain the extreme temperature sensitivity of this reaction (proportional to temperature to a power of around 16). The explanation lies in the usual Gamow peak, which provides an exponential dependence on temperature. The coefficient appearing in the exponent depends upon the product of the reactants atomic numbers and their atomic masses. Thus, the exponent is a larger (negative) number for the CN sequence because the rate controlling step involves nitrogen-14. The reason why these exponential temperature dependencies are often approximated as power laws should now be apparent – they facilitate approximate scaling results as derived above.

Annex

Why Is The Rate Of Reaction ${}^{14}_{7}N + p \to {}^{15}_{8}O + \gamma$ So Sensitive To Temperature?

The Tutorial Appendix A3, Section 4.1.3.1 has derived the energy dependence of the p-D capture cross-section. Appendix A3, Section 6 has also considered the temperature dependence of the p-D capture reaction in a thermal (Maxwellian) distribution of particle energies. We carry out here a similar analysis for the ${}^{14}_{7}N + p \to {}^{15}_{8}O + \gamma$ reaction.

The energy dependence of the matrix element derives from how much of the free state manages to "penetrate the Coulomb barrier" from outside. It is the magnitude of the free wavefunction within the region where the Coulomb potential is significant, ($a < r < r_E = Z_1 Z_2 \tilde{e}^2 / E$) that matters. This is the region within which the free state has negative kinetic energy, the exponential decay length being (the reciprocal of) $\kappa = \sqrt{2m(V(r) - E)} / \hbar$, noting that this varies with 'r', and where $V(r) = Z_1 Z_2 \tilde{e}^2 / r$. In the case of interest the two incident particles (p and ${}^{14}N$) have $Z_1 = 1$ and $Z_2 = 7$, and also $A_1 = 1$ and $A_2 = 14$.

Roughly speaking, we may expect the magnitude of the wavefunction to diminish from $r = r_E$ to $r = a$ by a factor of $e^{-\int_a^{r_E} \kappa dr}$. The integral may be evaluated exactly to give a factor,

$$X = \exp\left\{ -\frac{Z_1 Z_2 \tilde{e}^2 \sqrt{2m_R}}{\hbar} \cdot \frac{1}{\sqrt{E}} \left[\theta - \frac{1}{2} \sin 2\theta \right] \right\} \quad \text{where, } \theta = \tan^{-1} \sqrt{\frac{V_c}{E} - 1}, \qquad (A.1)$$

where, $V_c = Z_1 Z_2 \dfrac{\tilde{e}^2}{a}$ and m_R is the reduced mass, i.e. $m_R = \dfrac{A_1 A_2}{A_1 + A_2} M_p$. (A.2)

For energies sufficiently small compared with V_c, $\theta \to \pi / 2$ and hence the sin term is zero and we get, for the reaction ${}^{14}_{7}N + p \to {}^{15}_{8}O + \gamma$,

$$X \to \exp\left\{ -b / \sqrt{E} \right\} \quad \text{where, } b = \frac{\pi Z_1 Z_2 \tilde{e}^2}{2\hbar} \sqrt{\frac{2 A_1 A_2 M_p}{A_1 + A_2}} = 3.359 \sqrt{\text{MeV}} \qquad (A.3)$$

This gives the temperature dependence of the matrix element at sufficiently low energies.

We now assume that the expression for the cross-section of reaction ${}^{14}_{7}N + p \to {}^{15}_{8}O + \gamma$ has the same energy dependencies as that for the p + D capture analysed in Appendix A3. The temperature dependence of the reaction can therefore be found from the same analysis as Appendix A3, Section 6, which implies that the overall reaction rate is proportional to,

$$\exp\{-f_{min}\}/(kT)^{3/2} \quad \text{where, } f_{min} = \frac{3}{2^{2/3}} \cdot \frac{\tilde{b}^{2/3}}{(kT)^{1/3}} \qquad (A.4)$$

and $\tilde{b}$ is twice the value of b given above (i.e. it derives from the square of the matrix element), hence $\tilde{b} = 6.718$. If we equate (A.4) to a factor $\exp\{-\beta/\sqrt{E}\}$, where $E = 1.5kT$, we may therefore derive an effective β parameter:-

T (10^6 K)	**E = 1.5kT MeV**	**f_{min} dimensionless**	$\exp\{-f_{min}\}/(E)^{3/2}$ **$MeV^{-3/2}$**	**β √MeV**
10	0.00129	70.726	4.15×10^{-27}	1.624
14	0.00181	63.176	4.75×10^{-24}	1.705
20	0.00259	56.063	3.41×10^{-21}	1.775
25	0.00323	52.085	1.31×10^{-19}	1.826
30	0.00388	48.997	2.18×10^{-18}	1.883
40	0.00517	44.526	1.24×10^{-16}	1.946
50	0.00647	41.318	2.19×10^{-15}	-

Thus, the value of the β parameter estimated from first principles is in very good agreement with that resulting from the NACRE data (see the Table in Section 3.1). It follows that the temperature dependence expressed alternatively as a power law is also in good agreement, i.e. an exponent 'α' in Rate $\propto T^{\alpha}$ of 16-18 in the temperature range 18-30 million K.

It would appear that the temperature dependence of any binary nuclear reaction is found from the above analysis simply in terms of Z_1, Z_2, A_1 and A_2, at least at sufficiently low energies. Thus the synthesis of more massive nuclei requires ever higher temperatures.