Chapter 16 – The Opacity of the Stellar Medium

Last Update: 10 September

1. Introduction

The opacity, κ, of the stellar medium can be defined as the reciprocal of the mean free path of a photon, divided by the total mass density. Thus,

$$\kappa\rho = 1/\lambda \tag{1.1}$$

The opacity will depend upon the frequency of the photon in question. In this Chapter we shall be concerned only with the weighted average opacity over the spectrum of photon energies corresponding to a thermal equilibrium distribution at a given temperature (known as the Rosseland mean opacity).

The opacity is due to the presence of particles which interact with the electromagnetic radiation. Most obvious of these are free charged particles, i.e. free electrons, free protons and ionised nuclei. We will see that the opacity due to free electrons (Thompson scattering) is dominant at sufficiently high temperatures, typical of the centres of stars (i.e. ~10^7 K and above). In contrast, the opacity due to free protons or nuclei is always negligible.

However, at 'intermediate' temperatures, between ~10^4 K and ~10^7 K, the dominant contribution to opacity comes from electrons which are bound to a nucleus. The most obvious contribution is due to neutral atomic hydrogen. The mechanism is photon absorption causing the bound electron to be emitted, i.e. photodisintegration. This can occur only if the photon energy exceeds the ionisation potential. If the temperature is too low there will be few photons of such energy, and the opacity due to this mechanism must reduce to zero. On the other hand, this mechanism also requires that some electrons are bound to protons, or nuclei, and hence be available to absorb a photon. If the temperature is too high there will be vanishingly few such bound electrons, and again the opacity due to this mechanism will reduce to zero. Hence, we see that this mechanism will apply only between certain lower and upper temperature limits. We shall see that this temperature range is roughly ~10^4 K to ~10^7 K. We shall also see that the opacity is density dependent in this temperature range.

In this Chapter we shall derive approximate, closed-form, expressions for the opacity applicable above about 8,000 K. To do so we shall initially consider the opacity due to hydrogen and free electrons alone. We then compare our predictions with published data for stars of stellar composition. The opacity due to metals (which comprise ~2.5% of the Sun at the present time) is not addressed. Opacity at temperatures below ~8,000 K is also not derived, although we shall make some qualitative remarks about this regime in Section 11. Our hydrogen-based opacity displays the correct variation with temperature, and hence provides a good illustration of opacity generally. However, accurate calculations of opacity are a complex, specialist area which we cannot hope to reproduce here. In particular, accurate calculations would need to consider the contributions of neutral helium and metals.

2. Relationship Between Opacity and Cross Section

If the cross section of a particle x to absorb or deflect a photon is σ, and if the number density of x particles is ρ_N^x, then the reaction rate <u>per photon</u> is $\sigma\rho_N^x c = 1/t$, where t is the reaction time per photon. The mean free path of a photon is just λ = ct, hence,

$$\kappa\rho \equiv \frac{1}{\lambda} = \frac{1}{ct} = \sigma\rho_N^x \tag{2.1}$$

from the definition, (1.1). Note that the density on the LHS is the *total mass density* (of all types of particle present in the stellar medium), whereas on the RHS it is the *number density of the particle type x* in question only. The ratio of these two densities is found as follows:-

Suppose that free protons, plus protons in the form of hydrogen atoms, account for a mass fraction X of the stellar medium. Suppose that helium nuclei and atoms make up the remainder, and hence have a mass fraction Y = 1 – X. We will ignore the mass of the electrons for simplicity. We also approximate the mass of a helium nucleus, or atom, to that of four protons. Hence,

- For every X protons there are Y/4 helium nuclei/atoms;
- There are two electrons for each helium nucleus/atom, and one per proton, hence,
- For every X protons there are X + Y/2 electrons;

Consider scattering by electrons (Thompson scattering). The particle 'x' in (2.1) is therefore the electron. From the above observations, the number density of electrons is given in terms of the number density of protons by,

$$\rho_N^e = \frac{X + Y/2}{X}\rho_N^p \tag{2.2}$$

Similarly, the number density of helium nuclei/atoms is,

$$\rho_N^{He} = \frac{Y/4}{X}\rho_N^p \tag{2.3}$$

But the total mass density is given by,

$$\rho = M_p\rho_N^p + 4M_p\rho_N^{He} = \left(1 + \frac{Y}{X}\right)M_p\rho_N^p = \frac{M_p\rho_N^p}{X} \tag{2.4}$$

since X + Y = 1. Substituting (2.2) and (2.4) into (2.1) gives, for Thompson scattering by electrons, or for free-free electrons transitions,

$$\kappa = \frac{X + Y/2}{M_p}\sigma = \frac{1 + X}{2M_p}\sigma \tag{2.5}$$

Consider now the bound-free transition in hydrogen, for which the particle 'x' in (2.1) is a neutral hydrogen atom. Suppose that, excluding the protons bound in helium and metal nuclei, the fraction of protons which are in the form of neutral hydrogen atoms is f_H. Following similar reasoning to the above, we find that the relationship between opacity and cross-section for the bound-free transition in hydrogen is,

$$\kappa = f_H \frac{X}{M_p} \sigma \tag{2.6}$$

Finally, for the bound-free transition of He^+ we have,

$$\kappa = f_{He} \frac{Y}{4M_p} \sigma = f_{He} \frac{1 - X}{4M_p} \sigma \tag{2.7}$$

where f_{He} is the fraction of helium nuclei which have captured one electron to form He^+.

In the numerical evaluations which follow we shall use X = 0.71, which is representative of the Sun at the present time. Whilst we have derived (2.5-2.7) on the basis that X + Y = 1, i.e. that no metals are present in the star, the Sun has Y = 0.265 and hence about 2.5% metals. Our numerical evaluations implicitly assume Y = 0.29 and no metals.

3. The Thompson Opacity

Thompson scattering by free electrons has a cross-section,

$$\sigma_{Thompson} = \frac{8\pi}{3} \alpha^2 \left(\frac{\hbar}{m_e c} \right)^2 = 6.65 \text{ x } 10^{-29} \text{ m}^2 \tag{3.1}$$

(see, for example, Mandl & Shaw, Equs.1.70, 1.72). Using (2.5) the Thompson opacity is thus,

$$\kappa_{Thompson} = \frac{1 + X}{2M_p} \sigma = 0.020(1 + X) = 0.034 \text{ m}^2/\text{kg} = 0.34 \text{ cm}^2/\text{g} \tag{3.2a}$$

(NB: Astronomers tend to use cm^2/g as the *de facto* standard unit for opacity.)

This opacity assumes that all the electrons are free (i.e. not bound in atoms). This is a good approximation at high temperatures (above ~10^7 K). As a rough approximation we may take the fraction of electrons which are free to be the same as the fraction of protons which are free, i.e. 1 – f_H. (This would be correct if the star were all hydrogen, but the proportion of helium which is ionised, 1 – f_{He}, will differ). Thus, the contribution of Thompson scattering to the opacity in general is estimated from,

$$\kappa_{Thompson} \approx 0.20(1 - f_H)(1 + X) = 0.34(1 - f_H) \text{ cm}^2/\text{g} \tag{3.2b}$$

The determination of f_H is discussed below.

4. Finding the Fraction f_H of Neutral Hydrogen – The Saha Equation

The Saha equation, which determines the fraction (y) of protons which are free at a given temperature and density, has been derived in Chapter 8. It is,

$$\frac{y^2}{1-y} = \frac{(2\pi m_e kT)^{3/2}}{(2\pi\hbar)^3 \rho_N^p} e^{-B/kT} \tag{4.1}$$

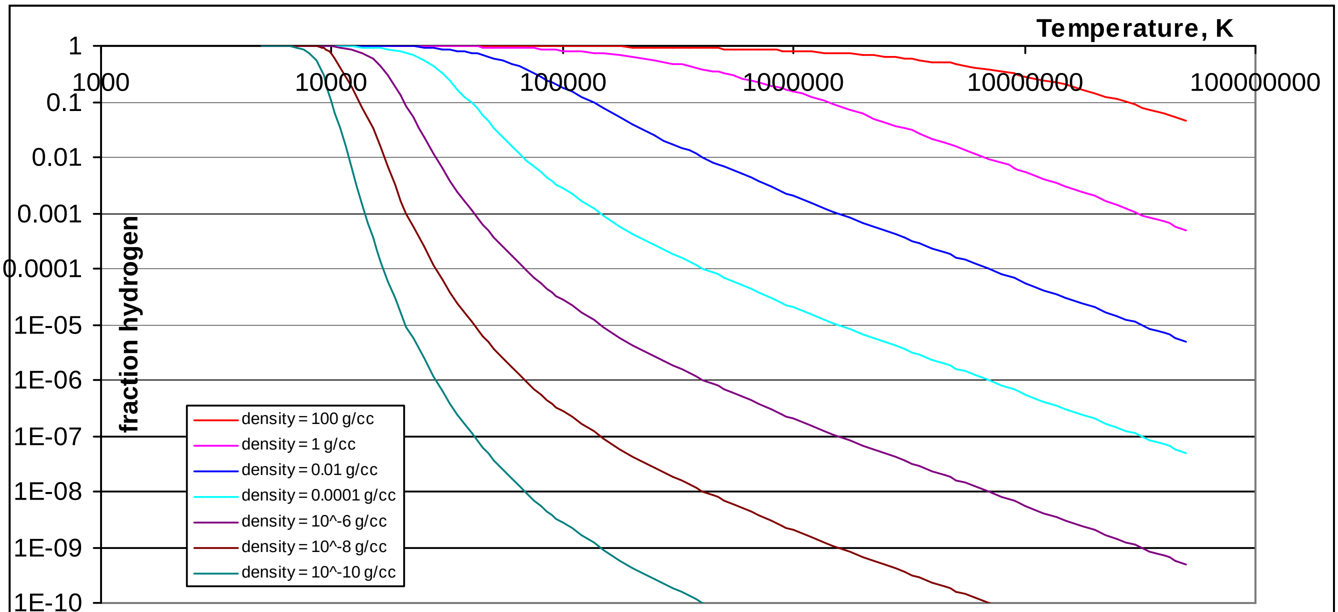
Recall that ρ_N^p is the number density of all protons, whether free or in the form of hydrogen, but excluding those in nuclei; y is the fraction of these protons which are free. Hence, $f_H = 1 - y$. In (4.1), B is the ionisation energy of hydrogen, i.e. B = 13.6 eV. For a nucleus of atomic number Z, the ionisation potential from the ground state s,

$$B_Z = Z^2 \cdot \frac{\alpha^2}{2} m_e c^2 \tag{4.2}$$

Numerical results for the fraction of neutral atomic hydrogen (f_H), using X = 0.71

T (K)	**Total Density (g/cc)**		
	1	**10^{-4}**	**10^{-10}**
10^4	0.99997	0.9972	0.102
10^5	0.825	0.00269	2.7×10^{-9}
10^6	0.150	2.1×10^{-5}	2.1×10^{-11}
10^7	0.0056	5.67×10^{-7}	5.7×10^{-13}

Further results are shown graphically below:-



Fraction of Protons in the form of Neutral Atomic Hydrogen Versus Temperature

Before leaving this Section we note that, when f_H is small compared with unity it is proportion to the density. This follows by writing (4.1) as $\frac{y^2}{1-y} = \zeta$ where ζ is inversely proportional to the density. Thus we have, $y = \left(\sqrt{\zeta^2 + 4\zeta} - \zeta\right)/2$. When $\zeta >> 1$ we may therefore approximate y by $1 + 1/\zeta$ and hence $f_H \approx 1/\zeta$ which is therefore proportional to density, as claimed. This is the reason why Kramer's Law (see Section 12) has a linear density dependence.

5. Cross-Section for Photodisintegration of an Atomic Ground State

The derivation of this cross-section is described in Schiff, leading to Equ.(46.9), i.e. a differential cross-section of,

$$\frac{d\sigma}{d\Omega} = \frac{32e^2a^3kk_x^2}{m_e c\omega(1+K^2a^2)^4} \tag{5.1}$$

where:

- ω is the frequency of the incoming photon;
- k is the wavenumber of the emitted electron;
- k_x is the component of the vector k in the direction of polarisation (x-axis), i.e.,

$$k_x = k\sin\theta\cos\phi \tag{5.2}$$

- a is the size scale of the ground-state wavefunction, i.e.,

$$a = \frac{\hbar}{Z\alpha m_e c} \quad \text{(Z = 1 for hydrogen)} \tag{5.3}$$

- hK is the momentum transferred to the electron, i.e. in terms of vectors,

$$\overline{K} = \frac{\omega}{c}\hat{z} - \overline{k} \tag{5.4}$$

which gives,

$$K^2 = \frac{\omega^2}{c^2} + k^2 - 2\frac{\omega}{c}k\cos\theta \tag{5.5}$$

Now we shall be interested in the total cross section, for which the $\cos\theta$ term in (5.5) averages to zero (crudely). Thus, roughly we have,

$$< K^2 > \approx \frac{\omega^2}{c^2} + k^2 \Rightarrow \frac{<\hbar^2K^2>}{2m} \approx \frac{(\hbar\omega)^2}{2mc^2} + \frac{(\hbar k)^2}{2m} = \frac{E_\gamma^2}{2mc^2} + E_e \tag{5.6}$$

where E_γ is the energy of the incident photon and E_e is the (non-relativistic) kinetic energy of the emitted electron. Conservation of energy requires,

$$E_e = E_\gamma - B \tag{5.7}$$

where B is the ionisation energy, given by (4.2). Hence, for photon energies which are small compared with the electron rest energy (i.e. for temperatures below ~10^9 K) the first term on the RHS of (5.6) is negligible compared with the second. Even for photon energies which are approaching B, so that the electron energy is tending towards zero, the first term on the RHS of (5.6) is only ~10^{-4} eV. The second term (the electron energy) will still dominate if the photon energy exceeds B by only ~0.01%. Hence we can safely approximate,

$$K^2 \approx k^2 \tag{5.8}$$

Equ.(5.8) is appropriate for the whole range of electron and photon energies of interest here. In particular it holds whether we are in the high energy regime for which $E_e \approx E_\gamma >> B$, or in the low energy regime in which $E_e << E_\gamma \approx B$.

Noting that the angular dependence of (5.2) when substituted into (5.1) and integrated over the whole solid angle yields a factor of 4π/3, we get a total cross-section from (5.1) of,

$$\sigma = \frac{32e^2a^3k^3}{m_e c\omega(1+k^2a^2)^4} \tag{5.9}$$

noting that we have used (5.8) to eliminate K. For Z = 1 (hydrogen) this can be written in terms of energies as,

(all energies)

$$\sigma = \frac{128\pi \cdot 2^{3/2}}{3\alpha^2} \cdot \frac{(\hbar c)^2 E_e^{3/2}}{\left(m_e c^2\right)^{5/2} E_\gamma} \cdot \left\{1 + \frac{2E_e}{\alpha^2 m_e c^2}\right\}^{-4} \tag{5.10}$$

For the high energy regime, when the second term in {…} is greater than unity and when $E_e \approx E_\gamma >> B$, (5.10) becomes,

(high energies)

$$\sigma = \frac{128\pi\alpha^6}{3 \cdot 2^{5/2}} \cdot \frac{(\hbar c)^2 \left(m_e c^2\right)^{3/2}}{E_\gamma E_e^{5/2}} \approx \frac{128\pi\alpha^6}{3 \cdot 2^{5/2}} \cdot \frac{(\hbar c)^2 \left(m_e c^2\right)^{3/2}}{E_\gamma^{7/2}} \tag{5.11}$$

and this is equivalent to Schiff's Equ.(46.11). Equ.(5.11) shows that in the high energy regime the cross-section, and hence the opacity, reduces with increasing photon energy as $E_\gamma^{-7/2}$. Thus, at temperature T, the opacity reduces as $T^{-7/2}$ in this regime. This is known as Kramer's Law (see Section 12). It applies only so long as the energy is high enough for (5.11) to be a good approximation for the bound-free transition, and provided that the opacity does not drop below the minimum provided by Thompson scattering. This generally coincides with the temperature range from about 50,000 K and up to 10^6 – 10^7 K depending upon density.

Equ.(5.10) shows that in the lower energy regime, when the {…} is about unity, the cross-section reduces with reducing electron energy (noting that the photon energy must be assumed no less than B or the cross section will be zero). Combining the behaviours from (5.10) and (5.11), the cross section must exhibit a maximum at some intermediate electron energy – which we will see below corresponds to a temperature in the range 10^4 to 10^5 K.

Remarkably, in the low energy regime, Equ.(5.10) shows that the cross-section is *inversely* related to the square of the fine structure constant. Thus, a weaker electromagnetic interaction would lead to a *larger* cross-section in the low energy regime. The reason for this is that the cross-section is related to the size scale, a, of the atom. A weaker electromagnetic interaction would clearly lead to a larger Bohr radius, as given by (5.3). The atom is therefore bigger and thus has a larger cross-section.

6. Proportion Of Photons With Energies Sufficient to Ionise An Atom

In Section 5 it was implicitly assumed that the photon energy was greater than the ionisation potential, B. For photons of lower energy, ionisation is not possible and the cross-section is zero. In principle, we can obtain the effective (mean) cross-section for a black body distribution of photons at a given temperature by averaging, that is,

$$\sigma_{effective} = \int \sigma(E_\gamma) P[T; E_\gamma] dE_\gamma \qquad (6.1)$$

where $P[T; E_\gamma]$ is the probability of a photon having energy between E_γ and $E_\gamma + dE_\gamma$ (i.e. the black body photon density spectrum normalised by the total photon density). In (6.1) the cross section is zero for photon energies below B. Consequently, if the temperature is so low that there are very few photons with energies greater than B, the integrand in (6.1) is non-zero only where P is very small. Consequently the effective cross section is small.

The approach taken in this Chapter is not to carry out the integral in (6.1). The reason is that this would not be tractable analytically. Whilst numerical evaluation would be simple, we wish to derive a simple closed-form expression – albeit approximate. Hence we shall estimate the effective cross-section by using that at a suitably chosen photon energy, but factoring this cross-section down if necessary to account for the sparsity of photons of the required energy. Thus we replace (6.1) with the approximation,

$$\sigma_{effective} = \sigma(E_{typical}) P[E_\gamma > B] \qquad (6.2)$$

where the probability $P[E_\gamma > B]$ that a photon has an energy greater than B will also be denoted more compactly as P_γ. We have already derived this probability in Chapter 6. The result is,

$$P_\gamma = 0.416 \left[2 + 2x_1 + x_1^2\right] e^{-x_1} \quad \text{where} \quad x_1 = B / kT \qquad (6.3)$$

7. Approximate Closed-Form Expression for Hydrogen and Electron Opacity

Putting together Equs.(2.6), (4.1,2) and (5.10) gives the overall result for the opacity due to hydrogen at a given temperature T,

$$\kappa_{hydrogen} = P_\gamma f_H \frac{X}{M_p} \cdot \frac{128\pi \cdot 2^{3/2}}{3\alpha^2} \cdot \frac{(\hbar c)^2 E_e^{3/2}}{(m_e c^2)^{5/2} E_\gamma} \cdot \left\{1 + \frac{2E_e}{\alpha^2 m_e c^2}\right\}^{-4} \qquad (7.1)$$

$$\text{where, } f_H = 1 - y \quad \text{and} \quad \frac{y^2}{1-y} = \frac{(2\pi m_e kT)^{3/2} M_p}{(2\pi\hbar)^3 X\rho} e^{-B/kT} \quad \text{and} \quad B = \frac{\alpha^2}{2} m_e c^2 \qquad (7.2)$$

and where P_γ is given by (6.3), above. To the hydrogen opacity, (7.1), must be added that due to Thompson scattering of free electrons,

$$\kappa_{Thompson} = (1 - f_H) \frac{8\pi}{3} \alpha^2 \left(\frac{\hbar}{m_e c} \right)^2 \frac{1+X}{2M_p} \qquad (7.3)$$

8. Numerical Results

In this Chapter numerical results will be derived using the following assumptions for the 'typical' energies:-

1) The average photon energy is 2.7kT. Consequently this is chosen for E_γ as long as it is greater than B.
2) If 2.7kT < B then we put E_γ = B. In reality the photon energy would be slightly bigger, so that the electron would have non-zero energy.
3) The electron energy is the photon energy less B. Hence, the electron energy is set to $E_e = E_\gamma - B$ when the photon energy is sufficiently big.
4) However, if this is less than 3kT/2, the thermal average electron energy, then this is used instead, i.e. if $E_\gamma - B < 3kT/2$ then $E_e = 3kT/2$.

The opacity due to (7.1) plus (7.3) is,

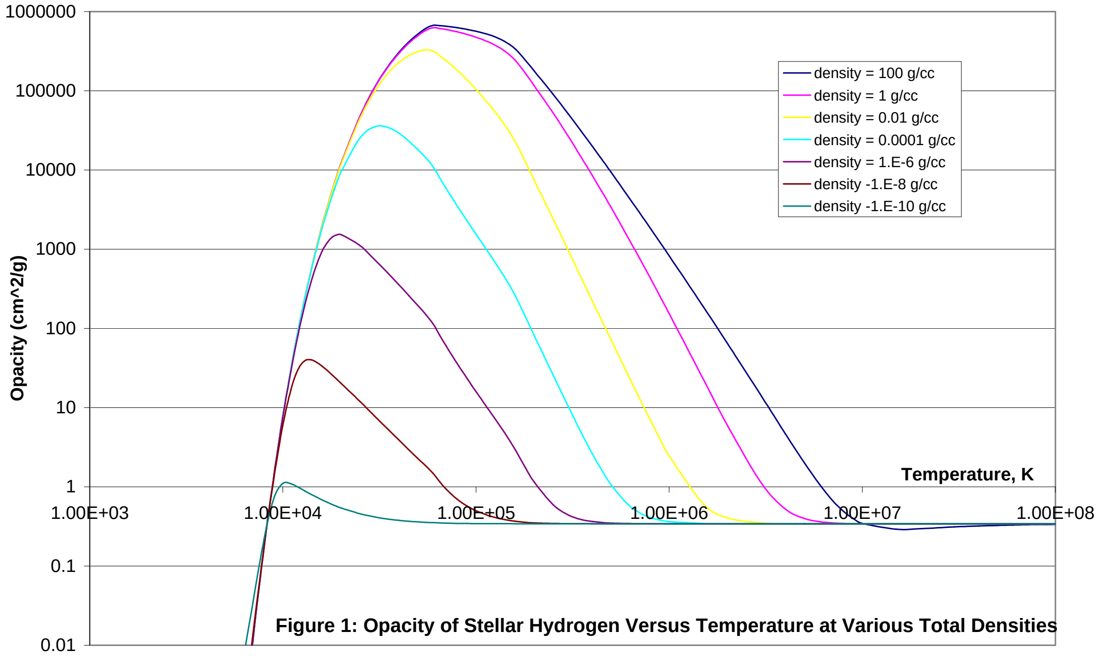
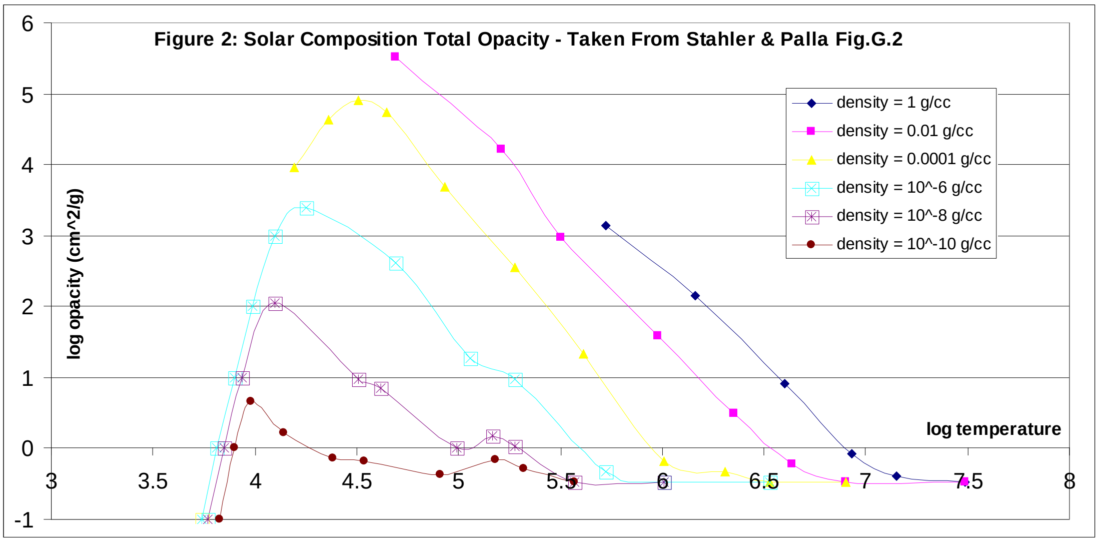
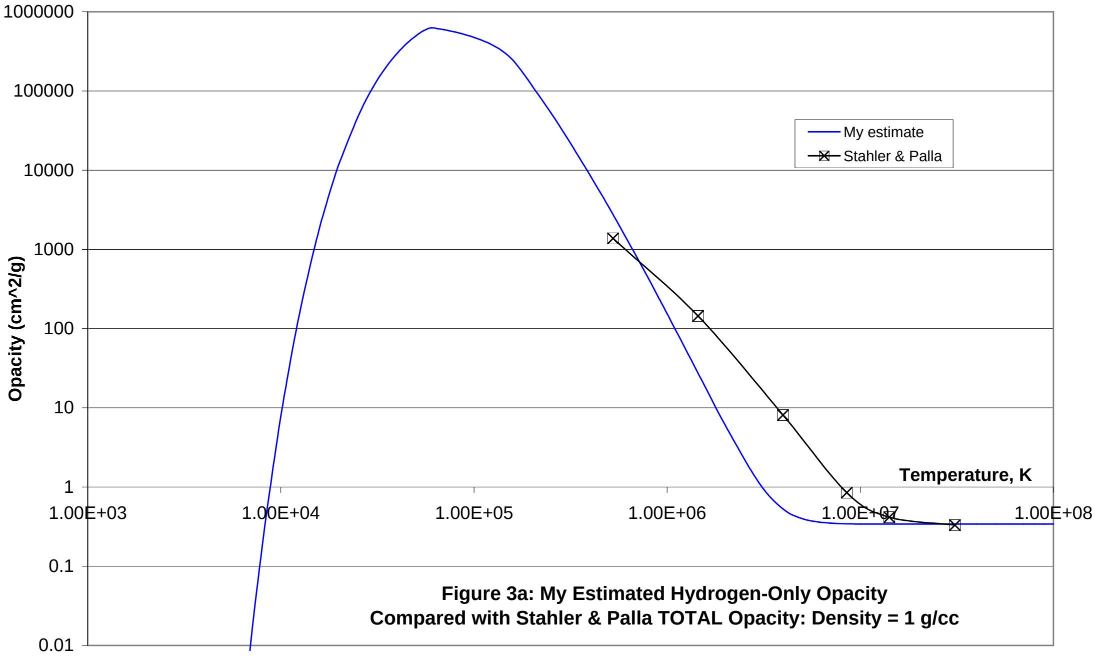


Figure 1: Opacity of Stellar Hydrogen Versus Temperature at Various Total Densities

By way of comparison, the following graph reproduces (roughly) Figure G.2 from Stahler & Palla, which gives the total opacity for a star of solar composition:-



The two graphs are qualitatively similar. To make the quantitative comparison clearer we have included below a series of graphs, one for each density, showing both our estimate for the hydrogen + electron opacity alone, and also the total opacity from Stahler & Palla:-



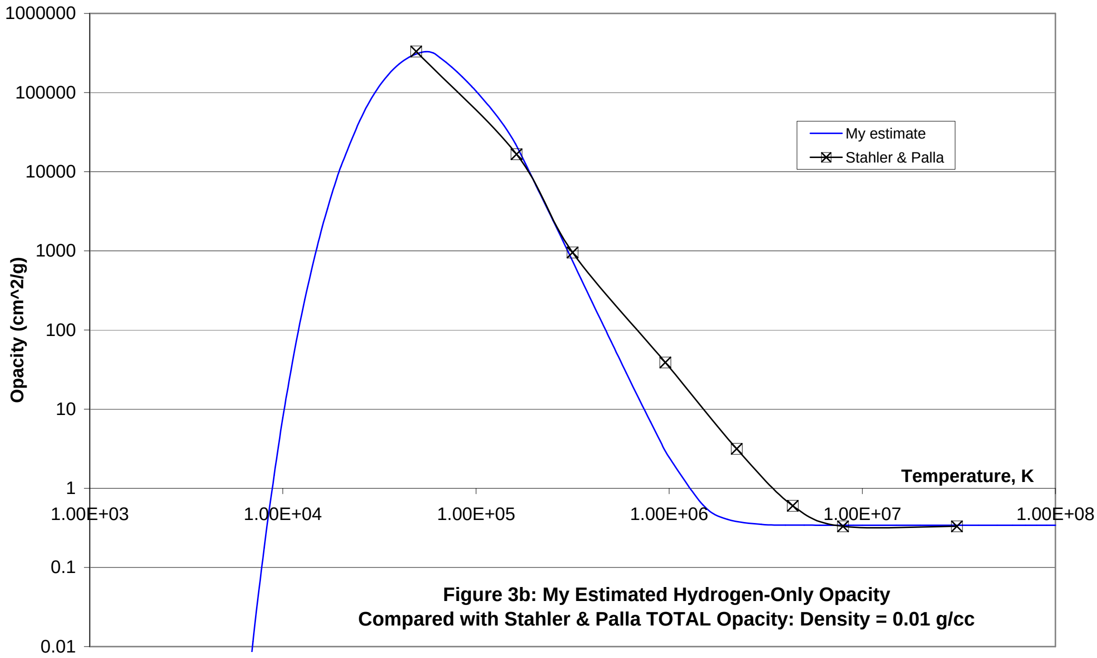


Figure 3b: My Estimated Hydrogen-Only Opacity Compared with Stahler & Palla TOTAL Opacity: Density = 0.01 g/cc

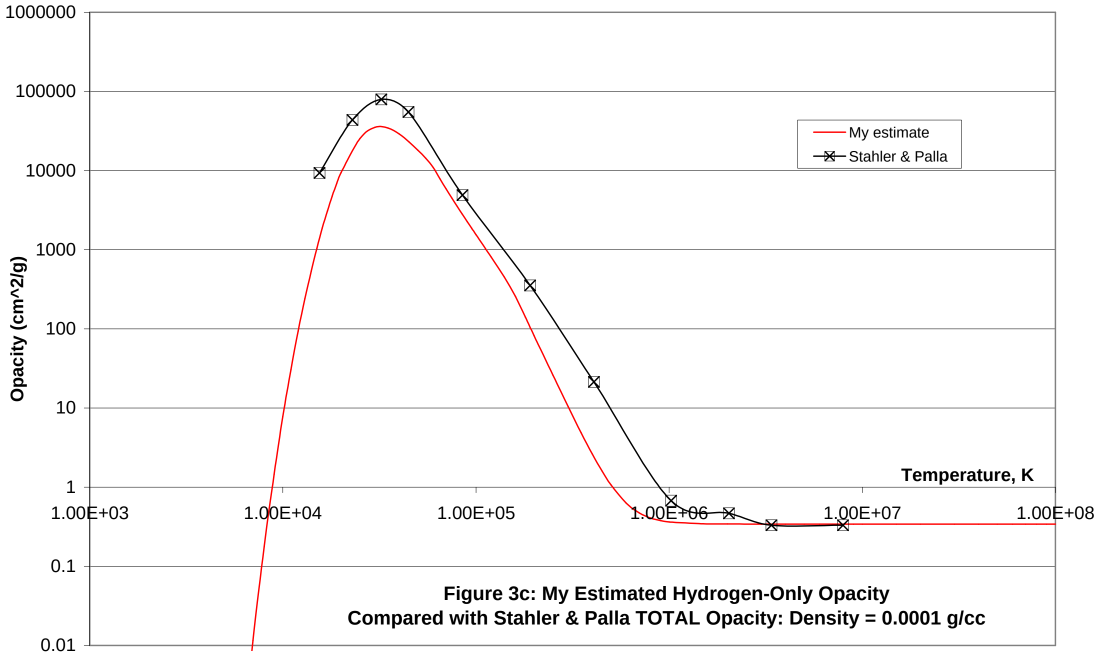


Figure 3c: My Estimated Hydrogen-Only Opacity Compared with Stahler & Palla TOTAL Opacity: Density = 0.0001 g/cc

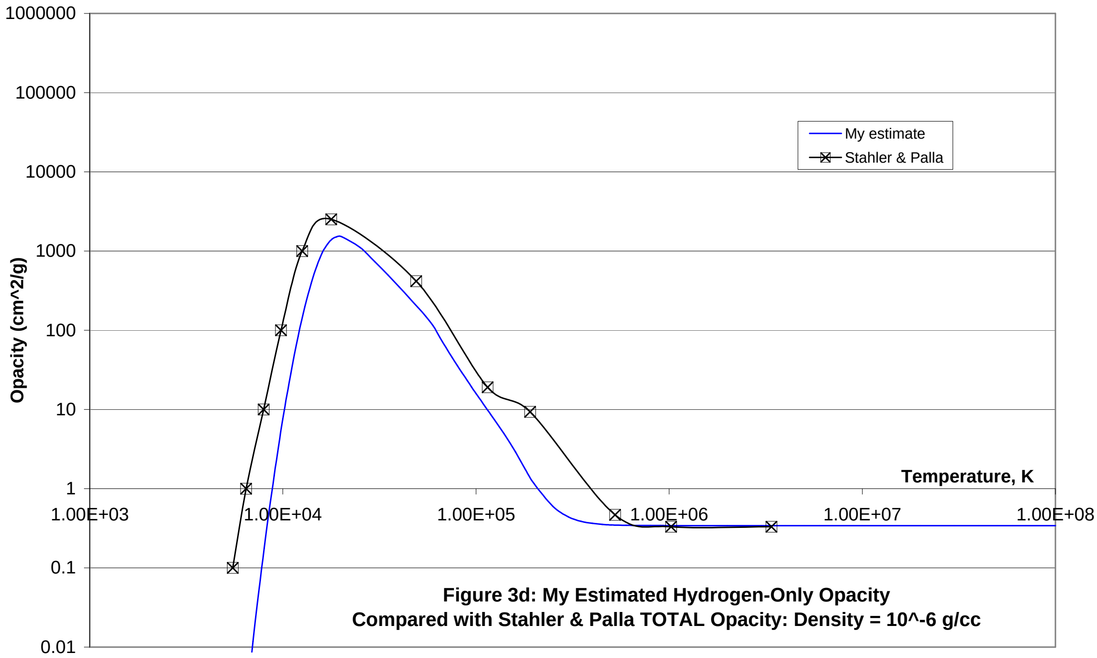


Figure 3d: My Estimated Hydrogen-Only Opacity
Compared with Stahler & Palla TOTAL Opacity: Density = 10^-6 g/cc

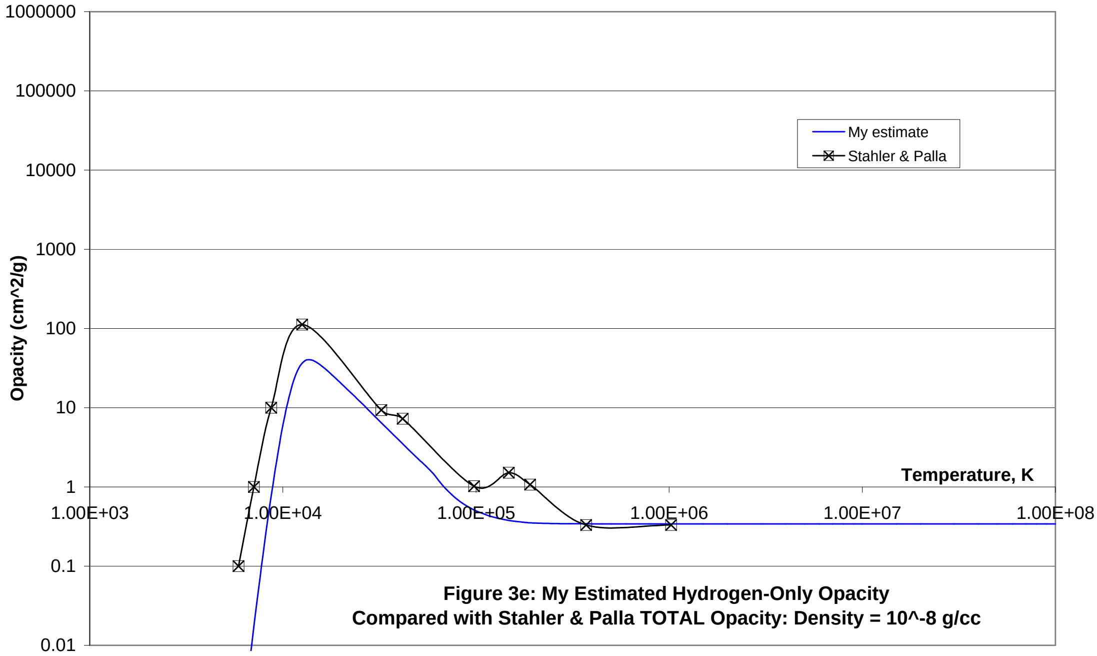


Figure 3e: My Estimated Hydrogen-Only Opacity
Compared with Stahler & Palla TOTAL Opacity: Density = 10^-8 g/cc

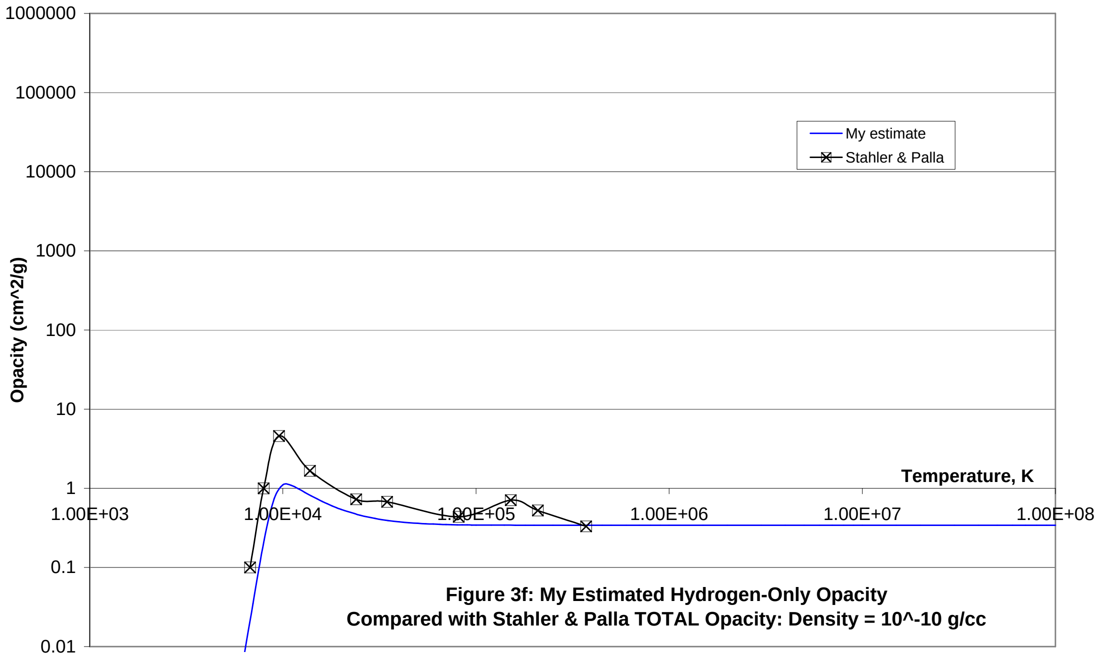


Figure 3f: My Estimated Hydrogen-Only Opacity
Compared with Stahler & Palla TOTAL Opacity: Density = 10^-10 g/cc

In all cases the qualitative agreement is very encouraging, particularly in view of the quite complex, and rapid, variations with temperature and density. The quantitative agreement is rather disguised by the logarithmic scales, which tends to make the agreement look better than it is. However, note that the total opacity from Stahler & Palla is always greater than our estimate for hydrogen (and electrons) alone. This, of course, is as is should be since the opacity can only be made greater when account is taken of the helium and metals. In the next Section we estimate the opacity due to helium ions He^+ in an attempt to improve the agreement.

9. Estimate of the Opacity Due to Helium Ions He^+

The advantage of confining attention to helium nuclei with a single captured electron, i.e. helium ions He^+, is that the theory developed above for hydrogen can simply be taken over unchanged except for:-

- The replacement of the atomic number Z = 1 by Z = 2 (i.e. for a doubly charged nucleus);
- The replacement of the nuclear mass M_p by $4M_p$;
- The replacement of the fraction X by the fraction Y = 1 – X.

Thus, the helium ion opacity is given by,

$$\kappa_{helium} = P_\gamma^{He} f_{He} \frac{1-X}{4M_p} \cdot \frac{128\pi \cdot 2^{3/2}}{3Z^3\alpha^2} \cdot \frac{(\hbar c)^2 E_e^{3/2}}{(m_e c^2)^{5/2} E_\gamma} \cdot \left\{1 + \frac{2E_e}{Z^2\alpha^2 m_e c^2}\right\}^{-4} \tag{9.1}$$

where, $$f_{He} = 1 - y \quad \text{and} \quad \frac{y^2}{1-y} = \frac{(2\pi m_e kT)^{3/2} 4M_p}{(2\pi\hbar)^3 (1-X)\rho} e^{-B_Z/kT} \tag{9.2}$$

and, $$B_Z = Z^2 \frac{\alpha^2}{2} m_e c^2 \tag{9.3}$$

and, $$P_\gamma^{He} = 0.416\left[2 + 2x_2 + x_2^2\right]e^{-x_2} \quad \text{where} \quad x_2 = B_Z / kT \qquad (9.4)$$

We have written Equs.(9.1-4) in terms of an arbitrary Z though they should be interpreted with Z = 2 for helium. Because the ionisation potential for helium (from He^+ to He^{++}) is four times that for hydrogen (4 x 13.6 = 54.4 eV), we expect that it will affect the opacity only at high temperatures.

Numerical evaluation of Equs.(9.1-4) shows that helium ions (He+) contribute little to the opacity, which appears dominated by hydrogen at all temperatures and densities. See EXCEL file "Stellar Hydrogen plus Helium Opacity" for graphs.

It could be that neutral He contributes more to the opacity, particularly since its first ionisation potential is significantly smaller (24.6 eV), though still larger than that of hydrogen (13.6 eV).

10. Opacity Due To Free-Free Transitions
We have assumed that the free-free transitions contribute negligibly to the total opacity.

11. Opacity Due To Metals and Other Mechanisms
There are several other mechanisms which can contribute to the opacity, and hence explain the larger opacity given in Stahler & Palla. These include:-

1) As noted above, neutral helium may contribute significantly.

2) The bound-bound transition in hydrogen, from the n =1 state to the n =2 state, has an energy requirement of 10.2 eV. Because this is a smaller energy requirement than the bound-free transition (13.6 eV), it can contribute more to the opacity at low temperatures.

3) The outer electrons in metals will generally have lower ionisation potentials than hydrogen. Consequently these will contribute significantly, and perhaps dominate, at sufficiently low temperatures. To dominate, their opacity per nucleus must be sufficiently large to overcome their numerical disadvantage. In the case of the Sun, metals comprise only ~2.5% of the stellar medium.

4) At temperatures below ~7000 K, hydrogen can form negative ions by capturing a second electron. This has a very low ionisation energy and hence will contribute significantly to the opacity in the regions approaching the photosphere.

5) Also at the lowest temperatures, approaching the photosphere, there will be an equilibrium concentration of molecules. Bound-bound electron transitions in molecules have low energy thresholds and hence will contribute significantly to the low temperature opacity. The behaviour of the opacity at these lowest temperatures is therefore extremely complex.

12. Kramer's Law Opacity

We have remarked above that, in a certain range, the opacity may be expected to vary roughly proportionally to density and proportional to temperature raised to the power –7/2. This type of dependence is known as Kramer's Law. This behaviour is expected to be roughly valid for temperatures greater than that at which the opacity peaks, but less than that at which the Thompson scattering becomes dominant. In other words, for the right hand side of the opacity peak, where the opacity is reducing with temperature, and before it attains its minimum value.

A fit to this form produces a coefficient of proportionality of 1.777 x 10^{24} when density is in g/cc and the opacity in cm^2/g. (We have used Z = 0.025 to derive the above value). The comparison between this fit and the opacity data from Stahler & Palla follows:-

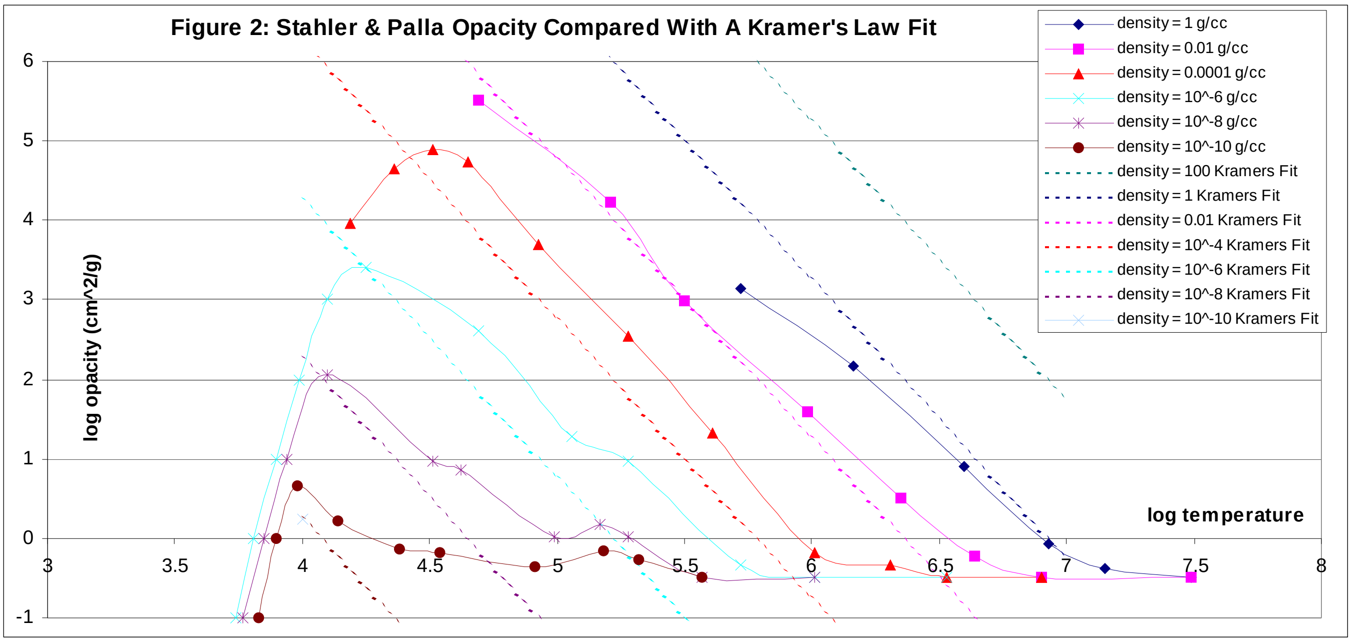


Figure 2: Stahler & Palla Opacity Compared With A Kramer's Law Fit

Thus, the Kramer's Law fit is indeed crudely representative. Incidentally, individual fits for bound-free and free-free transitions imply that the latter are negligible.