Chapter 14 – The Rate of the First Stellar Nuclear Reaction

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1. Introduction

In Appendices A2, A3 and A4 we derive from first principles the rates of some of the lowest mass nuclear reactions. These are all reactions mediated by the electromagnetic interaction and resulting in composite particles (nuclei) bound by the strong nuclear force. As such they are typical of the subsequent stellar reactions[1] which result in successively more massive nuclei. However, the first and most crucial reaction is exceptional[1]. This is the reaction which fuses two protons to form deuterium, $p + p \to D + e^+ + \nu_e$. We have already seen in Chapter 13 that this reaction is dramatically slower than the subsequent reactions. The reason is that it occurs via the weak nuclear interaction rather than the electromagnetic interaction. We have already noted in Appendix A3 that, whilst the Coulomb barrier plays a significant part in determining the absolute rate of this reaction, this is also true of the subsequent reactions. For example, $p + D \to {}^3He + \gamma$ involves overcoming the same Coulomb barrier, but is some 17 orders of magnitude faster than $p + p \to D + e^+ + \nu_e$. The discriminating factor between the two is not the Coulomb barrier but the relative strengths of the electromagnetic and weak nuclear forces at the relevant energy.

The approach of this Chapter is rather idiosyncratic. Its purpose was to convince me that I had some degree of understanding of why this reaction proceeds at the rate that it does. In particular, that the relative rates of $p + p \to D + e^+ + \nu_e$ and $p + D \to {}^3He + \gamma$ is attributably to the weakness of the weak force, and not the Coulomb barrier. The latter does, of course, have a dramatic affect on the absolute reaction rates, but this applies to both reactions.

In this Chapter we provide a heuristic argument which results in a simple analytic expression for the $p + p \to D + e^+ + \nu_e$ reaction rate. The four ingredients are:-

1) The Schrodinger wavefunctions in the presence of the strong nuclear potential, and the resulting matrix elements;
2) The Coulomb barrier and how this affects (1);
3) The competition between the Coulomb barrier, which is more easily surmounted at high energies, and the Maxwell distribution of particle energies at a given temperature, which makes energetic particles scarce;
4) Finally, and most importantly, the effective weak nuclear coupling strength.

In Section 2 we address the first three of these, using as a vehicle a hypothetical reaction $p + \wp \to \Im + \gamma$ which is supposed to proceed via the electromagnetic interaction. In this reaction $\wp$ is a hypothetical spin ½ fermion with a single positive charge, distinct from the proton. $\Im$ is a hypothetical doubly charged particle, considered to be a composite state of p + $\wp$, bound by the strong nuclear force. This hypothetical reaction differs from the (equally hypothetical!) formation of a diproton, $p + p \to {}^2_2He + \gamma$, as considered in Appendix A4, in that the initial states does not

[1] Other reactions in the pp sequences are also weak-force mediated, not just reaction 1 - though the electro-strong reactions are more common.

involve a pair of identical particles. In Appendix A4 we saw that this forced the diproton formation reaction to proceed via a quadrupole interaction. In contrast, our hypothetical reaction, $p + \wp \to \Im + \gamma$, is more like the proton-neutron capture reaction $p + n \to D + \gamma$, except with a 'charged neutron'.

It then only remains, in Section 3, to address the difference caused by the interaction being via the weak nuclear force rather than the electromagnetic force. Finally, in Section 4, we discuss briefly the rather glaring simplifications which make our 'derivation' of the $p + p \to D + e^+ + \nu_e$ reaction rate only heuristic. Nevertheless, we produce a simple closed form expression for the reaction rate in terms of the basic constants which compares quite well with published reaction rates.

2. Rate of the Hypothetical Electromagnetic Reaction $p + \wp \to \Im + \gamma$

We shall assume that the reaction proceeds by the electric dipole interaction. Appendices A2 and A3 have shown that, for the similar reaction which does not involve a Coulomb barrier, $p + n \to D + \gamma$, the cross section can be written,

$$\sigma_{E-dipole}^{np \to D\gamma} = \frac{3\pi}{4} \alpha \frac{M_R c r_{max}}{\hbar} \cdot \frac{(E+B)^3}{(M_p c^2 E)^{3/2}} \left| \left\langle {}^3S(bound) \middle| r\cos\theta \middle| {}^3P(free) \right\rangle \right|^2 \qquad (1)$$

where the matrix element is given by,

$$\left\langle {}^3\mathrm{S(bound)} \middle| r\cos\theta \middle| {}^3\mathrm{P(free)} \right\rangle = \frac{4k^2}{\sqrt{3}(k^2+\beta^2)^2} \sqrt{\frac{\beta}{r_{max}}} \cdot \sqrt{\eta} \qquad (2)$$

where the correction factor is $\eta = \left(1 + \frac{8}{\pi^2}\beta a\right) \approx 1$ (3)

Henceforth we shall take $\eta = 1$. In (1, 2):

- M_R is the reduced mass of the proton-plus-neutron system;
- E is the sum of the kinetic energies of the initial state neutron and proton in the CoM frame;
- B is the binding energy of the deuteron (2.225 MeV);
- The momenta of each of the incoming neutron and proton are $\hbar k$ and we thus have, $(\hbar k)^2 = M_R E$;
- $(\hbar\beta)^2 = M_R B = 0.2312$ fm^{-1};
- $\alpha = 1 / 137$ (the fine structure constant);
- r_{max} is the normalisation volume radius, which cancels from the cross-section.

Equs.(1, 2) thus give,

$$\sigma_{E-dipole}^{pn \to D\gamma} = 4\pi\alpha \left(\frac{\hbar}{M_R c}\right)^2 \frac{\sqrt{EB}}{E+B} \qquad (4)$$

Provided that we assume our hypothetical reaction $p + \wp \to \Im + \gamma$ also proceeds via the electric dipole interaction, the only difference is the Coulomb barrier.

2.1 Accounting For The Coulomb Barrier

We have already seen in Appendix A3, Section 4.1.3.1, and more generally in the Annex to Chapter 18, how the effect of the Coulomb barrier can be estimated. For completeness we reiterate the general argument here:-

The energy dependence of the matrix element derives from how much of the free state manages to "penetrate the Coulomb barrier" from outside. It is the magnitude of the free wavefunction within the region of the Coulomb potential which is significant, (i.e. radii 'r' such that, a < r < $r_E = Z_1 Z_2 \tilde{e}^2 / E$). This is the region within which the free state has negative kinetic energy, the exponential decay length being (the reciprocal of) $\kappa = \sqrt{2m(V(r) - E)} / \hbar$, noting that this varies with 'r', and where V(r) = $Z_1 Z_2 \tilde{e}^2 / r$.

Roughly speaking, we may expect the magnitude of the wavefunction to diminish from r = r_E to r = a by a factor of $\exp\left\{-\int_a^{r_E} \kappa dr\right\}$. The integral may be evaluated exactly to give a factor,

$$X = \exp\left\{-\frac{Z_1 Z_2 \tilde{e}^2 \sqrt{2M_R}}{\hbar} \cdot \frac{1}{\sqrt{E}}\left[\theta - \frac{1}{2}\sin 2\theta\right]\right\} \quad \text{where, } \theta = \tan^{-1}\sqrt{\frac{V_c}{E} - 1}, \tag{5}$$

where, $V_c = Z_1 Z_2 \frac{\tilde{e}^2}{a}$ and M_R is the reduced mass, i.e. $M_R = \frac{A_1 A_2}{A_1 + A_2} M_p$. (6)

For energies sufficiently small compared with V_c, $\theta \to \pi / 2$ and hence the sin term is zero and we get,

$$X \to \exp\left\{-b / 2\sqrt{E}\right\} \quad \text{where, } \frac{b}{2} = \frac{\pi Z_1 Z_2}{2} \alpha \sqrt{\frac{2A_1 A_2}{A_1 + A_2} M_p c^2} \tag{7}$$

[Noting that we have changed our convention from Chapter 18 and Appendix A3, calling the expression in (7) b/2 rather than b]. In the case of interest the two incident particles have $Z_1 = Z_2 = A_1 = A_2 = 1$, and so b / 2 = 0.351 √MeV.

Hence, for our hypothetical reaction, $p + \wp \to \Im + \gamma$, the matrix element is given by (4) multiplied by the factor X, and the cross section thus becomes,

$$\sigma_{E-dipole}^{p\wp \to \Im\gamma} = 4\pi\alpha \left(\frac{\hbar}{M_p c}\right)^2 \frac{\sqrt{EB}}{E + B} e^{-b/\sqrt{E}} \tag{8}$$

Since we shall ultimately interpret $\Im$ as the deuteron, B is still the deuteron binding energy. Similarly, the derivation only holds if $\wp$ has the same mass as the proton, but this is also OK since we will ultimately interpret $\wp$ to be a proton.

The reaction rate per proton is $\rho_N^{\wp} v \sigma_{E-dipole}^{p\wp \to \Im\gamma}$, where the first term is the number density of $\wp$ particles, and v is the closing velocity of the incoming particles, i.e.

$$v = 2\hbar k / M_p = 2\sqrt{E / M_p} \tag{9}$$

assuming non-relativistic speeds. Assuming the $\wp$ particle density is 1 mole per cm^3, i.e. A per m^3, where A = 6.02 x 10^{29}, then the reaction rate per proton is,

$$\begin{aligned} \text{Reaction Rate}_{E-dipole}^{p\wp \to \Im\gamma} &= 8\pi A\alpha \left(\frac{\hbar}{M_p c} \right)^2 \frac{E}{E+B} \sqrt{\frac{B}{M_p}} e^{-b/\sqrt{E}} \\ &\approx 8\pi A\alpha \left(\frac{\hbar}{M_p c} \right)^2 \frac{E}{\sqrt{BM_p}} e^{-b/\sqrt{E}} \end{aligned} \tag{10}$$

where the last expression assumes E << B (i.e. T << 2 x 10^{10} °K). Equ.(10) gives the reaction rate assuming all pairs of colliding particles have the same CoM frame kinetic energy E. Clearly this is not true for a thermal distribution of particle speeds and random directions of motion. The influence of the actual (Maxwellian) distribution of energies is considered next.

2.2 Reaction Rate at a Specified Temperature – The Maxwell Distribution
The derivation is identical to that already considered in Appendix A3, Section 6. For completeness we re-iterate it in full:-

For the non-relativistic energies of interest, the kinetic energies of the reacting particles are given by the Maxwell distribution,

$$P[\varepsilon]d\varepsilon = \frac{2}{\sqrt{\pi}} \sqrt{\varepsilon} e^{-\varepsilon} d\varepsilon \tag{11}$$

where $\varepsilon = E / kT$. We denote the reaction rate at temperature T by R[T]. If one proton has speed v_1 in the lab frame, and the other speed v_2, and if the two velocities were at an angle θ, the resulting reaction rate is denoted R[v_1, v_2, θ]. If we know the latter we can find R[T] from,

$$R[T] = \int_0^\infty P[\varepsilon_1]d\varepsilon_1 \int_0^\infty P[\varepsilon_2]d\varepsilon_2 \int_0^\pi \frac{d(\cos\theta)}{2} \cdot R[v_1, v_2, \theta] \tag{12}$$

where, $\varepsilon_{1,2} = \frac{M_p v_{1,2}^2}{2kT}$. Because the random motions are isotropic, all solid angles are equally likely, hence the uniform probability density in cosθ. Now R[v_1, v_2, θ] will be given by Equ.10 provided that we find the CoM system total kinetic energy (E) in terms of v_p, v_D and θ.

We now derive an approximate analytic expression for the temperature dependence of the reaction rate from (10) and (12). We shall ignore the complication of the θ dependence, simply replacing the integral of d(cosθ) by unity. [NB: This leaves a residual factor of ½ due to the θ integration. This is because the component of the second proton's velocity in the direction of the first's averages to zero. Thus, on average, their relative speed is just the speed of the first proton, rather than twice this as we have used in Equ.(9), above]. Hence we get,

$$R[T] = \frac{4}{\pi} R_0 \int_0^\infty \int_0^\infty d\varepsilon_1 d\varepsilon_2 \sqrt{\varepsilon_1 \varepsilon_2} (\varepsilon_1 + \varepsilon_2) \exp - \left\{ \frac{\tilde{b}}{\sqrt{\varepsilon_1 + \varepsilon_2}} + \varepsilon_1 + \varepsilon_2 \right\} \qquad (13)$$

where,

$$\tilde{b} = b / \sqrt{kT} \qquad (14)$$

and,

$$R_0 = 4\pi\alpha A \left(\frac{\hbar}{M_p c} \right)^2 \cdot \frac{kT}{\sqrt{BM_p}} \qquad (15)$$

and where we have used the dimensionless variables,

$$\varepsilon_1 = E_1 / kT, \ \varepsilon_2 = E_2 / kT, \ \varepsilon = \varepsilon_1 + \varepsilon_2 \qquad (16)$$

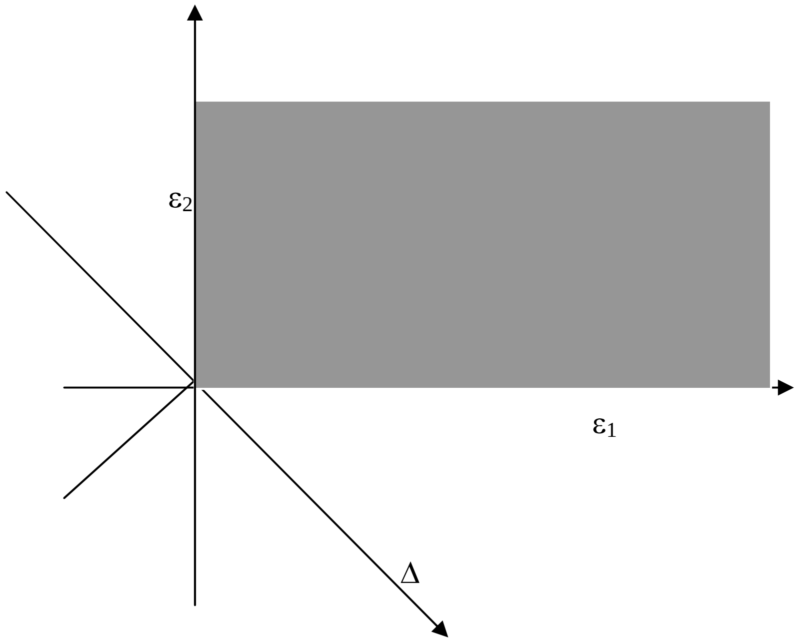
We now change the integration variable to ε and $\Delta = \varepsilon_1 - \varepsilon_2$, noting that,

$$\varepsilon_1 = (\varepsilon + \Delta)/2 \ \text{and} \ \varepsilon_2 = (\varepsilon - \Delta)/2 \qquad (17)$$

so that the Jacobean ($d\varepsilon_1 d\varepsilon_2 / d\varepsilon d\Delta$) takes the value ½. Hence we have,

$$R[T] = \frac{R_0}{\pi} \int_0^\infty \varepsilon \cdot \exp - \left\{ \frac{\tilde{b}}{\sqrt{\varepsilon}} + \varepsilon \right\} \cdot d\varepsilon \int_{-\varepsilon}^{+\varepsilon} \sqrt{\varepsilon^2 - \Delta^2} \cdot d\Delta \qquad (18)$$

Note that the range of integration $\varepsilon_1 \in [0, \infty]$ and $\varepsilon_2 \in [0, \infty]$ maps into $\varepsilon \in [0, \infty]$ and $\Delta \in [-\varepsilon, +\varepsilon]$, as may be seen from the following (where the region of integration is shaded)…



The second integral is just $\pi\varepsilon^2/2$, so (19) becomes,

$$R[T] = \frac{R_0}{2}\int_0^\infty \varepsilon^3 d\varepsilon \cdot \exp - \left\{ \frac{\tilde{b}}{\sqrt{\varepsilon}} + \varepsilon \right\} \quad (19)$$

Calling the exponent,

$$f(\varepsilon) = \frac{\tilde{b}}{\sqrt{\varepsilon}} + \varepsilon \quad (20)$$

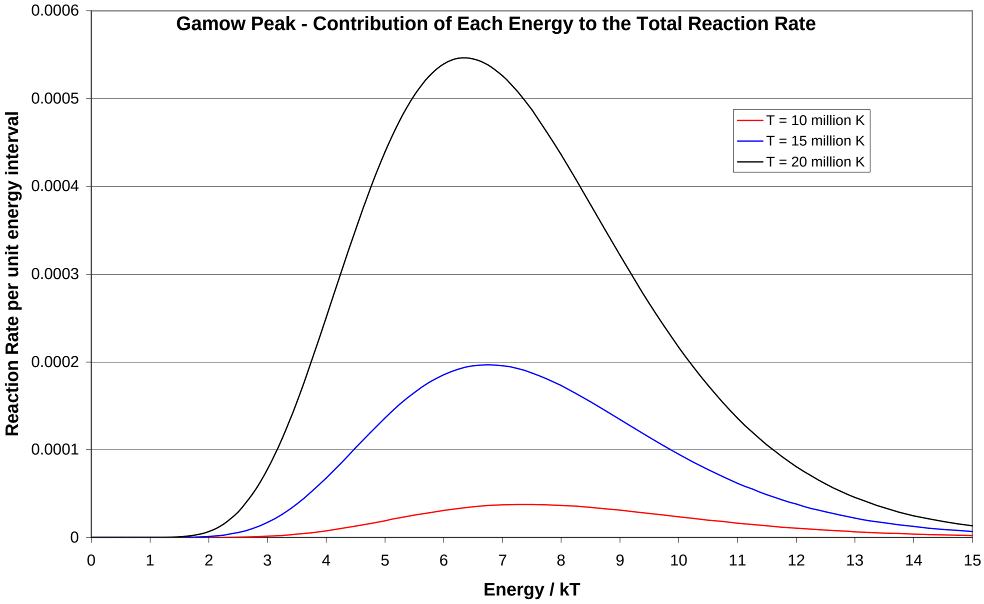
we note that it attains a minimum when,

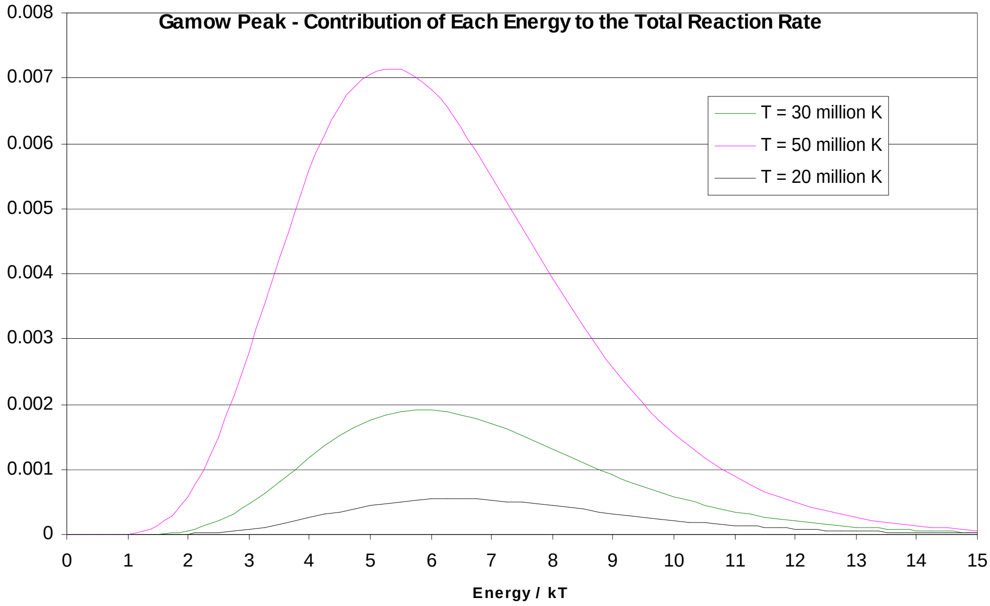
$$\varepsilon_0 = (0.5\tilde{b})^{2/3} = \frac{(0.5b)^{2/3}}{(kT)^{1/3}} = E_0 / kT \Rightarrow E_0 = (0.5bkT)^{2/3} \quad (21)$$

The minimum value of the exponent is,

$$f_{min} = 3\left(\frac{\tilde{b}}{2}\right)^{2/3} \quad (22)$$

The minimum in f is due to the competition between the energy dependence of the matrix element and the Maxwell distribution. At a given temperature, the Maxwell distribution means that there are far more protons with low energies than with higher energies. However, for the lower energy protons, the matrix element is much smaller, and hence the reaction rate much slower, due to the greater difficulty low energy protons have in penetrating the Coulomb barrier. Conversely, at the same temperature, the Maxwell distribution ensures that there are fewer protons with energies sufficiently great to penetrate the Coulomb barrier easily and hence to result in a faster reaction rate. The result of the competition between the Coulomb barrier and the Maxwell distribution is illustrated by plotting the integrand in (19) against normalised energy, ε:-





Note that the energy which contributes most to the reaction rate is about 4 or 5 times the mean thermal energy (i.e. 4 or 5 times 3kT/2 ~ 6 to 7.5 times kT). Note also that this reaction peak is at a rather higher energy than E_0. The latter gives the maximum of e^{-f}, but does not account for the factor E^3 in (19) which pushes the peak to a higher energy

These graphs also illustrate how the overall reaction rate varies with temperature, since this is given by the area under the curves (times a constant factor, namely $0.5R_0$). We now find an approximate closed-form expression for the total reaction rate, carrying out the integral in (19) by expanding the exponent as a power series, i.e.,

$$f(\varepsilon) \approx f_{min} + 0.5 f_0''(\varepsilon - \varepsilon_0)^2 + ... \tag{23}$$

where,
$$f_0'' = \frac{3}{2^{1/3}\tilde{b}^{2/3}} \tag{24}$$

Hence, (19) gives,

$$\begin{aligned} R[T] &\approx \frac{R_0}{2}\int_0^\infty \varepsilon^3 d\varepsilon \cdot \exp-\left\{f_{min} + 0.5f_0''(\varepsilon-\varepsilon_0)^2\right\} \\ &\approx \frac{R_0}{2}\varepsilon_o^3 e^{-f_{min}} \int_0^\infty d\varepsilon \cdot \exp-\left\{0.5f_0''(\varepsilon-\varepsilon_0)^2\right\} \\ &= \frac{R_0}{2}\varepsilon_o^3 e^{-f_{min}} \int_{-\varepsilon_0}^\infty dy \cdot \exp-\left\{0.5f_0''y^2\right\} \\ &\approx \frac{R_0}{2}\varepsilon_o^3 e^{-f_{min}} \int_{-\infty}^\infty dy \cdot \exp-\left\{0.5f_0''y^2\right\} \\ &= \frac{R_0}{2}\varepsilon_o^3 e^{-f_{min}} \sqrt{\frac{2\pi}{f_0''}} \end{aligned} \tag{25}$$

The derivation of (25) requires that $\varepsilon_0 >> 1$, which is equivalent to the requirement that the temperature is less than about 50 million K ($\varepsilon_0 = 3.06$), and hence is reasonable for stars not too much more massive than the Sun in their hydrogen burning phase. (In helium burning at 100 MK, $\varepsilon_0 = 2.43$, so the approximation is still not so bad). However, if this approximation fails, the error is a most a factor of 2. Substituting (21) and (24) into (25) gives,

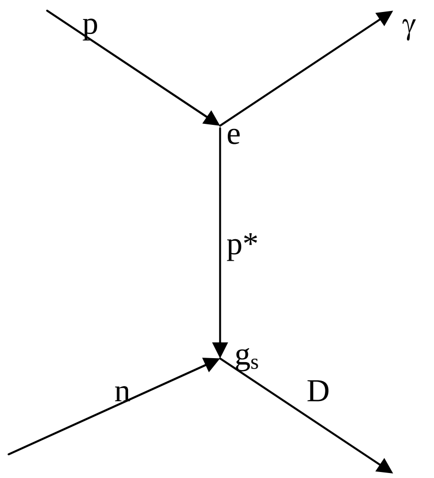
$$\begin{aligned} R[T] &\approx R_0\left(\frac{\tilde{b}}{2}\right)^{7/3} \exp\{-f_{min}\} \\ &= 4\pi\alpha A\left(\frac{\hbar}{M_p c}\right)^2\left(\frac{c}{\sqrt{BM_p c^2}}\right)\left(\frac{b}{2}\right)^{7/3} \cdot \frac{\exp\{-f_{min}\}}{(kT)^{1/6}} \end{aligned} \tag{26}$$

where f_{min} is given by (22) and (14) and b/2 = 0.351. Note that the temperature dependence of the reaction rate is given almost entirely by the exponent factor, since kT in the denominator is raised to a small power.

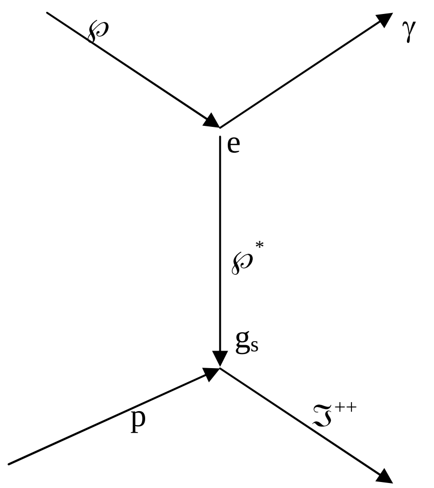
We cannot confront the calculated reaction rate, (26), with experimental data since the reaction in question, $p + \wp \to \Im + \gamma$, is hypothetical. However, the reaction $p + D \to {}^3_2He + \gamma$ is very similar, in particular it proceeds by the electric dipole interaction. The only differences are the masses involved, and the spin factor. Appendix A3, Section 6 has already shown that the above estimation procedure gives good results compared both with an accurate numerical integration of (the equivalent of) Equ.19, and also with published reaction rate data (Hoffman et al).

3. The Actual Reaction $p + p \rightarrow D + e^+ + \nu_e$ - The Weak Interaction Coupling

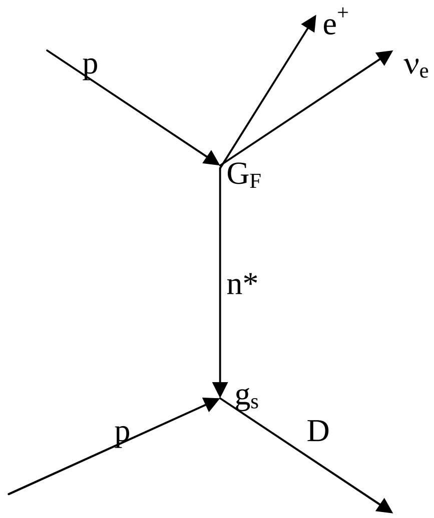
We must now replace our hypothetical electromagnetic interaction with the true weak nuclear interaction. At this point our 'derivation' becomes very heuristic. The proton-neutron capture reaction can be envisaged in terms of a pseudo-Feynman diagram as follows,



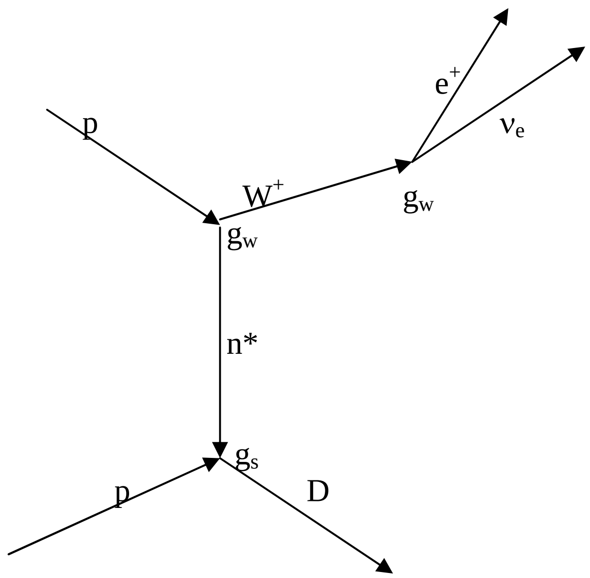
where, as indicated, the upper vertex is electromagnetic and the lower vertex is the strong nuclear force which binds the deuteron. The virtual proton (p*) is off-mass-shell, having emitted a photon with energy equal to the deuteron binding energy, B, plus the initial (n+p) kinetic energy, E. Our hypothetical reaction $p + \wp \rightarrow \Im + \gamma$ can be written similarly,



If we want the final state to be a deuteron, it is necessary that the virtual particle be an off-mass-shell neutron. Hence, in place of the emitted photon we must have emitted particles which carry away the positive charge. Thus, for the reaction of interest, $p + p \rightarrow D + e^+ + \nu_e$, we have,



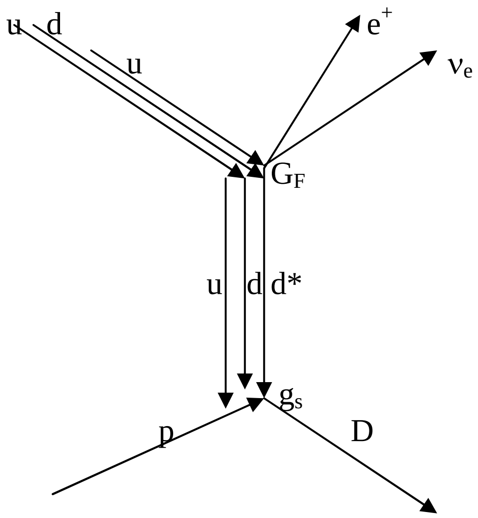
where the Fermi constant, G_F, indicates a weak interaction coupling at the top vertex. In terms of the more modern gauge theory of the weak interactions, the above graph can equivalently be drawn as,



where the weak gauge coupling strength is related to the Fermi constant by,

$$\frac{G_F}{\sqrt{2}} = \frac{g_w^2}{M_W^2} \tag{27}$$

and M_W is the mass of the gauge vector boson W, i.e. roughly 80 GeV. However, the gauge theory diagram adds nothing to our argument and is not considered further. Of more importance is that the conversion of the two nucleon types by the weak interaction can be interpreted as the conversion between 'up' and 'down' quarks. Thus, the proton is uud and the conversion of one of the 'up' quarks to a 'down' quark results in a neutron, which is udd. Hence, the diagram becomes,



Hence, two quarks, u and d, essentially "go along for the ride", and only the third quark (u) actually interacts and converts to a virtual d quark. The importance of considering the weak interaction to be one between quarks and leptons, rather than between nucleons and leptons, will be made evident below.

To find the effective weak vertex coupling strength we look at standard weak force theory. For example, for a muon decaying into an electron, an anti-electron neutrino and a muon neutrino, the Fermi interaction gives a matrix element,

$$\text{M.E.(muon decay)} = \frac{G_F}{\sqrt{2}}\left[\bar{u}_e \gamma_\alpha (1+\gamma^5) v_{\nu_e}\right]\left[\bar{u}_{\nu_\mu} \gamma^\alpha (1+\gamma^5) u_\mu\right] \tag{28}$$

(see any standard text on weak interactions, for example, Commins, Equ.2.12). Reading from right to left, the four Dirac spinors in (28) have the following effects: annihilate the muon; create a muon-neutrino; create an anti-electron-neutrino; and, create an electron, i.e., $\mu^- \rightarrow \nu_\mu + \bar{\nu}_e + e^-$ We can therefore turn a muon decay matrix element into the quark-lepton vertex term $u^+ \rightarrow d + e^+ + \nu_e$ by the exchanges as follows,

$$\mu^- \rightarrow u^+ \qquad \nu_\mu \rightarrow d \qquad \bar{\nu}_e \rightarrow e^+ \qquad e^- \rightarrow \nu_e \tag{29}$$

Hence, the absolute square of the matrix element can be evaluated (see, for example, the derivation leading to Commins Equ.2.26) as,

$$|\text{M.E.}|^2 = 32 G_F^2 \left[(p_\nu - m_\nu s_\nu)\bullet(p_d - m_d s_d)\right]\left[(p_u - m_u s_u)\bullet(p_e - m_e s_e)\right] \tag{30}$$

where the subscripts denote the particle type, and each p and s is a 4-vector – of momentum and spin respectively. [NB: e = anti-electron = positron]. The terms in […] are 4-vector dot products. The neutrino mass is negligible compared with the energies in question, so the first spin term disappears. Since we are not concerned with polarisation, we make the rather crude simplifying assumption that the other spin factors can be ‘averaged to zero’. The accuracy of this is not clear, but it has the advantage of great simplification of (30), namely to,

$$|M.E.|^2 = 32G_F^2[p_\nu \bullet p_d][p_u \bullet p_e] \quad (33)$$

Now, because the final deuteron is so much more massive than the neutrino and the positron (= e), it carries away a negligible proportion of the released energy. Consequently the energy of the ‘down’ quark must be very small, i.e. negligible compared with its mass. Furthermore, since we are interested in temperatures of (say) up to ~50 million K, even the Gamow peak is at an energy (~7.5kT) of only ~0.022 MeV. This is also negligible compared with the ‘up’ quark mass. Thus, the 4-momenta of both quarks approximates to just (mass, 0). Hence (33) becomes,

$$|M.E.|^2 = 32G_F^2 m_u m_d E_\nu E_e \quad (34)$$

where, in obvious notation, (34) involves the up and down quark masses and the total energies of the neutrino and the positron (including their rest mass). But the magnitude of the Fermi constant is,

$$G_F = \frac{1.03 \text{ x } 10^{-5}}{M_p^2} \quad (35)$$

Substituting into (34) gives,

$$|M.E.|^2 = 34\text{x}10^{-10} \frac{m_u m_d E_\nu E_e}{M_p^4} \quad (36)$$

The up and down quark masses are not known with precision. They are roughly,

$$m_d \sim 6 \text{ MeV}; \quad m_u \sim 3 \text{ MeV} \quad (37)$$

As for the energies of the neutrino and positron, these will vary with the energy of the initial protons. However this variation is very slightly. This is because the energy released by the fusion into deuterium is $B - (M_n - M_p)c^2 = 2.225 - 1.2933 = 0.932$ MeV. Now, even at the Gamow peak for a temperature of 50 million K, the proton kinetic energy is only 0.022 MeV. Hence, the total available energy is not more than ~0.95 MeV, and is little affected by the original proton kinetic energy. The deuteron will take a negligible share of this energy. It is easily shown that the energies of the neutrino and positron are 0.641 MeV and 0.82 MeV (including the rest mass) respectively. Hence, we find the dimensionless coupling strength of the weak vertex in this reaction from (37) to be,

$$|M.E.|^2 = 34\text{x}10^{-10} \frac{6 \text{ x } 3 \text{ x } 0.641 \text{ x } 0.82}{(938.273)^3} = 4.14 \text{ x } 10^{-20} \quad (38)$$

This vertex coupling strength takes the place of the electromagnetic fine structure constant, α, in our expression Equ.(26) for the reaction rate. Thus the weak interaction is about 0.5 x 10^{-17} times weaker than the electromagnetic interaction *in this case*, noting that this comparison is energy dependent. This factor is all that is necessary to explain the huge difference in timescale between the stellar "reaction No.1", i.e., $p + p \to D + e^+ + \nu_e$ and the second reaction, i.e., $p + D \to {}^3_2He + \gamma$, i.e. this factor takes us from a reaction time of 1.4 x 10^{10} years (at solar temperatures) to just a few seconds for the second reaction.

Finally, our closed form expression for the rate of "reaction No.1", i.e., $p + p \to D + e^+ + \nu_e$, is just (26) with α replaced by (38), i.e.,

$$R[T] = \left(4\pi \times 4.14x10^{-20}\right) A \left(\frac{\hbar}{M_p c}\right)^2 \left(\frac{c}{\sqrt{BM_p c^2}}\right) \left(\frac{b}{2}\right)^{7/3} \cdot \frac{\exp\{-f_{min}\}}{(kT)^{1/6}} \qquad (39)$$

In (39) we have used the explicit magnitude of the weak vertex coupling *in this universe*. Since we may be interested in alternative universes in which the Fermi constant, or the masses of the quarks, are different, the general expression is obtained by substituting (34) into (26), giving,

$$R[T] = 128\pi G_F^2 m_u m_d E_\nu E_e A \left(\frac{\hbar}{M_p c}\right)^2 \left(\frac{c}{\sqrt{BM_p c^2}}\right) \left(\frac{b}{2}\right)^{7/3} \cdot \frac{\exp\{-f_{min}\}}{(kT)^{1/6}} \qquad (40)$$

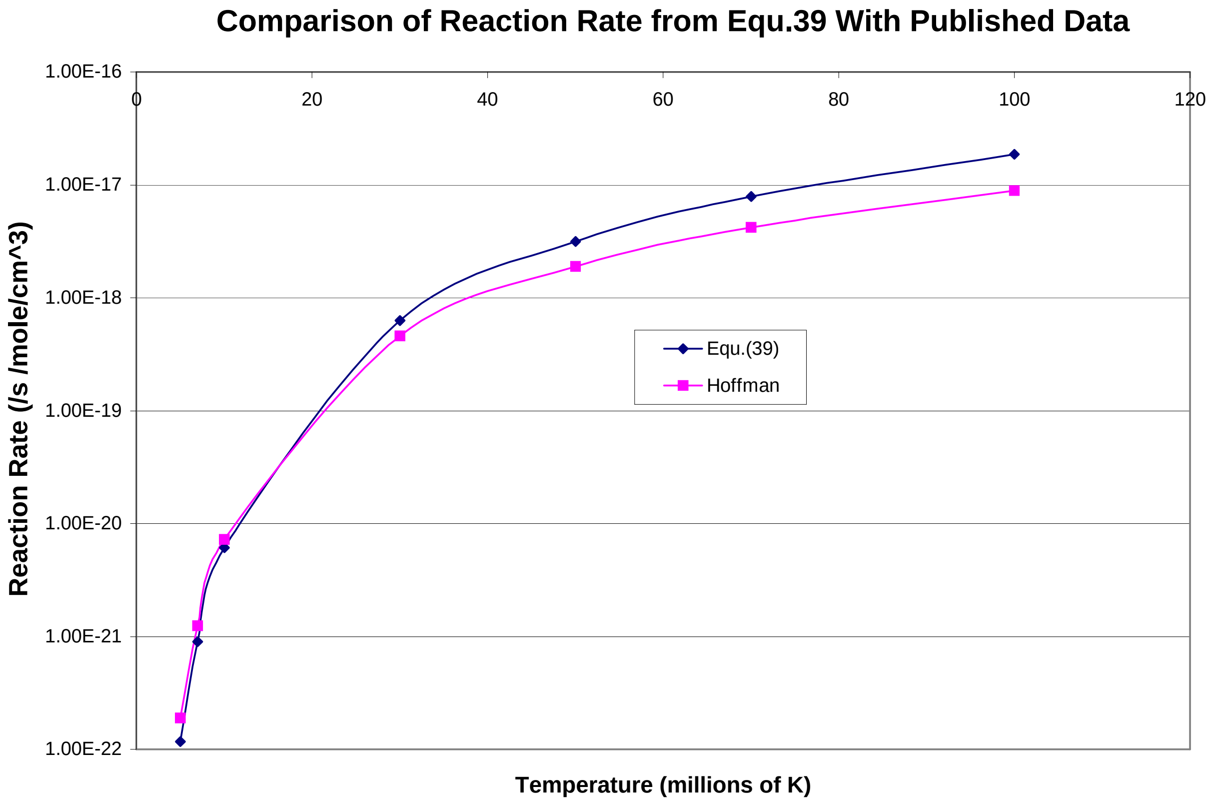
where we may approximate the lepton energies by $E_\nu E_e \sim [B - (M_n - M_p)c^2]m_e c^2$.

However, back in this universe, we may check that (39) is reasonably accurate by comparing it with the published results of Hoffman et al. Substituting the values of the temperature independent parts, (39) gives,

$$R[T] = 1.225x10^{-14} \frac{\exp\{-f_{min}\}}{(kT)^{1/6}} \qquad (s^{-1}) \qquad (41)$$

where kT must be in MeV. The comparison at stellar temperatures is as follows,

T (10^6 K)	**kT (MeV)**	$\tilde{b}/2$	**f_{min}**	$e^{-f_{min}}/(kT)^{1/6}$	**Equ.39 (= 41)**	**Hoffman**
5	4.31E-04	1.69E+01	1.98E+01	9.55E-09	1.17E-22	1.89 x 10^{-22}
7	6.04E-04	1.43E+01	1.77E+01	7.35E-08	9.00E-22	1.24 x 10^{-21}
10	8.63E-04	1.20E+01	1.57E+01	5.01E-07	6.14E-21	7.21 x 10^{-21}
30	2.59E-03	6.90E+00	1.09E+01	5.12E-05	6.27E-19	4.59 x 10^{-19}
50	4.31E-03	5.34E+00	9.17E+00	2.58E-04	3.16E-18	1.90 x 10^{-18}
70	6.04E-03	4.52E+00	8.20E+00	6.45E-04	7.90E-18	4.21 x 10^{-18}
100	8.63E-03	3.78E+00	7.28E+00	1.52E-03	1.87E-17	8.87 x 10^{-18}



Thus the comparison is very good, given the rapid variation with temperature. Our estimate is within a factor of 2 of Hoffman's data over the whole temperature range (being smaller at lower temperatures and higher at high temperatures).

4. Shortcomings of Our Approximation - INCOMPLETE

The possible pp states are ^1S, ^3P, ^1D, etc. The diproton would be a ^1S state, bound by the singlet nuclear force. Its formation reaction would be forced to proceed via a quadrupole interaction because,

- The magnetic interaction is proportional to the difference between the magnetic dipole moments, which is clearly zero for identical particles (and, in any case, causes a spin flip which would give a zero matrix element between two ^1S states);
- The electric dipole moment between ^3P and ^1S is zero by orthogonality of the spin states (which the electric interaction does not flip).

The formation of a deuteron from two protons differs because the deuteron is a triplet state, ^3S. Thus, it is feasible that there could be non-zero contributions from both S and P wave initial free states, i.e. from matrix elements $\left\langle {}^1S(\text{free}) \middle| H_1 \middle| {}^3S(\text{bound}) \right\rangle$ and $\left\langle {}^3P(\text{free}) \middle| H_2 \middle| {}^3S(\text{bound}) \right\rangle$, provided that H_1 causes a spin flip whereas H_2 does not. Thus, H_2 could be simply a term proportional to $r.\cos\theta$, as assumed in our derivation. Since the interaction is not electromagnetic, the term H_1 must be a coupling between particle spins. This is indeed the form of the Fermi coupling, which involves four fermion fields interacting at a point…… What this amounts to is that expansions like that given by Commins, Equ.2.38, suggest that both spin-flip and no-spin-flip contributes in the Fermi interaction. Our derivation has used the no-spin-flip only, i.e. a free P wave with a r.cosθ interaction. It is not clear why the spin-flip S-wave does not contribute – it would be expected to be larger (since S waves generally contribute more than P-waves, which contribute more than D-waves, etc – unless they are forbidden).