Chapter 12 – Stellar Timescales

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In this brief chapter we use the simplest of crude estimates to derive various characteristic timescales for stellar behaviour.

1. Gravitational Potential (Binding Energy)

A crude estimate of the gravitational (binding) energy of a star can be made assuming uniform density (although we know from Chapter 11 that this is grossly at variance with the requirement for hydrostatic equilibrium). Thus,

$$|P.E.| = \int_0^R \frac{Gm(r)\rho(r)}{r} dV \approx \frac{4\pi G\rho^2}{3}\int_0^R 4\pi r^4 dr = \frac{(4\pi)^2 G\rho^2 R^5}{15} = \frac{3}{5}\frac{GM^2}{R} \qquad (1)$$

Thus, for the sun with mass 2 x 10^{30} kg and radius 7 x 10^8 m, we get (using G = 6.67 x 10^{-11} $m^3 kgs^{-2}$) a potential energy of (minus) 2.3 x 10^{41} J.

Of course we could improve on this crude estimate by using the density distribution from the Clayton model of Chapter 11. We would then get a gravitational potential energy given by,

$$P.E. = -\eta\frac{GM^2}{R}, \quad \text{where} \quad \eta = \frac{R\int_0^{R/a} x^2 e^{-x^2} dV}{\sqrt{6}\, a \int_0^{R/a} \frac{x^3 e^{-x^2}}{\Phi(x)} dV}, \quad \text{where} \quad x = r/a \qquad (2)$$

and we must choose some specific value for 'a' (see Chapter 11, which also gives the expression for the function Φ). Equ.(2) will clearly lead to a larger binding energy than (1), since it involves much larger densities near the centre, i.e. it involves additional gravitational collapse. In other words, $\eta > 3/5$. Numerical integration gives, for R/a = 5.4 (a value for which the Clayton model was reasonably accurate for the sun, for $x < 1$), $\eta = 1.79461$, i.e. close to 9/5.

If a more accurate expression like (2) is used for the gravitational potential energy, for completeness we also need to appeal to the virial theorem to estimate the energy converted to heat. This is because not all the binding energy appears as heat as the gas collapses. The virial theorem says that, in gravitational collapse between two gravitationally bound states in hydrostatic equilibrium, the heat produced is just half the decrease in the gravitational potential energy. Thus,

$$\text{Heat Energy} = \frac{\eta}{2}\frac{GM^2}{R} \approx 0.9\frac{GM^2}{R} \qquad (3)$$

(at least in the Clayton model approximation and for R/a=5.4). Hence, for a star of the same size and mass as the sun, the heat generated by collapse is **~3.4 x 10^{41}J**.

2. How Hot Does A Star Get Due To Its Initial Collapse?

Knowing the heat generated, we need only the specific heat of the gas to estimate the average temperature of the collapsed gas (=star). Perfect gases have a specific heat at constant volume equal to the universal gas constant, i.e. k_B per particle. Clearly the process is not at constant volume. The upper bound specific heat is probably that at constant pressure, which is expected to be around 5/3 times that at constant volume. We shall use this value here as an estimate, although the process is clearly not really at constant pressure either (except at the surface of the star). Thus, the specific heat is $5/3 \times 1.38 \times 10^{-23} = 2.2 \times 10^{-23}$ J/^{o}K per particle.

The number of particles is the mass of the star divided by the average mass of the particles. The latter was evaluated in Chapter 11 as 0.99×10^{-27} kg (for the case of the primordial composition of 75% H + 25% He, and not forgetting that the electrons roughly halve the average mass). Hence the number of particles is,
$2 \times 10^3 / 0.99 \times 10^{-27} = 2.0 \times 10^{57}$ particles.

Hence, the thermal capacity of the whole star is 2.2×10^{-23} J/^{o}K per particle x 2.0×10^{57} particles = 4.44×10^{34} J/^{o}K.

The average temperature of the collapsed gas (assuming from a cold start) is thus about ~3.4×10^{41}J / 4.44×10^{34} J/^{o}K = **7.7×10^6 ^{o}K**.

Since we know that the requirements of hydrostatic equilibrium lead to temperatures on the surface of the star which are very low compared with those in the centre, the temperature in the centre will be much greater than this average value. For a Clayton model temperature distribution, with R/a = 5.4, the average temperature (by numerical integration of the distribution given in Chapter 11) is 0.0306 times the central temperature. Thus, when a star of the size and mass of the sun first forms, if its temperature distribution conformed to that of a Clayton model, there would be sufficient heat available to raise the central temperature to ~$7.7 \times 10^6 / 0.0306 \cong 250 \times 10^6$ ^{o}K. This is some 18 or so times greater than the actual temperature at the centre of the sun (from standard solar models).

It is not surprising that the actual central temperature is lower. We will see in the next Section that the initial heat inventory due to gravitational collapse is expected to have dissipated long ago. The actual central temperature today is therefore determined by the heat balance between nuclear energy production and radiative heat loss at the surface (together with the constraint of hydrostatic equilibrium). The initial gravitational heat inventory is now irrelevant to the temperature of the sun. Nevertheless, the above estimate of the central temperature immediately after the initial collapse is extremely important. Its large magnitude confirms that gravitational heating would have been sufficient to ignite the nuclear fuel in the first place (this requires temperatures in excess of about 10×10^6 K).

3. The Kelvin Timescale

The sun presented Victorian scientists with a conundrum. The geologists of that era were already estimating the age of rocks on the earth in the order of billions, rather than millions, of years. However, the Victorians could not understand how the sun could burn for so long.

The Kelvin timescale is the lifetime one derives for the sun based on the assumption that the only heat source is that of the initial gravitational collapse. We have estimated this to be 3.4 x 10^{41} J, above. The rate of loss of heat from the surface of the sun is simply calculated from Stefan's law in terms of the surface temperature and area. The former is known from the spectrum (colour) of the sun and is 5780 ^{o}K. The latter is $4\pi(7 \times 10^8 m)^2 = 6.15 \times 10^{18}$ m^2. Hence, the sun's total energy loss per second is,

$$\text{Luminosity} = \sigma T^4 A = 5.66 \times 10^{-8} \times 5780^4 \times 6.15 \times 10^{18} = \mathbf{3.9 \times 10^{26}\ J/sec}$$

Hence, the Kelvin timescale is,

$$t_{Kelvin} = 3.4 \times 10^{41} \text{ J} / 3.9 \times 10^{26} = 8.7 \times 10^{14} \text{ sec} = 28 \text{ Myrs}$$

The sun is roughly 200 times older than this. After about 28 Myrs the initial 'transient' temperature distribution due to gravitational heating may be assumed to have given way to a steady state distribution determined by nuclear heating.

4. A Silly Chemical Timescale

There are two reasons why the source of the sun's heat cannot be chemical burning. Firstly, there is no significant admixture of elements with which the initial 75% hydrogen and 25% helium mixture can combine. However, even if there were, chemical burning would not be possible. This is simply because the sun's temperature is such that the gases are fully ionised. There are no atoms to chemically combine! Thus, even if the sun comprised two parts hydrogen to one part oxygen, they would be in the form of fully ionised nuclei and no chemical heat source would be provided by them.

Nevertheless, it is interesting to calculate what energy would be available if the sun comprised this optimal ratio of hydrogen and oxygen in di-atomic molecular form (i.e. cold). The calorific value of hydrogen burning in oxygen is 57.8 kcal/mole, and a mole of H_2O is 18g. The number of such moles in the sun's mass is thus 2 x 10^{30} kg / 0.018 kg = 1.1 x 10^{32} moles, so the total calorific energy is 6.4 x 10^{33} kcal = 2.7 x 10^{37} J.

Hence, we see that even with this vast, star-sized, store of molecular gases, the chemical energy available is about four orders of magnitude smaller than the initial gravitational energy. The associated timescale is therefore also four orders of magnitude smaller, i.e. chemical burning would keep the sun going for a mere 2,000 years or so.

It is initially rather surprising that this chemical energy is so much smaller than the gravitational energy. One normally regards gravity as being very weak compared with the electromagnetic forces which underlie chemical bonding. The problem is elucidated by examining the energy per molecule (of H_2O, say). The chemical energy of burning per molecule is 57.8 kcal divided by Avogadro's number (~6 x 10^{23}), i.e. about 4 x 10^{-19} J = 2.5 eV per molecule. The gravitational energy per molecule (with mass about 18 times the proton mass, i.e. ~3 x 10^{-26} kg) is, grav. energy per molecule ~ $\frac{GMm}{R} = 6.67 \times 10^{-11} \times 2 \times 10^{30} \times 3 \times 10^{-26} / 7 \times 10^{8}$

$\sim 5.7 \times 10^{-15}$ J = 36,000 eV.

which confirms that the gravitational energy is roughly four orders of magnitude larger. We can now see the origin of the difference. In the expression for the gravitational energy the entire mass of the star contributes to the energy of each molecule. Whilst the chemical (electromagnetic) energy is determined purely by a one-on-one interaction between two particles (well three – two H atoms and one O, in this case), in the gravitational case every particle comprising the star interacts with the molecule in question and contributes to its total gravitational energy. Thus, it is the scale of the problem, in the sense of the number of particles involved (2.0×10^{57}), which permits gravity to become dominant.

This silly example also serves to illustrate how vastly larger must the true source of the sun's energy be than a typical chemical, i.e. atomic, source. If the gravitational energy falls short by x200, and the chemical energy is four orders of magnitude smaller still, the true energy source must be around 2×10^{6} times larger than the chemical energy, i.e. of the order of 5 MeV per proton.

5. The Nuclear Timescale

In this Section we estimate the timescale for the sun to run out of energy, radiating power as it is, given the supply of fusable hydrogen. For simplicity we consider only the formation of helium-4 from hydrogen. More nuclear energy is available, of course, from the subsequent formation of ever heavier elements (up to iron). The H-to-He timescale is therefore a lower bound.

The binding energy of the helium-4 nucleus is 7.074 MeV per nucleon, i.e. a total binding energy of 28.296 MeV, (and thus compliant with our rough estimate in Section 4, above). Thus each initial proton causes the release of 7.074 MeV of energy when it is eventually captured in a helium-4 nucleus (as either a proton or a neutron). Since the mass of a proton is 938.272 MeV, the total energy released is thus a fraction 7.074 / 938.272 = 0.00754 of the mass of hydrogen converted. For a new star formed from primordial material, 75% of its mass will be hydrogen (see Chapter 6B). Hence, the total energy released by H-to-He fusion if the whole inventory of hydrogen was used up would be,

Total H-to-He fusion energy $= 0.75 \times 0.00754 \times 2 \times 10^{30}$ kg $\times (3 \times 10^{8}$ m/s$)^{2}$

$= 1.0 \times 10^{45}$ J

This is about 3,000 times larger than the initial gravitational energy. Dividing by the luminosity of 3.9×10^{26} J/sec gives a nuclear timescale of 2.6×10^{18} sec = **83 Byrs.**

This nuclear timescale exceeds the age of the universe (by about a factor of 6) and so nuclear burning can easily account for a star's energy source. If the lifetime of a solar mass star is 10 Byrs, and taking account of the additional energy available from subsequent nuclear reactions burning helium, carbon, oxygen, etc all the way to iron, we see that only about 10% of the star's material need take part in nuclear fusion to account for all the energy radiated.

6. The Free-Fall Timescale

If the gas pressure suddenly disappeared, so that the star collapsed freely under gravity to an infinitely dense point, how long would this take? This is left as an exercise for the reader but the answer is,

$$\text{free fall time} \approx \sqrt{\frac{R^3}{GM}} \qquad (4)$$

For the sun this free-fall timescale is ~1600 secs (about half an hour).

7. The Diffusion Timescale

If the sun were transparent it would take just 7 x 10^8 m / 3 x 10^8 m/s = 2.3 seconds for the high-frequency photons from the interior at ~14 million ^{o}K to escape to the surface. It this were the case, we would be bathed in high intensity X-rays here on earth – and the sun would loose energy at a prodigiously faster rate than it does.

The sun is not transparent. Of course it is not – because it consists of very dense fully ionised plasma. Photons do not go very far before they interact with the free electrons. As the photons diffuse outwards they equilibrate thermally with each radial shell. How long does it take a photon to diffuse out from the centre? (We shall conveniently ignore the fact that it is entirely fictional to imagine that the same photon emerges at the surface as set off from the centre). An extremely crude order of magnitude estimate can be made by considering, (a)how many photons there are within the volume of the star, and, (b) how many photons per second are radiated away. Dividing one by the other gives the average time that each photon has to wait within the star before it gets released at the surface.

We can equally perform the calculation in terms of energy rather than numbers of photons (the same thing modulo the average photon energy). Thus,

$$\text{diffusion timescale} = \frac{\int U dV}{4\pi R^2 \sigma T_s^4} = \frac{\frac{4\sigma}{c}\int T^4 dV}{4\pi R^2 \sigma T_s^4} = \frac{4}{cT_s^4}\int \frac{T^4 dV}{4\pi R^2} \qquad (5)$$

where T_s is the temperature at the surface (5780^{o}K for the sun). A crude approximation is to use an ‘average’ temperature of $T_c/2$, where T_c is the temperature at the centre (~14 million ^{o}K). This approximation gives,

$$\text{diffusion timescale} \sim \frac{R}{12c}\left(\frac{T_c}{T_s}\right)^4 = \frac{7x10^8}{12x3x10^8}\left(\frac{14x10^6}{5780}\right)^4 = 6x10^{12}\text{ s} \sim 200{,}000\text{ years}$$

However, this argument seems spurious because $T_c/2$ is a very bad approximation to the effective average temperature. The average temperature from the Clayton model of Chapter 11 (for R/a = 5.4) is far smaller at $0.0306T_c$. Moreover, we are actually interested in the average of the fourth power of T, and this turns out, from the same

Clayton model, to be $0.014\,T_c^4$, which is a factor of 0.22 smaller than $(T_c/2)^4$. Thus, our improved estimate of the diffusion timescale is,

diffusion timescale $\sim \frac{4}{3}.\frac{0.014R}{c}\left(\frac{T_c}{T_s}\right)^4 = \frac{0.0187x7x10^8}{3x10^8}\left(\frac{14x10^6}{5780}\right)^4$

$= 1.5x10^{12}$ s ~ **48,000 years**