# Chapter 11
# Stellar Structure Part 1: Pressure, Density and Temperature Distributions in Spherically Symmetric, Main Sequence Stars – The Clayton Model

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## 1. Introduction

This is the first of a sequence of Chapters considering the structure of stars. Stars are not simple beasts. In their lives they undergo several qualitatively different phases. The formation of a star from a diffuse gas cloud is considerably more problematical to understand than might be supposed. The idea that the gas cloud merely collapses under gravity until the centre is hot and dense enough to initiate nuclear reactions, whilst essentially correct, rather misses the point. Paradoxically, in order to get hot enough by collapse, the gas cloud must have an effective cooling mechanism. Without a cooling mechanism, the gas cloud is supported against collapse by its pressure, or by radiation pressure. If a cooling mechanism is available, the collapse of a very large gas cloud may be subject to instabilities leading to fragmentation into several regions each with their own distinct collapse centre.

Once nuclear burning has initiated, the star goes through a transient period whilst steady conditions are set up, the so-called 'Hayashi track'. Typical stars then spend the bulk of their lives burning hydrogen into helium. Stars burning hydrogen in a steady state are called 'main sequence' stars. Observationally, main sequence stars are identified by the fact that they lie on a characteristic curve on a plot of luminosity against surface temperature (i.e. colour), called a Hertzsprung-Russell diagram. Whilst burning their central stock of hydrogen, stars do not change their position on the H-R diagram very much. Stars lying at different points on the H-R main sequence curve have different mass. Hotter, bluer, stars have greater luminosity and are more massive. When hydrogen becomes exhausted in the centre of a star, things get more complicated.

A star supports itself against gravitational collapse by virtue of the pressure of the (ionised) gas of which it is composed. The pressure is therefore greatest at the centre. Because the star is radiating its heat away into space, if the nuclear reactions ceased its temperature would drop. As a consequence, its pressure would also drop. The result of a star running out of hydrogen fuel in its central core region is, therefore, that it can no longer support itself against gravitational collapse. Consequently gravitational collapse occurs until the central temperature is high enough to trigger the nuclear burning of helium, at about 100 MK. Crudely speaking, a new equilibrium condition can then be set up in which the rate of nuclear heat production balances against the radiant loss. However, some complex physics is going on. For one thing, hydrogen burning has not ceased. In a shell around the core, hydrogen is still being burnt. Hydrogen was only exhausted in the core region. The further gravitational collapse has created temperatures sufficient to burn hydrogen (>10 MK) to arise in regions which were previously too cool – and hence which still contain hydrogen. Between the helium burning core and the hydrogen burning shell there will be a region which is not producing nuclear heating, being without hydrogen but not hot enough to burn helium.

The track of a helium burning star on the H-R diagram is interesting. Such stars leave the main sequence. Unlike main sequence stars, for which ‘luminous’ means ‘blue’, helium burning stars initially become more luminous whilst also becoming redder. Since ‘redder’ means a lower surface temperature, and because this implies a reduced radiant power per unit area, the star can only be more luminous if it is far bigger in size. In short, the star becomes a red giant. During the helium burning period, the star moves in a wide zigzag on the H-R diagram, which we will not attempt to describe further here.

The eventual exhaustion of the core helium fuel leads to further gravitational collapse. The composition of the core will now be predominantly carbon and oxygen. The next steady phase is that of carbon burning in the core, around which there is an annular shell burning helium, and around that another shell burning hydrogen. The hydrogen burning shell feeds helium fuel into the shell below, whilst the helium burning shell in turn feeds carbon fuel into the core. I suppose this could be called a fusion-breeder reactor. Sufficiently massive stars ultimately have five concentric regions burning, from outside inwards: hydrogen, helium, carbon, oxygen and silicon. By this time the core is iron. Because iron has the highest nuclear binding energy per nucleon of all the nuclides, it is the stable end point of the chain of nuclear reactions.

Each phase of burning occurs at an accelerating rate. Whilst hydrogen burning in a solar mass star may last for 10 Byrs, the burning of silicon into iron in massive stars occurs in a timescale of days. It is rather as if the Creator has decided to get a move on. The impression is confirmed by the subsequent, rather crass, display of power. The supernova which occurs in sufficiently massive stars creates all the elements beyond iron in a split second.

How is the above picture of the evolution of stars known? The answer is a combination of calculation and observation. What cannot be observed, of course, is the evolution of a single star as it moves around the H-R diagram. That would be ideal, but unfortunately the human lifespan is too pathetically short. However, the whole population of stars can be used as an indication of the trajectories of individual stars. Their relative sparsity in some regions of the H-R diagram implies relatively brief periods of time spent by individual stars in these regions. As regards calculations, the underlying physics of stellar evolution is generally quite well understood and computer models are sophisticated. Whilst different research groups will differ on the details, the more significant observation is the broad agreement regarding stellar evolution from the start of the main sequence to the supernova progenitor condition.

However, considering stellar evolution in this level of detail is well beyond the scope of these notes. Take, for example, just one issue – that of heat transport. A star must transport heat as fast as it is produced if it is to remain in a steady state. There are two mechanisms of heat transport available: radiation and convection. The effectiveness of the former depends upon the degree of opacity of the ionised gas to the radiation. We shall say more about this in later chapters. If the radiation mechanism is not sufficiently effective, the resulting temperature gradients will drive a convective mechanism. As well as transporting heat, convective mechanisms also transport matter. Thus, regions of convection will mix the stellar composition. The products of nuclear fusion reactions do not, therefore, stay put. They can be transported to outer regions where conditions are too cool for them to have been produced directly. In practice most stars, at most times, have some regions which are radiative and some which are convective. Moreover, the boundaries between these regions changes. At key times, called ‘dredge-ups’, the convective region transiently penetrates deep into the star towards its core. As the name implies, this has the effect of dredging up nuclear fusion products – the chemical elements – into the outer regions of the star. Modelling convection, particularly with rapidly moving convection-radiative boundaries, and hence the resulting changes in the chemical composition of the stellar medium at any position, requires sophisticated computer codes.

To give one further example, in the red giant phase – at least for some stellar masses – stars exhibit thermal instabilities. Thus, we are not really dealing with a steady state but a dynamical condition of oscillating power densities. Once again we see that models capable of reproducing all the main features of stellar evolution are beyond the reach of amateur dabblers. In any case, the attempt to produce a detailed stellar model would obscure the essential features which we wish to emphasise. In this Chapter, and up to Chapter 19, we shall try to bring out the essential physics without undue complexity. We shall concentrate upon main sequence, hydrogen-burning stars. In this chapter we shall start by presenting a simple, if incomplete, model of the pressure, density and temperature distribution within a star.

Rather remarkably, the problem of the pressure, density and temperature distributions within stars in a quasi-static state can be decoupled from the issues of nuclear energy production and heat transport. Or, rather, such decoupling is possible provided that we are able to specify a boundary condition value, such as a central density, which is not determined by the model itself. This would not be necessary, and the complete solution could be found for a star of a given mass, if compatibility with nuclear heat generation and heat transport were imposed. This Chapter considers stellar models using this simplified, but restricted, approach. Chapter 18 will derive a slightly more complete model incorporating the nuclear heating ingredient. We shall not, though, develop a complete model which also addresses heat transport thoroughly. There will be some discussion of this in Chapter 18, i.e. how the complete problem is formulated and simple closed-form solutions (the homologous models) under highly idealised assumptions.

Thus, the problem considered here is a body of ‘gas’ which is in static equilibrium under its own gravity. We will not consider here the formation of the star, i.e. the period of gravitational collapse. We wish to determine the distributions of pressure, density and temperature when the gas is in equilibrium. We shall understand by the term ‘gas’ either a gas of atoms or molecules or a gas of ionised plasma. Henceforth we shall drop the inverted commas, but in most cases the gas in question will be a plasma.

In this Chapter we shall initially derive a non-linear differential equation for the density which arises from just two ingredients: hydrostatic equilibrium and an assumed polytropic equation of state for the ionised gas comprising the star. This non-linear differential equation, the structure equation, will be used in Chapter 18.

However, the simplified approach of this Chapter postulates an analytical form for the pressure distribution and imposes only the requirements of hydrostatic equilibrium (i.e. the polytropic equation of state is not used). The advantage of this approach is that simple expressions for the pressure, density and temperature distributions result, and their origin is clear from the algebraic derivation. The method is reasonably accurate (perhaps within ~15%?) sufficiently near the centre of a star. Its predictions are compared with those from more complete stellar structure models. However, not all the relevant physical constraints are imposed by the model, e.g. the specific equation of state for the ‘gas’, compatibility with nuclear heating rates, and compatibility with heat transport rates. As a consequence, there are some undetermined parameters in this simplified approach which have to be supplied as ‘givens’. In addition, the model becomes poor away from the centre of the star, and cannot be used even approximately near the star’s surface.

## 2. Hydrostatic Equilibrium

Only two forces act within the body of ionised gas envisaged: gravity and pressure. In this Section we consider what distributions of pressure will maintain the gas in static equilibrium against the gravitational forces which tend to cause it to collapse. We consider only the spherically symmetric problem (hence a star which is not rotating – an unrealistic simplification in truth). Call the pressure P(r) at any radius r. Consider an element of gas of radial thickness $\delta r$ and subtending a solid angle $\delta\Omega$ about the centre. Considering firstly only the pressure forces acting on the spherical surfaces of this element, the net <u>outward</u> force is,

$$\delta F^{spherical}_{pressure} = P(r)r^2\delta\Omega - P(r+\delta r)(r+\delta r)^2\delta\Omega = -\left(r^2\frac{dP}{dr} + 2\,Pr\right)\delta r\delta\Omega \qquad (1)$$

The net radial force due to pressure acting on the ‘sides’ of the element (i.e. the faces for which one of the polar angles $\theta$ or $\phi$ is constant) can easily be deduced. Imagine gravity being switched off, so that the pressure was also uniform. The dP/dr term in (1) is then zero, and, since there is no gravitational force to balance, the ‘sides’ must contribute a force which cancels with the term -2Pr in (1). Hence, (turning gravity on again) the net pressure force acting outwards is,

$$\delta F_{pressure} = -\left(r^2\frac{dP}{dr}\right)\delta r\delta\Omega \qquad (2)$$

The gravitational force acting <u>inwards</u> on the same element of gas is,

$$\delta F_{gravity} = \frac{Gm(r)}{r^2}\rho(r)r^2\delta r\delta\Omega \qquad (3)$$

The second factor in (3) is simply the mass of the element of gas, where $\rho(r)$ is its density at radius r. The term m(r) means the mass of gas within radius r, i.e.,

$$m(r) = \int_0^r \rho(r')4\pi r'^2 dr' \qquad (4)$$

The equation for hydrostatic equilibrium follows from equating (2) and (3), i.e.,

$$Gm(r)\rho(r) = -r^2 \frac{dP}{dr} \tag{5}$$

It follows from (5) that the pressure must fall monotonically away from the centre (since the LHS is positive) – which is not surprising. Equ.(5) may also be written in a form which eliminates m(r). Firstly we differentiate (5), and then use (4) to substitute for dm/dr, giving,

$$G\frac{dm}{dr} = -\frac{d}{dr}\left(\frac{r^2}{\rho}.\frac{dP}{dr}\right) = 4\pi r^2 G\rho \tag{6}$$

Another form of the hydrostatic equation which can be useful is,

$$Gmdm = -\frac{dP}{dr}\frac{r^2}{G\rho}.4\pi r^2 G\rho dr = -4\pi r^4 dP \tag{7}$$

This form is useful because the LHS is an exact differential and hence can be integrated explicitly to give,

$$Gm(r)^2 = -8\pi\int_o^r r'^4 dP \tag{8}$$

## 3. The Structure Equation

In Section 2, the equations derived for hydrostatic equilibrium, i..e. Equs.(5-8), apply whatever the source of the pressure. In general this will be a combination of gas pressure and radiation pressure. In this Section we shall assume that the gas pressure dominates.

The hydrostatic equation [say, Equ.(6)] contains two independent functions of radius: the density and the pressure. To achieve a soluble system we therefore require an additional equation. This is provided by the equation of state of the ionised gas. In some circumstances this can be approximated as a polytropic equation of the form,

$$P = K\rho^\gamma \tag{9}$$

where K and $\gamma$ are constants, i.e. the same at all positions in the star. This equation of state is analogous to the adiabatic equation for perfect gases. We may expect $\gamma$ to be something like the ratio of specifics heats and hence lie between 4/3 or 5/3. It turns out that stability does indeed require $4/3 < \gamma \le 5/3$ though we will not justify this here. However, the coefficients in the polytropic equation are not rigorously uniform throughout the star. Hence, the use of Equ.(9) is a convenient approximation only.

Differentiating (9) and substituting the result into (6) gives,

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{\gamma K r^2}{\rho^{2-\gamma}}.\frac{d\rho}{dr}\right) = -4\pi\rho G \tag{10}$$

This second order DE in density provides the basis for our simplified stellar structure determination. It would be linear only if $\gamma = 2$, which is not possible. Actually $\gamma < 2$ and the equation is non-linear. The boundary conditions are,

1) $\rho(r \to \infty) \to 0$ (condition for a finite sized star) (10a)

2) $\left.\frac{d\rho}{dr}\right|_{r=0} = 0$ (10b)

The second b.c. follows from (9) and our observation, following (5), that the pressure must be maximum at the centre. We have,

$$m(r \to 0) \to \frac{4}{3}\pi r^3 \rho_c \tag{11}$$

$$\left.\frac{dP}{dr}\right|_{r\to 0} \to -\frac{4}{3}\pi r G \rho_c^2 \tag{12}$$

Thus, if we knew the polytropic parameters K and $\gamma$, Equ.(10) could be solved (numerically) to find the density distribution for the boundary conditions (10a) and (10b), together with some assumed central density, $\rho_c$. From the density, the pressure distribution is then found from the polytropic equation of state, Equ.(9). Finally, the temperature would be found from the ideal gas law,

$$kT = \frac{\langle \text{mass} \rangle P}{\rho} \tag{13}$$

where <mass> is the average mass per particle (including nuclei, ions and electrons).

In the above description, the only things which distinguish one star from another are the values of the polytropic constants, K and $\gamma$, and the assumed central density, $\rho_c$. Thus, we may expect that K depends on the star’s total mass, for example. Thus, K is not only a property of the gas. The value of K is therefore an undetermined parameter which must be deduced for the particular case.

For the rest of this Chapter, we drop the polytropic equation and consider only hydrostatic equilibrium.

## 4. Clayton Model

The Clayton model consists of assuming a specific analytic form for the pressure distribution, P(r), which respects the requirement of hydrostatic equilibrium, and which is also consistent with virtually the whole mass, M, of the star being contained within a specified radius, R. The latter means that,

$$m(R) \approx M \tag{14}$$

to a very good approximation. Using this in (5) gives,

$$\left.\frac{dP}{dr}\right|_{r=R} \approx -\frac{GM\rho(R)}{R^2} \qquad (15)$$

The form chosen by Clayton was,

$$\frac{dP}{dr} = -\frac{4\pi}{3} G\rho_c^2 r e^{-r^2/a^2} \qquad (16)$$

where ‘a’ is an initially unknown length scale. Equ.(16) is consistent with Equ.(12) as $r \to 0$. Equating (15) and (16) at r = R gives,

$$\rho(R) = \frac{4\pi}{3} . \frac{R^3 \rho_c^2 e^{-R^2/a^2}}{M} \qquad (17)$$

The significant feature of (17) is that, if the length scale parameter, a, is small compared with R (say, if a < R/5), then the gas density at the surface of the star is vanishingly small. This is, of course, what is required physically since the surface of a star interfaces with interstellar space, and hence is a virtually perfect vacuum (at least by terrestrial standards).

We note that the analytic form (16) is convenient because it may be integrated explicitly to give the pressure,

$$P(x) = \frac{2\pi}{3} G a^2 \rho_c^2 e^{-x^2}, \qquad \text{where} \quad x = \frac{r}{a} \qquad (18)$$

and where we have appealed to the boundary condition that pressure falls to zero at infinity. (This differs from the original Clayton model which imposed zero pressure at the surface, r = R, of the star).

So far we have noted only that the Clayton form, Equ.(16), has sensible boundary behaviour. We now also impose hydrostatic equilibrium. The form Equ.(7) is convenient. Integrating it and inserting (16) gives,

$$G\frac{m^2}{2} = -4\pi \int r^4 \left(\frac{-4\pi}{3}\right) G\rho_c^2 r e^{-r^2/a^2} dr$$

and hence,

$$m = \frac{4\pi}{3}\rho_c a^3 \Phi, \qquad \text{where} \quad \Phi^2 = 6\int x^5 e^{-x^2} dx = 6 - 3(x^4 + 2x^2 + 2)e^{-x^2} \quad (19)$$

where we have imposed the boundary condition m(0) = 0, i.e. that the density at the centre is finite (i.e. Equ.11). The density distribution may now be found from m(x) using the definition of m, Equ.(4), which gives,

$$\rho(x) = \frac{\rho_c x^3 e^{-x^2}}{\Phi} \qquad (20)$$

where we have used $\frac{d\Phi}{dx} = \frac{1}{2\Phi}.\frac{d}{dx}\Phi^2 = \frac{6x^5 e^{-x^2}}{2\Phi}$. Noting that we also need to obey the boundary condition Equ.(17), we equate (2) at r = R with (17) which simplifies to an expression for the length scale, a, as follows,

$$a = \left( \frac{\sqrt{3}}{4\sqrt{2}\pi} . \frac{M}{\rho_c} \right)^{1/3} \qquad (21)$$

where we have assumed that R is sufficiently larger than a (e.g. R > 3a) so that $\Phi \approx \sqrt{6}$. The same expression for the length scale ‘a’ results from imposing the condition Equ.(14) on the explicit expression Equ.(19) for m, i.e. from requiring that the whole of the star’s mass is contained within r = R to a very good approximation.

In summary, the Clayton model of stellar structure gives the pressure distribution Equ.(18) and the density distribution Equ.(20), where the parameter ‘a’ is given by Equ.(21) and the function Φ is given by Equ.(19). These distributions of pressure and density contain just two undetermined parameters, the total mass M of the star and the density $\rho_c$ at its centre. Given the mass and the density at the centre, the pressure and density are determined everywhere.

It is useful to note that, from (18) and (21), the pressure at the centre of the star is given by,

$$P_c = \left( \frac{\pi}{36} \right)^{1/3} G M^{2/3} \rho_c^{4/3} \qquad (22)$$

Hence, we see that, at the centre of the star, the Clayton model would be consistent with a polytropic equation of state, Equ.(9), with $\gamma = 4/3$ and $K = 0.444GM^{2/3}$. As surmised above, the coefficient K is mass dependent. [Note, however, that assuming an $\gamma = 4/3$ polytropic relation held everywhere would actually result in a central density about 18% lower than that given by Eq.(22)].

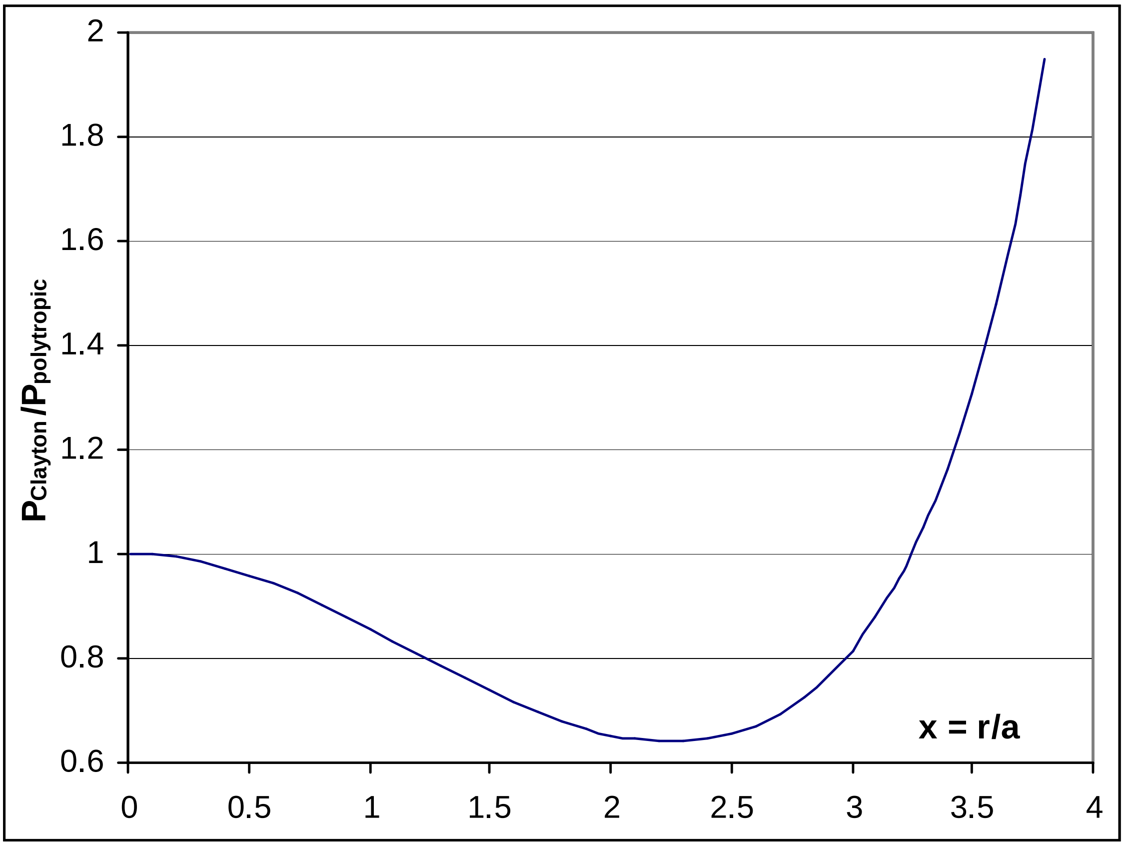
However, at other locations the pressure is not given in terms of the density by a power-law like Equ.(9), i.e. Equs.(18) and (20) are not related by a power law. In as far as Equ.(9) is a good approximation to the equation of state, this exposes the approximation inherent in the Clayton models. It is of interest to see how far adrift the Clayton model is from the polytropic state. To do this we insert the density given by Equ.(20) into the following polytropic expression for the pressure,

$$P_{polytropic} = \left( \frac{\pi}{36} \right)^{1/3} G M^{2/3} \rho^{4/3} \qquad (23)$$

and compare the result with Equ.(18), the Clayton expression for pressure. The two agree at the centre, as shown by Equ.(22). In general their ratio is,

$$\frac{P_{Clayton}}{P_{polytropic}} = \left(\frac{\rho_c}{\rho}\right)^{4/3} e^{-x^2} = \frac{\Phi^{4/3} e^{+x^2/3}}{x^4} \qquad (24)$$

This is plotted against x below,

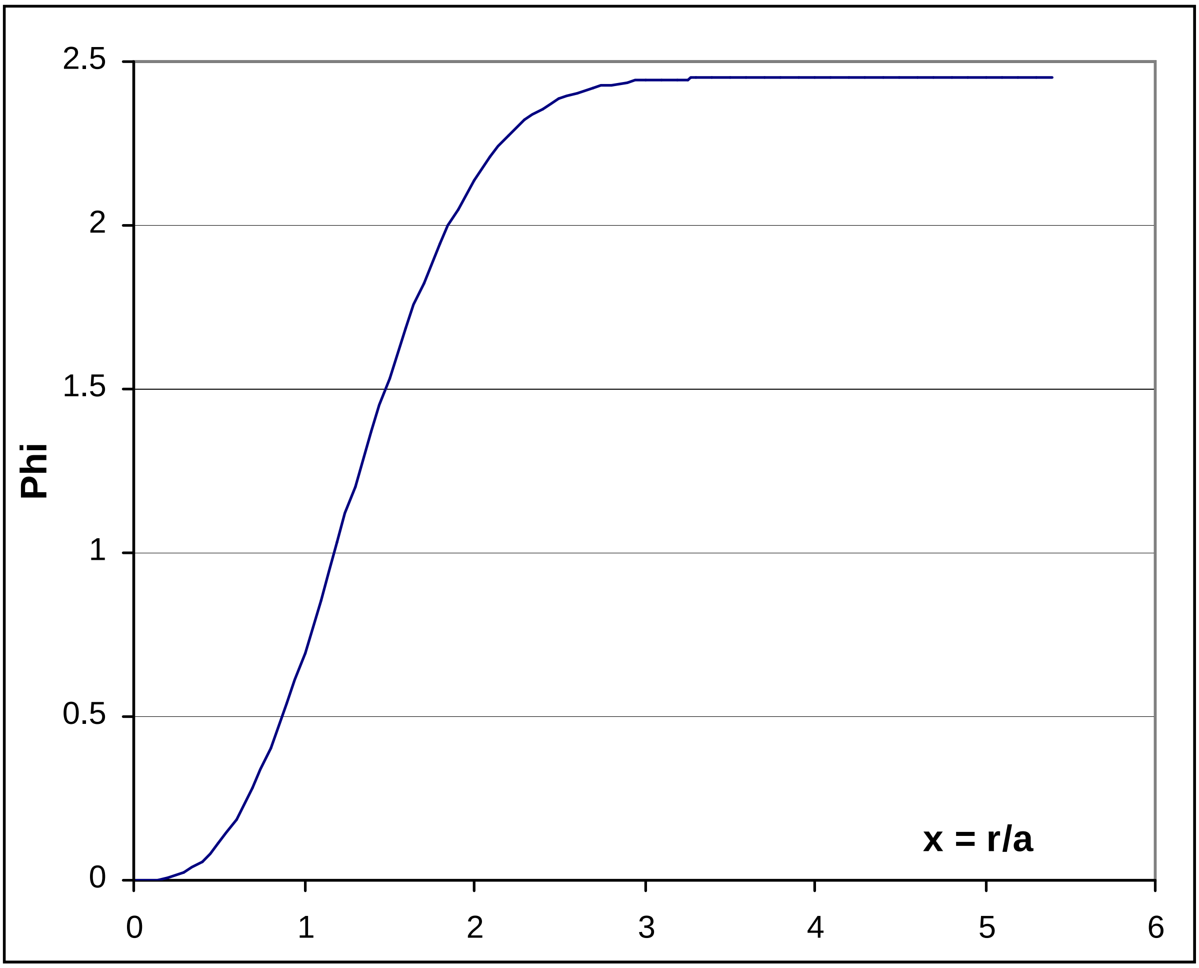



This would seem to imply that the Clayton model pressure might be quite seriously in error except near the centre. For x <0.5 the error appears to be <4%. However, the Clayton model pressure is not even indicative of the order of magnitude at the surface of a star. The star surface may be at x values approaching ~6, for which the above ratio exceeds ~100. Having said this, the polytropic assumption is not strictly correct either.

Example of the Clayton Model – The Sun:
We note from Equ.(19) that, at sufficiently large 'x', $\Phi^2$ tends to 6. Hence the whole mass of the star is given by $M = \sqrt{6}(4\pi/3)a^3\rho_c$. Thus we may re-write (19) simply as $m(x) = \Phi(x)M/\sqrt{6}$. Hence $\Phi/\sqrt{6}$ is the fraction of the total mass within the radius in question.

Since Equ.(19) gives $\Phi(1) = 0.694$ we can interpret the length scale 'a' as the radius within which the fraction $0.694/\sqrt{6}$ = 28.3% of the star's total mass resides.

To apply the Clayton model to the sun we first choose a value of R/a which we are content to treat as the surface:-




From this graph, $\Phi$ appears to have virtually reached its asymptote ($\sqrt{6}$) by x ~ 3.4. Closer inspections reveals that the deviation of $\Phi/\sqrt{6}$, the mass fraction, from unity is as follows,

| | | | | | | |
|---|---|---|---|---|---|---|
| x = | 3.4 | 4.1 | 4.5 | 4.6 | 4.9 | 5.4 |
| error in decimal place | 3 | 5 | 7 | 8 | 9 | 10 |

Consequently it's a little difficult to be precise as regards the choice of x to associate with the star's surface. For illustration we shall use both R/a = 3.4 and R/a = 5.4, corresponding to 99.9% of the mass and all but a fraction $10^{-10}$ of the mass respectively. (The 'surface' is best defined as that region of finite thickness such that the thickness equals the mean free path of a photon – i.e. the photosphere).

To determine the two degrees of freedom we impose the actual values of the mass and size of the sun, i.e. 2 x $10^{30}$ kg and 6.96 x $10^{8}$ m respectively. Hence the two values for the 'a' parameter are:-

1) R/a = 3.4: a = 2.05 x $10^{8}$ m $\rho_c$ = 22,700 kg/m$^3$ $P_c$ = 3.0 x $10^{15}$ Pa
2) R/a = 5.4: a = 1.29 x $10^{8}$ m $\rho_c$ = 91,000 kg/m$^3$ $P_c$ = 1.9 x $10^{16}$ Pa

where the central density from Equ.(21) has also been given for these values of a and M. The larger of these densities is 11.6 times the density of terrestrial steel. Standard solar models give the central density as ~9 x $10^4$ kg/$m^3$, so the Clayton model with R/a = 5.4 is the best choice.

The central pressure from Equ.(22) is also given above (using G = 6.67 x $10^{-11}$ in MKSA units). Since 1 atm = $10^5$ Pa, the larger of the above pressures is a huge 1.9 x $10^{11}$ atmospheres pressure. Standard solar models give the central pressure as 1.65 x $10^{16}$ Pa, so the Clayton model with R/a ~ 5.4 is reasonably close.

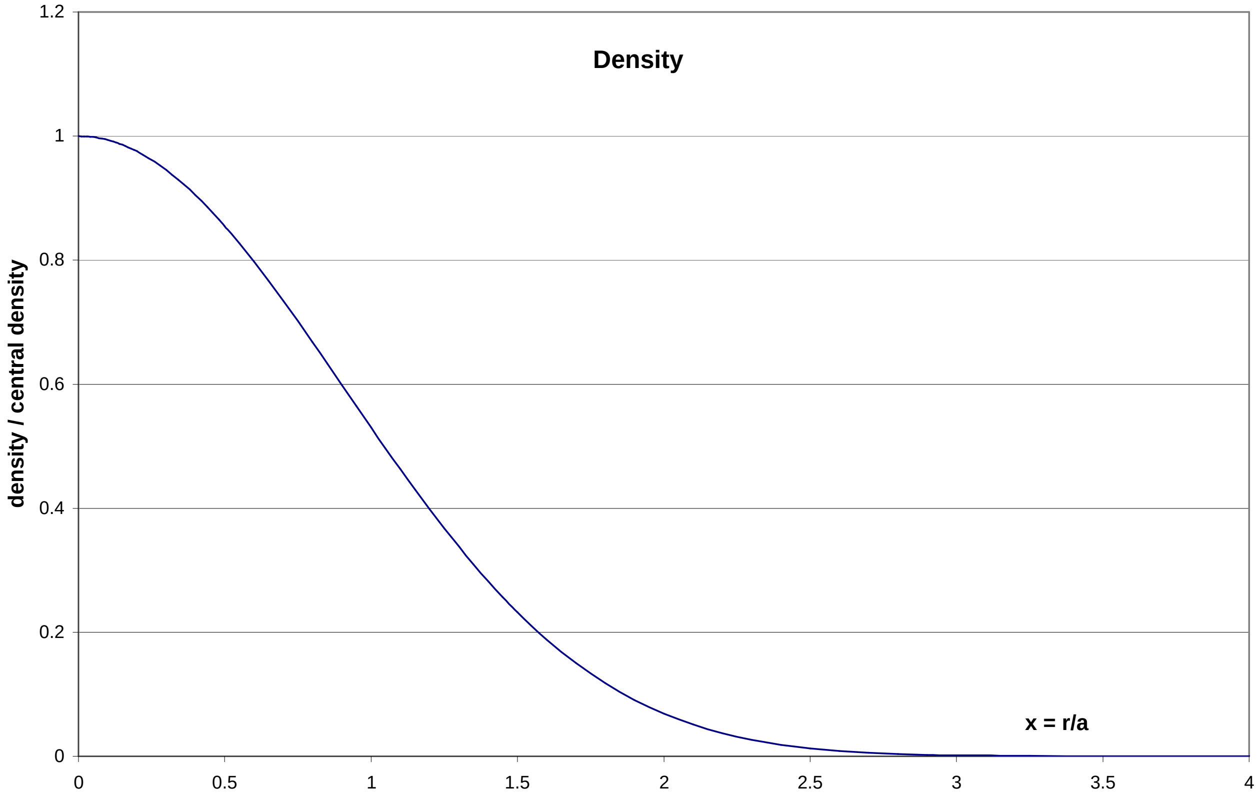
To derive the temperature from Equ.(13) we need the average mass of the particles which give rise to the pressure. We shall assume that the contributing particles are protons, electrons and helium-4 nuclei. Other nuclei total only a small fraction. We assume that neutrinos, which will also be present in large numbers due to the nuclear reactions, interact sufficiently weakly to make negligible contribution to the pressure (or the temperature of other species). Helium-4 is ~25% of the nuclei by mass. (This assumes a newly formed star with a composition equal to that after the Big Bang). Since helium is 4 times heavier than a proton, this means that for every 75 protons there are 6.25 helium nuclei. There are also 75 electrons to match the protons, plus 12.5 electrons to match the helium nuclei. Hence, the average mass can be evaluated from,

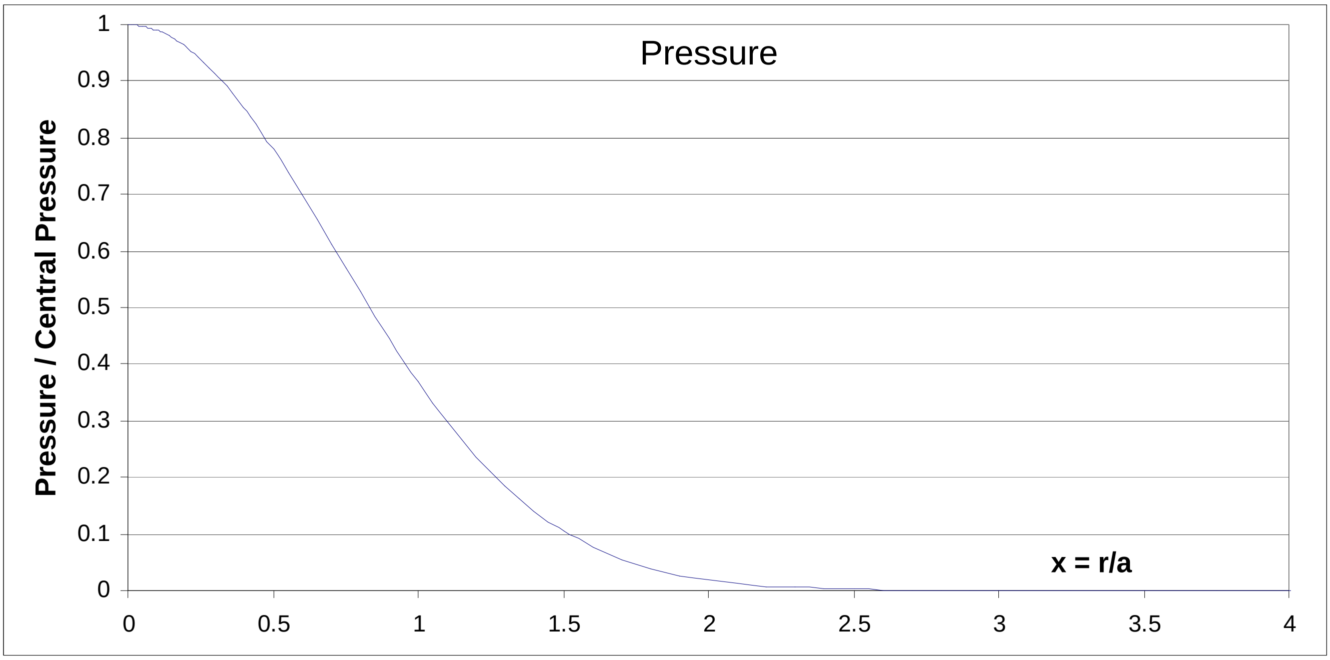
| **species** | **relative number** | **percent number** | **mass(MeV)** | **contribn.** |
|---|---|---|---|---|
| protons | 75 | 44.4% | 939 | 417 |
| helium-4 | 6.25 | 3.7% | 4 x 939 | 139 |
| electrons | 87.5 | 51.9% | 0.511 | 0.265 |
| Total | | 100% | | 556.265 |

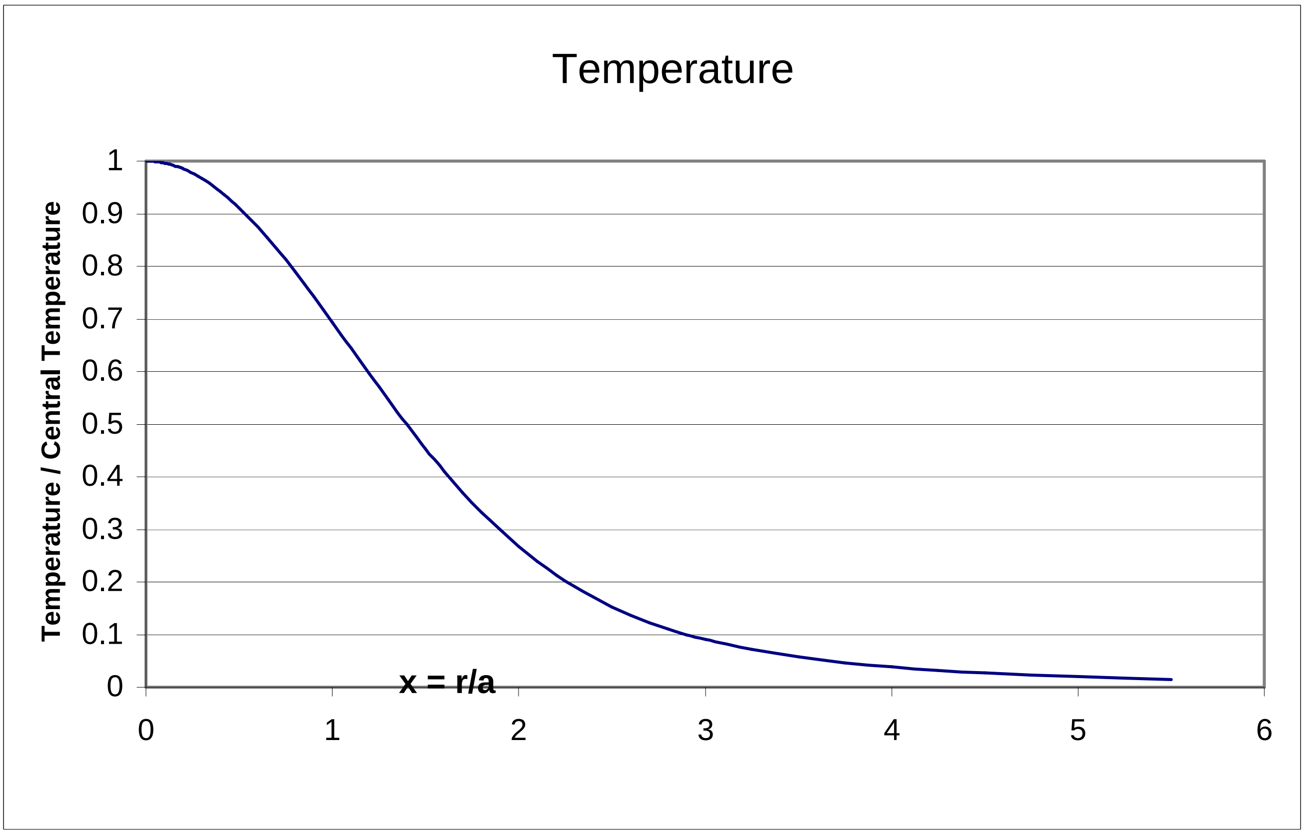
Hence average mass = 556MeV = 0.592 x proton mass = 0.592 x 1.67 x $10^{-27}$ kg = 9.9 x $10^{-28}$ kg. Use of this mass in Equ.(13) together with the above results for the pressure and density at the centre of the sun give estimates for the temperature at the centre of the sun of 9.5 x $10^6$ $^oK$ (for R/a = 3.4) and 15 x $10^6$ $^oK$ (for R/a = 5.4). Standard solar models give the central temperature as 13.7 x $10^6$ $^oK$, so the Clayton model with R/a ~ 5.4 is again quite close. The R/a = 3.4 model is poor since the predicted temperature is too low for the required nuclear reaction rates.

We have seen above that the Clayton model is expected to be seriously in error for values of ‘x’ greater than unity, and certainly grossly wrong at the surface of the star. Nevertheless, it is of interest to see what numerical values are predicted at the surface. For this we use R/a = 5.4, which gives the density, pressure and temperature at the surface to be 1.3 x $10^{-6}$ kg/$m^3$, 4.1 x $10^3$ Pa (=0.041 atm) and 233,000$^o$K respectively. The density and pressure are at least small - though whether they are small enough is not obvious without comparison with the correct result. The density is equivalent to about 7 x $10^{20}$ protons per $m^3$, and a pressure of 0.041atm is a very poor vacuum. The temperature is certainly wrong by a large factor, the correct value being about 6,000$^o$K. Hence, for reasonable predictions near the surface of the star we need to solve the full structure equation, (10), for which a polytopic equation of state is required. This is the subject of Chapter 18. Accurate values of the surface temperature are best obtained by balancing the luminosity against the rate of nuclear heat production in the centre of the star. This will also be dealt with in Chapter 18.

The following graphs plot the density, pressure and temperature distributions from the Clayton model, normalised by the value in the centre (and we must bear in mind that the model is inaccurate for $x > 1$).






These curves have been generated from the expressions,

$$\rho = \rho_c \frac{x^3 e^{-x^2}}{\Phi(x)}, \quad P = P_c e^{-x^2}, \quad T = T_c \frac{\Phi(x)}{x^3} \qquad (25)$$

where $\Phi(x)$ is given by Equ.(19). Thus, we see that the density and pressure fall off in a manner that looks qualitatively like a Gaussian, being dominated by the $e^{-x^2}$ term. The temperature, however, falls off much more slowly since the exponential term cancels in the ratio of pressure and density which enters the expression for temperature.

## 5. Conclusion

We have derived a non-linear differential equation in density which embodies the requirement for hydrostatic equilibrium, Equ.(10), for a gas obeying a polytropic equation of state. We have deferred until Chapter 18 a full numerical solution of this equation. In this Chapter we have explored a simple analytical approximation to this solution, the Clayton model, which is accurate near the centre of a star but not at its surface. The value of this model is that it gives simple closed-form expressions for the density, pressure and temperature distributions which have the correct qualitative form. In particular, they fall off rapidly away from the centre, becoming very small fractions of their central values at just a fraction of the star's radius. Because the model is based on hydrostatic equilibrium alone, it shows that this qualitative feature of the pressure, density and temperature distributions is expected for any stars under any conditions, i.e. P, T and $\rho$ will always reduced rapidly away from the centre.