Estimation Formulae for Creep Fracture Parameter C(t) and Creep Crack Growth under Primary and Secondary Creep Including Plasticity and Mixed Primary plus Secondary Loadings

Last Update: Circa 2000 (apart from note in red below, added March 2008)

These Notes predate the period when C(t) estimation formulae of R5V4/5-type became the standard in assessments. This occurred around the time of Dean & Ainsworth, 2003, which facilitated the adoption of this type of C(t) estimation. In the R5V4/5, or Dean & Ainsworth, approach the secondary stresses enter through an effective pseudo-reference stress. In principle this provides a more accurate estimation because relaxation of the secondary stresses can be accounted for, in addition to redistribution. Cases in which secondary stresses dominate are common, and hence it is highly desirable that assessment procedures have the facility to account for relaxation. However, the accuracy of the R5V4/5 method will depend upon the appropriateness of the choice of pseudo-reference stress formulation. This has been a source of contention, especially when there are large out-of-plane secondary stresses in the application. The situation is not helped by the absence of any definition of the pseudo-reference stress. The accuracy of the pseudo-reference stress formulation adopted can only be judged by the accuracy of the resulting C(t) – which is generally unknown. In practice, guidance has been provided based on a very few specimen geometries which have been analysed with finite elements. It is possible that part of the benefit of the R5V4/5 approach is being lost through the need to adopt rather conservative pseudo-reference stress formulations due to the absence of more accurate, general guidance.

The advantage of the more old-fashioned, R5V7, approach is that no pseudo-reference stress is required. This is because this approach does not address relaxation, but only stress redistribution. It is, perhaps, unlikely that the R5V7 approach will regain widespread application to austenitic materials. However, it is worth bearing in mind since it is far easier to use than the R5V4/5 methodology. Moreover, since it ignores relaxation, it could be argued to be an upper bound to C(t). Thus, in cases where the R5V4/5 approach is suspected of being overly conservative due to a pessimistic pseudo-reference stress formulation, the R5V7 approach may provide alleviation. This Note considers only the R5V7 type of estimation.

1. INTRODUCTION

The derivations of various estimation formulae for C(t) are spread amongst a number of different reports and papers (eg.Refs.1,2). It is desirable to bring them together in order to clarify their regimes of applicability. Most derivations are explicitly, or implicitly, dependent upon the assumption of a simple Norton power law (secondary) creep behaviour. It is not clear whether the resulting estimation formulae are valid for more general creep constitutive laws, in particular those including primary creep behaviour. The objectives of this Note are;

i) to derive estimation formulae applicable to primary and secondary creep behaviour, and including plastic strain effects, and,

ii) to combine the effects of plasticity with that of mixed primary plus secondary loading (not addressed previously), and,

iii) to clarify the regime of applicability of the formulae.

Limitations are that;

- The crack tip fields used relate to stationary cracks, and hence the formulae derived are applicable only for sufficiently slowly growing cracks. This restriction may be assessed using the R5 λ–criterion (Ref.3).
- The crack is assumed to be present at 'creep time zero', ie. no creep strains have accumulated prior to the time at which the crack is postulated. This corresponds to an original sin defect, rather than a service initiated defect.
- The plasticity and creep indices, m and n, are assumed the same.

2. CREEP CONSTITUTIVE LAW

We assume the following creep strain rate behaviour,

$$\dot{\varepsilon}^c = B(t)\sigma^n \qquad \text{(uniaxial)} \qquad \text{(Equ.1a)}$$

$$\dot{\varepsilon}^c_{ij} = \frac{3}{2}B(t)\bar{\sigma}^{n-1}S_{ij} \qquad \text{(multiaxial)} \qquad \text{(Equ.1b)}$$

where S_{ij} are the deviatoric stress components and $\bar{\sigma}$ is the Mises equivalent stress. Throughout this Note the superscripts c, e, p and ep will refer to 'creep', 'elastic', 'plastic' and 'elastic-plus-plastic' strains.

Equs.1 reduce to the usual description of secondary creep when B(t) is a constant, and to the most common description of primary creep when,

$$B(t) = pB_o t^{p-1} \quad \text{, where } p < 1 \qquad \textit{(primary creep)} \qquad \text{(Equ.2)}$$

Equ.2 gives divergent strain rates at t = 0, and creep strains which increase as $\varepsilon_c = B_o t^p \sigma^n$. However most of the derivations apply for arbitrary B(t).

3. DEFINITIONS OF J(t) AND C(t)

The fracture parameter J(t) is defined through a contour integral around some contour Γ which encloses the crack tip and starts and finishes on opposite faces of the crack,

$$J(t) = \int_{\Gamma} \left\{ W_{epc}dy - \sigma_{ij}\frac{du_i}{dx}n_j ds \right\} \qquad \text{(Equ.3)}$$

where,

$$W_{epc} = \int \sigma_{ij} d\varepsilon_{ij} \qquad \textit{(summation over repeated indices)} \qquad \text{(Equ.4)}$$

and the total strain, ε_{ij}, is the sum of the elastic, plastic and creep parts,

$$\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij} + \varepsilon^c_{ij} \qquad \text{(Equ.5)}$$

For example, if a power-law plastic behaviour were assumed,

$$\varepsilon_{ij}^{p} = \frac{3}{2} D\bar{\sigma}^{m-1} S_{ij} \qquad \text{(Equ.6)}$$

then,

$$W_{epc} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}^{e} + \frac{m}{1+m}\sigma_{ij}\varepsilon_{ij}^{p} + W_c \qquad \text{(Equ.7)}$$

and W_c is the creep term,

$$W_c = \int \sigma_{ij} d\varepsilon_{ij}^{c} \qquad \textit{(constant time)} \qquad \text{(Equ.8)}$$

This creep “energy-density” integral is carried out at constant time (i.e. constant B(t)). Creep deformation described by Equs.1 is equivalent under time hardening to creep strains,

$$\varepsilon^{c} = \tilde{B}(t)\sigma^{n} \qquad \text{(uniaxial)} \qquad \text{(Equ.9a)}$$

$$\varepsilon_{ij}^{c} = \frac{3}{2}\tilde{B}(t)\bar{\sigma}^{n-1} S_{ij} \qquad \text{(multiaxial)} \qquad \text{(Equ.9b)}$$

where $\tilde{B}(t)$ is the time-integral of B(t). Substituting into Equ.8 these give,

$$W_c = \frac{n}{1+n}\sigma_{ij}\varepsilon_{ij}^{c} \qquad \text{(Equ.10)}$$

since $\tilde{B}(t)$ can be factored out of the constant time integral. Hence Equ.7 becomes,

$$W_{epc} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}^{e} + \frac{m}{1+m}\sigma_{ij}\varepsilon_{ij}^{p} + \frac{n}{1+n}\sigma_{ij}\varepsilon_{ij}^{c} \qquad \text{(Equ.7b)}$$

We note that Equ.3 for J(t) defines a parameter which depends upon the elastic, plastic and creep strains at time t, and upon the (redistributed) stresses at time t. The usual PYFM fracture parameter, J, which depends only upon elastic-plastic strains, is obtained at time zero, ie. J = J(0).

Contour integrals of the form Equ.3 are rigorously path independent for non-linear elasticity. An incremental elastic-plastic-creep material which undergoes no stress reversals is identical to a non-linear elastic material, and hence also has a rigorously path independent J(t). This will be true for elastic-plastic materials undergoing monotonic loading. Hence J(0) is path independent for monotonic loading. However, the introduction of creep will lead to stress reduction (unloading) near the crack tip, and hence path independence of J(t) is not rigorous. The literature suggests that approximate path independence of Equ.3 is expected for constant loads. In particular, path independence of J(t) will recur for sufficiently large times, $t \to \infty$, if the material then behaves under secondary creep, since stresses will become constant.

Unlike J(t), the contour integral definition of C(t) applies only for sufficiently small contours, and will not be path independent in the general case unless steady state creep conditions have been attained. The definition is,

$$C(t) = \mathrm{LIM}\,\Gamma \to 0\left[\int_{\Gamma}\left\{\tilde{W}_c dy - \sigma_{ij}\frac{d\dot{u}_i}{dx}n_j ds\right\}\right] \qquad \text{(Equ.11)}$$

where $\dot{u}$ represents displacement rates, just as $\dot{\varepsilon}$ represents strain rates, and $\tilde{W}_c$ is defined analogously to Equ.8 but in terms of strain rates, i.e.,

$$\tilde{W}_c = \int \sigma_{ij} d\dot{\varepsilon}^c_{ij} \qquad \textit{(constant time)} \qquad \text{(Equ.12)}$$

and hence, for constitutive Equs.1,

$$\tilde{W}_c = \frac{n}{1+n}\sigma_{ij}\dot{\varepsilon}^c_{ij} \qquad \text{(Equ.13)}$$

Thus, for creep behaviour given by Equs.1, C(t) is given by,

$$C(t) = \mathrm{LIM}\,\Gamma \to 0\left[\int_{\Gamma}\left\{\frac{n}{1+n}\sigma_{ij}\dot{\varepsilon}^c_{ij}dy - \sigma_{ij}\frac{d\dot{u}_i}{dx}n_j ds\right\}\right] \qquad \text{(Equ.11b)}$$

We can define a quantity C* as the long-time limit of C(t),

$$C^* = \mathrm{LIM}\; t \to \infty\{C(t)\} = \mathrm{LIM}\; t \to \infty\left\{\frac{dJ}{dt}\right\} \qquad \text{(Equ.14)}$$

This limit exists only if the long-time limit of the creep law is secondary (ie. constant strain rates, $B(t) \to B(\infty)$ = constant) since stresses are then also constant in the long-time limit. In this case, C* is time independent because Equ.11 depends only upon strain rates, displacement rates and stresses which all become constant in the long-time limit. If B(t) does not become constant, then C* is not defined. Hence C* is not defined if primary creep continues indefinitely or if the long term behaviour is a form of tertiary creep which appears explicitly in the creep constitutive relation.

Where the long-time limits exist, C(t) becomes the same as dJ/dt. This is because, in steady creep, the only terms in Equ.3 which are time dependent are the creep strain and displacement, which become strain/displacement rates upon differentiation. Hence the derivative of Equ.3 (using Equ.7b) is just Equ.11b. Prior to steady creep, C(t) does not equal dJ/dt since the latter contains terms in stress-rates.

It is important for later purposes to draw a distinction between C* as defined through Equ.14, which must be time independent if it exists at all, and the R5 reference stress based quantity, which we will denote C^*_{ref}. The latter can be evaluated at any time and retains time dependence by virtue of the time dependence of the strain rate (primary creep) and the potential time dependence of the reference stress if the crack is growing.

4. CRACK TIP FIELDS

For the elastic-plastic case with no creep (ie. at time zero), and when the plastic zone around the crack tip is sufficiently small to be encompassed within a larger region in which the fields are of LEFM form, the elastic-plastic crack tip fields are uniquely determined by the LEFM boundary condition. They are therefore uniquely determined by K and otherwise independent of geometry. In the case of power law hardening materials (ie. for Equ.6) these crack tip fields are known as the HRR fields and are given by,

$$\sigma_{ij} = \left[\frac{J(0)}{DI_m r}\right]^{\frac{1}{m+1}} \hat{\sigma}_{ij}(\theta, m) \qquad \text{(Equ.15)}$$

$$\varepsilon^{ep}_{ij} \sim \varepsilon^{p}_{ij} = D\left[\frac{J(0)}{DI_m r}\right]^{\frac{m}{m+1}} \hat{\varepsilon}_{ij}(\theta, m) \qquad \text{(Equ.16)}$$

where r, θ are polar coordinates around the crack tip and $I_m, \hat{\sigma}_{ij}, \hat{\varepsilon}_{ij}$ are dimensionless parameters dependent on the hardening index, m. These parameters have been tabulated by Fong Shih and others (eg. Ref.4).

The utility of the parameter J in PYFM lies in the fact that, in many cases, the crack tip fields are given accurately by Equs.15 and 16 even well beyond the point of general yield. In such cases, the parameter J uniquely characterises the crack-tip fields and therefore must control any fracture process which depends only upon the near-tip fields.

Laboratory toughness specimens are said to exhibit ‘J-dominance’, or simply to be ‘valid’, if their crack tip fields are given by Equs.15, 16 with J defined by Equ.3 (but without creep). The more constrained the specimen, the more likely it is to be valid. Other things being equal, the larger the specimen the more likely it is to be valid. Hence, the J-dominance of the crack tip fields in large structures is ensured by that of smaller specimens for a similar type of loading and crack configuration.

Equs.15, 16 relate to the fields sufficiently close to the crack tip ($r \to 0$). The elastic strains for sufficiently small ‘r’ are negligible compared with the plastic strains and hence we have written $\varepsilon^{ep}_{ij} \sim \varepsilon^{p}_{ij}$ in Equ.16. Because J(0) is path independent we can choose a very small contour, Γ, in Equ.3, so that the elastic strains in Equ.7b are negligible. At time zero there are no creep strains, so only the plastic strains remain in Equ.7b. Hence, Equ.3 implies,

$$J(0) = \int_{\text{small}\,\Gamma} \left\{ \frac{m}{1+m} \sigma_{ij} \varepsilon_{ij}^{p} dy - \sigma_{ij} \frac{du_i}{dx} n_j ds \right\} \qquad \text{(Equ.17)}$$

We now wish to find expressions for the near-tip fields when creep is occurring. Comparing Equ.17 for J(0) with Equ.11b for C(t) shows them to be formally the same except that the plastic strains (displacements) are replaced by creep strain rates (displacement rates) and the plastic hardening index 'm' is replaced by the creep index 'n'. This suggests that the near-tip stress and strain-rate fields can be written, in analogy with Equs.15,16,

$$\sigma_{ij}(t) = \left[\frac{C(t)}{B(t) I_n r} \right]^{\frac{1}{n+1}} \hat{\sigma}_{ij}(\theta, n) \qquad \text{(Equ.18)}$$

$$\dot{\varepsilon}_{ij}^{c}(t) = B(t) \left[\frac{C(t)}{B(t) I_n r} \right]^{\frac{n}{n+1}} \hat{\varepsilon}_{ij}(\theta, n) \qquad \text{(Equ.19)}$$

That these expressions are correct can be checked by substituting them into Equ.11b for C(t). Only terms which are products of stress and strain rate (displacement rate gradient) occur, and it is readily seen that B(t) cancels in such terms. The integral becomes identical to that obtained by substituting Equs.15,16 into Equ.17, except that C(t) replaces J(0) – and consequently the integral must evaluate simply to C(t), as required.

Equs.18,19 describe the stress and strain rate fields only sufficiently near the crack tip, and, in particular, only within the creep zone around the crack tip. Equs.15,16 cease to describe the crack tip fields as soon as creep strains occur, although they may continue to have some approximate validity outside the creep zone for as long as the creep zone is sufficiently small. This is analogous to an LEFM field persisting around a sufficiently small plastic zone. Hence, for small scale yielding and for sufficiently early creep times, there is a hierarchy of size scales,

$$r_{HRR}^{c} < r_c < r_p < r_{LEFM} < L$$

where, r_{HRR}^{c} is the size of the region in which the fields are given by Equs.18,19,
r_c is the size of the creep zone,
r_p is the size of the plastic zone (fields given approximately by Equs.15,16),
r_{LEFM} is the size of the region in which fields may approximate to LEFM form,
L is the ligament size.

At longer times, creep will tend to obliterate both the PYFM and LEFM regions.

5. DERIVATION OF THE C(t) ESTIMATION FORMULAE

The derivation of estimation formulae for C(t) proceeds in two steps,

i. firstly by finding an expression for C(t) in terms of J(t), and then,
ii. using an estimation formula for J(t) to find C(t) from (i).

5.1 The Expression for C(t) in Terms of J(t)

Equ.19 may be integrated to give the creep strains near the crack tip,

$$\varepsilon_{ij}^{c}(t) = \int_0^t dt' B(t')\left[\frac{C(t')}{B(t')I_n r}\right]^{\frac{n}{n+1}} \hat{\varepsilon}_{ij}(\theta, n) \qquad \text{(Equ.20)}$$

From which the ratio of creep strains to strain rates is found to be,

$$\frac{\varepsilon_{ij}^{c}(t)}{\dot{\varepsilon}_{ij}^{c}(t)} = \frac{\int_0^t dt' B(t')^{\frac{1}{n+1}} C(t')^{\frac{n}{n+1}}}{B(t)^{\frac{1}{n+1}} C(t)^{\frac{n}{n+1}}} = f(t) \quad \textit{(near crack tip)} \qquad \text{(Equ.21)}$$

where the function f(t) is defined by the above expression. Equ.3 for J(t), together with Equ.7b, can be rewritten by separating out the parts with time independent and time dependent strains/displacements,

(Equ.22)

$$J(t) = \int_\Gamma \left\{\left(\frac{1}{2}\sigma_{ij}\varepsilon_{ij}^{e} + \frac{m}{1+m}\sigma_{ij}\varepsilon_{ij}^{p}\right)dy - \sigma_{ij}\frac{du_i(0)}{dx}n_j ds\right\} + \int_\Gamma \left\{\frac{n}{1+n}\sigma_{ij}\varepsilon_{ij}^{c}dy - \sigma_{ij}\frac{du_i^{c}}{dx}n_j ds\right\}$$

where we have defined the creep displacements $u_i^c = u_i(t) - u_i(0)$ the gradients of which give the creep strains. The ratio of the creep displacement gradients to their rates will clearly be the same as that for strains, ie.,

$$\frac{\frac{du_i^c}{dx}}{\frac{d\dot{u}_i^x}{dx}} = f(t) \qquad \textit{(near crack tip)} \qquad \text{(Equ.21b)}$$

The first term in Equ.22 is <u>not</u> the same as J(0) because the stresses in Equ.22 are evaluated at time t. Hence this first term is time dependent. Providing we confine attention to sufficiently small contours, Γ, the first term can be written in terms of J(0) by using the ratio of stresses at time t and time zero given by Equs.15 and 18. In the special case that the plastic and creep indices are the same, m = n, these give,

$$\frac{\sigma_{ij}(t)}{\sigma_{ij}(0)} = \left(\frac{DC(t)}{B(t)J(0)}\right)^{\frac{1}{n+1}} \qquad \textit{(for m = n only)} \qquad \text{(Equ.23)}$$

Substituting this into Equ.22, and noting that the above time-dependent factor may be taken outside the integral because it does not depend upon the spatial coordinates, gives the first term in Equ.22 to be,

$$J(t)\textit{(first term only)} = \left(\frac{DC(t)}{B(t)J(0)}\right)^{\frac{1}{n+1}} J(0) \qquad \textit{(for m = n only)} \qquad \text{(Equ.24)}$$

In the general case, when m ≠ n, the ratio of stresses at times t and zero retains a dependence upon 'r' and 'θ', and hence cannot be factored out of the contour integral to provide an expression like Equ.24. This is the reason for restricting attention to the special case m = n.

The second term in Equ.22 is the same as that defining C(t), Equ.11b, except that creep strains (displacement gradients) occur in place of creep strain rates (displacement gradient rates), and Equ.11b applies only for sufficiently small Γ. Equs.21,21b are appropriate in this limit of small Γ. Hence, substituting for creep strains and creep displacements in Equ.22 using Equs.21,21b, and noting that f(t) does not depend upon spatial coordinates and so may be taken outside the integral, we get simply,

$$J(t)\textit{(second term only)} = f(t)C(t) \qquad \text{(Equ.25)}$$

This, together with Equ.24, gives,

$$J(t) = \left(\frac{DC(t)}{B(t)}\right)^{\frac{1}{n+1}} J(0)^{\frac{n}{n+1}} + f(t)C(t) \qquad \text{(Equ.26)}$$

Substituting for f(t) from Equ.21 gives,

$$J(t) = \left(\frac{DC(t)}{B(t)}\right)^{\frac{1}{n+1}} J(0)^{\frac{n}{n+1}} + B(t)^{-\frac{1}{n+1}} C(t)^{\frac{1}{n+1}} \int_0^t B(t')^{\frac{1}{n+1}} C(t')^{\frac{n}{n+1}} dt' \qquad \text{(Equ.27)}$$

This expression may be simplified using the following scaling transformation,

$$\tilde{J}(t) = B(t)^{\frac{1}{n}} J(t), \quad \tilde{C}(t) = B(t)^{\frac{1}{n}} C(t) \qquad \text{(Equ.28)}$$

Equ.27 then becomes,

$$\tilde{J}(t) = \left(D\tilde{C}(t)\right)^{\frac{1}{n+1}} J(0)^{\frac{n}{n+1}} + \tilde{C}(t)^{\frac{1}{n+1}} \int_0^t \tilde{C}(t')^{\frac{n}{n+1}} dt' \qquad \text{(Equ.29)}$$

Equ.29 may be inverted to give $\tilde{C}(t)$ in terms of $\tilde{J}(t)$ as follows; differentiating gives,

$$\frac{d\tilde{J}}{dt} = \tilde{C} + \left(\frac{\tilde{J}}{(n+1)\tilde{C}}\right)\frac{d\tilde{C}}{dt} \qquad \text{(Equ.30)}$$

which simple manipulation shows to imply,

$$\frac{d}{dt}\left(\frac{\tilde{C}}{\tilde{J}^{n+1}}\right)+(n+1)\left(\frac{\tilde{C}}{\tilde{J}^{n+1}}\right)^{2}\tilde{J}^{n}=0 \qquad \text{(Equ.31)}$$

which can be integrated directly to give,

$$\frac{\tilde{J}^{n+1}}{\tilde{C}}=(n+1)\int_0^t \tilde{J}(t')^n\,dt'+\text{constant} \qquad \text{(Equ.32)}$$

Equ.29 shows that the constant, ie. the LHS evaluated at time zero, is just $DJ(0)^n$. Hence Equ.32 gives,

$$\tilde{C}(t)=\frac{\tilde{J}(t)^{n+1}}{(n+1)\int_0^t \tilde{J}(t')^n\,dt'+DJ(0)^n} \qquad \text{(Equ.33)}$$

Finally, Equ.33 can be re-expressed as the equation we have been seeking for C(t) in terms of J(t) and B(t) by using Equs.28, i.e.,

$$C(t)=\frac{B(t)J(t)^{n+1}}{(n+1)\int_0^t B(t')J(t')^n\,dt'+DJ(0)^n} \qquad \text{(Equ.34)}$$

In the case of secondary creep (constant B(t)), Equ.34 becomes identical to Joch & Ainsworth (Ref.2) Equ.26.

5.2 Estimation Formulae for J(t) and Hence C(t)

The estimation formula for J(t) which has been introduced at this stage by previous authors (Refs.1,2) is,

$$J(t) \approx J(0) + C^*t \qquad \text{(Equ.35)}$$

Our derivation avoids the use of this approximation, not least because it is not clear that it is true for primary creep. In fact, for primary creep, it is not clear what is meant by C^* in Equ.35. If creep described by Equs.1 prevails for the whole time of interest, then no quantity C^* is defined by Equ.14. On the other hand, if the R6 based reference stress formula were used to define C^* (see Equ.37 below) then C^* is time dependent, which undermines the linear time dependence intended by Equ.35.

Instead of Equ.35 we shall introduce the reference stress based estimation formula for J. Considering firstly the case of no creep, the usual reference stress formula for J(0) is,

$$J(0) = \frac{\varepsilon_{ref}^{ep}}{\sigma_{ref}} K_{TOT}^2 \qquad \text{(Equ.36)}$$

where σ_{ref} is the primary reference stress, $\varepsilon_{ref}^{ep} = \left(\varepsilon_{ref}^{e} + \varepsilon_{ref}^{p}\right)$ is the corresponding elastic-plastic reference strain, and K_{TOT} is the sum of the primary and secondary elastic stress intensity factors. Equ.36 ignores the small scale yielding correction. The assumption implicit in Equ.36 is that the effects of plasticity in enhancing J(0) above its LEFM value can be attributed to the primary load induced strains alone. In R6 terminology this is equivalent to assuming that the secondary stress plasticity correction factor, ρ, is negligible. These approximations are not essential, but simplify the algebra and are unlikely to introduce serious error as regards the eventual estimation of C(t).

We now wish to find an estimation formula for J(t), which must also include creep strains. The R5 estimation formula for C* is,

$$C_{ref}^{*} = \frac{\dot{\varepsilon}_{ref}^{c}}{\sigma_{ref}} K_P^2 \qquad \text{(Equ.37)}$$

where K_P is the primary stress intensity factor. However, in steady state creep we have $C^* = dJ/dt$, so Equ.37 gives,

$$J(t) = J(0) + \frac{\varepsilon_{ref}^{c}}{\sigma_{ref}} K_P^2 \qquad \textit{(steady creep)} \qquad \text{(Equ.38)}$$

Our estimation formula for J(t) is simply to assume that Equ.38 holds at all times, i.e. that J increases above its elastic-plastic value by an amount which is proportional to the creep strain, irrespective of any possible non-linear accumulation of the creep strain with time. Hence,

$$J(t) = \frac{\varepsilon_{ref}^{ep}}{\sigma_{ref}} K_{TOT}^2 + \frac{\varepsilon_{ref}^{c}(t)}{\sigma_{ref}} K_P^2 \qquad \text{(Equ.39)}$$

The estimation formula for C(t) is found by substituting Equ.39 into Equ.34. We note that Equ.1a gives,

$$B(t) = \dot{\varepsilon}_{ref}^{c}(t) \Big/ \sigma_{ref}^{n} \qquad \text{(Equ.40)}$$

Hence the integral in the denominator of Equ.34 becomes, using Equs.39,40,

$$\int_0^t B(t')J(t')^n dt' = \int_0^t \left[\frac{\varepsilon_{ref}^{ep}}{\sigma_{ref}} K_{TOT}^2 + \frac{\varepsilon_{ref}^{c}(t')}{\sigma_{ref}} K_P^2\right]^n \frac{\dot{\varepsilon}_{ref}^{c}(t')}{\sigma_{ref}^{n}} dt'$$

$$= \int_0^{\varepsilon_{ref}^c(t)} \left[\frac{\varepsilon_{ref}^{ep}}{\sigma_{ref}} K_{TOT}^2 + \frac{\varepsilon_{ref}^c(t')}{\sigma_{ref}} K_P^2\right]^n \frac{d\varepsilon_{ref}^c(t')}{\sigma_{ref}^n}$$

$$= \frac{1}{n+1}\left(\frac{K_{TOT}}{\sigma_{ref}}\right)^{2n}\left(\frac{K_{TOT}}{K_P}\right)^2\left[\left(\varepsilon_{ref}^{ep} + \left(\frac{K_P}{K_{TOT}}\right)^2 \varepsilon_{ref}^c\right)^{n+1} - \left(\varepsilon_{ref}^{ep}\right)^{n+1}\right] \qquad \text{(Equ.41)}$$

Note that in deriving Equ.41, σ_{ref} and K_p can be taken out of the integral because they depend only upon the primary stresses, and hence are time independent. K_{TOT} can be taken outside the integral because we are ignoring creep relaxation of secondary stresses.

The numerator of Equ.34 is,

$$B(t)J(t)^{n+1} = \frac{\dot{\varepsilon}_{ref}^c}{\sigma_{ref}^n}\left(\frac{\varepsilon_{ref}^{ep}}{\sigma_{ref}} K_{TOT}^2 + \frac{\varepsilon_{ref}^c}{\sigma_{ref}} K_P^2\right)^{n+1}$$

$$= \frac{K_{TOT}^{2(n+1)}}{\sigma_{ref}^{2n+1}} \dot{\varepsilon}_{ref}^c \left(\varepsilon_{ref}^{ep} + \left(\frac{K_P}{K_{TOT}}\right)^2 \varepsilon_{ref}^c\right)^{n+1}$$

$$= \left(\frac{K_{TOT}}{\sigma_{ref}}\right)^{2n}\left(\frac{K_{TOT}}{K_P}\right)^2 C_{ref}^* \left(\varepsilon_{ref}^{ep} + \left(\frac{K_P}{K_{TOT}}\right)^2 \varepsilon_{ref}^c\right)^{n+1} \qquad \text{(Equ.42)}$$

where we have used the R5 estimation formula for C* (Equ.37). Hence, substituting Equs.41,42 into Equ.34 gives,

$$\frac{C(t)}{C_{ref}^*} = \frac{(1+\tau)^{n+1}}{(1+\tau)^{n+1} - 1 + \phi} \qquad \text{(Equ.43)}$$

where, $\tau = \left(\frac{K_P}{K_{TOT}}\right)^2 \frac{\varepsilon_{ref}^c}{\varepsilon_{ref}^{ep}}$ (Equ.44)

and, $\phi = \frac{DJ(0)^n}{\left(\varepsilon_{ref}^{ep}\right)^{n+1}}\left(\frac{\sigma_{ref}}{K_{TOT}}\right)^{2n}\left(\frac{K_P}{K_{TOT}}\right)^2 = \frac{\varepsilon_{ref}^p}{\varepsilon_{ref}^{ep}}\left(\frac{K_P}{K_{TOT}}\right)^2$ (Equ.45)

where the last step has been accomplished by substituting for D and J(0) from Equs.6 and 36 respectively, and noting that the multiaxial Equ.6 implies the uniaxial relation,

$\varepsilon_{ref}^p = D\sigma_{ref}^m$ *(and we are assuming that m = n)* (Equ.6b)

We note the following features of Equs.43-45,

i) ϕ is a plasticity correction term. The larger the plastic strains, the larger is ϕ and hence the smaller is C(t).

ii) For plastic strains much larger than the elastic strain, and when there is no secondary loading, $\phi \to 1$ and C(t) = C^*_{ref} .

iii) Equ.45 for ϕ shows how the effects of plasticity and secondary loading interact. In particular, for large plastic strains, the maximum value of C(t), at sufficiently early times, is $\left(\frac{K_{TOT}}{K_P}\right)^2 C^*_{ref} = \frac{\dot{\varepsilon}^c_{ref}}{\sigma_{ref}} K^2_{TOT}$.

iv) Equs.44,45 show, as expected, that it is conservative to ignore the creep relaxation of the secondary stresses (i.e. C(t) is larger if K_{TOT} is over-estimated).

v) The definition of τ, Equ.44, involves the elastic-plus-plastic strains in the denominator, not just the elastic strain. This will make τ smaller than would be the case if elastic strains only were used, and hence leads to a larger C(t). (Overall, however, plasticity will decrease C(t) due to the ϕ term).

vi) It is inappropriate to refer to τ as a dimensionless time in the case of primary creep since Equ.44 shows that τ is not linearly related to time when Equ.1 is used for strain rates. Hence τ must <u>not</u> be approximated as t/t_{red} in the primary creep regime.

Equs.44, 45, 46 provide the sought-for estimation formula for C(t). It is remarkable that this estimation formula can be derived for arbitrary primary creep behaviour, B(t). Despite the fact that B(t) does not appear explicitly in Equs.44-46, primary creep affects the value of C(t) through the magnitude of the creep strain, ε^c_{ref} , in the definition of the normalised time, τ (Equ.44). The more rapid the accumulation of primary creep strain, the larger is τ, and hence the smaller is the ratio C(t)/C*, at any given time. [It does not follow that C(t) is smaller, because C* is larger].

In the particular case of secondary creep, Equ.43 is the same as the estimation formula derived by Joch & Ainsworth (Ref.2, Equ.28). In particular, their definition of the dimensionless time is,

$$\tau = \frac{C^* t}{J(0)} = \left(\frac{K_P}{K_{TOT}}\right)^2 \frac{\dot{\varepsilon}^c_{ref} t}{\varepsilon^p_{ref}} = \left(\frac{K_P}{K_{TOT}}\right)^2 \frac{\varepsilon^c_{ref}}{\varepsilon^p_{ref}} \quad \textit{(for secondary creep)} \qquad \text{(Equ.46)}$$

in agreement with Equ.44 (although Joch & Ainsworth considered only the case of zero secondary loading, $K_P = K_{TOT}$).

For realistic plastic hardening behaviour, as opposed to the idealised power-law assumed in the above derivation, the plasticity correction factor, ϕ, will be estimated relatively crudely by Equ.45. In particular, if there is a proportionality limit stress, then when the reference stress is below this stress, the plastic reference strain is zero and hence the plastic correction factor given by Equ.45 is zero. C(t) is then given by Equ.43 to be the same as that which would obtain if plastic strains were zero everywhere. This is physically unreasonable because there will be plastic straining near the crack tip at arbitrarily small reference stresses.

Small scale yielding terms, eg. in R6, vary as $(\sigma_{ref}/\sigma_y)^2 = L_r^2$. Since $\varepsilon_{ref}^p = 0.2\%$ at $L_r = 1$ a reasonable suggestion for an effective plastic strain to employ in Equs.44 and 45, when the strict value of ε_{ref}^p is zero, might be,

$$\varepsilon_{ref}^p\,(\text{small scale yielding}) = 0.2\% L_r^2 \qquad \textit{(for } L_r < \sigma_L/\sigma_y\textit{)} \qquad \text{(Equ.47)}$$

where σ_L is the proportionality limit stress. The use of Equ.47 thus allows crack-tip plasticity effects to be taken into account for arbitrarily small levels of applied load. On reflection, this seems rather a crude estimate. It would be better to use the small scale yielding correction compatible with the R6 FAD in the estimate of J(0) inserted into Equ.(39) and work the implications of this through. Perhaps it amounts to much the same thing? (RAWB, 26/3/08).

5.3 Illustration of Special Cases:

(A) No Plasticity

The estimation formulae become,

$$\frac{C(t)}{C_{ref}^*} = \frac{(1+\tau)^{n+1}}{(1+\tau)^{n+1} - 1} \qquad \text{(Equ.43c)}$$

where, $$\tau = \left(\frac{K_P}{K_{TOT}}\right)^2 \frac{\varepsilon_{ref}^c}{\varepsilon_{ref}^e} \qquad \text{(Equ.44c)}$$

and, $$\phi = 0 \qquad \text{(Equ.45c)}$$

In this case C(t) is divergent as $t \to 0$ since Equ.43c gives,

$$C(t) \to \frac{C_{ref}^*}{(n+1)\tau} = \frac{1}{n+1}\left(\frac{\dot{\varepsilon}_{ref}^c}{\varepsilon_{ref}^c}\right)\frac{K_{TOT}^2}{E} \quad \textit{(as } t \to 0\textit{)} \qquad \text{(Equ.48)}$$

For secondary creep the ratio of strain rate to creep strain is just 1/t, whereas for primary creep with a power-law time dependence, i.e. for $\varepsilon^c = At^p\sigma^n$ with p<1, this ratio is p/t, so,

$$C(t) \to \frac{K_{TOT}^2}{(n+1)Et} \qquad \textit{(for secondary creep as } t \to 0\textit{)} \qquad \text{(Equ.49)}$$

$$C(t) \to \frac{pK_{TOT}^2}{(n+1)Et} \qquad \textit{(for primary creep as } t \to 0\textit{)} \qquad \text{(Equ.50)}$$

Hence, the strength of the divergence in C(t) as $t \to 0$ is $\propto 1/t$ for both secondary and primary creep, although the amplitude is a factor 'p' smaller in the latter case. This appears to be consistent with Riedel (Ref.5) whose equations (25.6) and (25.8) permit the identification of C(t) with Riedel's C*, and equation (25.6) implies that this quantity varies $\propto 1/t$ for constant applied load.

(B) With Small Scale Plasticity

This case is defined by a reference stress which is less than the proportionality limit stress, and hence strictly has $\varepsilon^p_{ref} = 0$. In this case we use Equ.47 to define an effective plastic strain appropriate for the crack tip. The estimation formulae are as Equs.43-45. The short time limit is,

$$C(t) \to \frac{C^*_{ref}}{\phi} = \frac{\varepsilon^{ep}_{ref}\dot{\varepsilon}^c_{ref}(t)}{\varepsilon^p_{ref}\sigma_{ref}} K^2_{TOT} \qquad \textit{[for } \tau << \phi/(1+n)] \qquad \text{(Equ.51)}$$

which is finite for secondary creep (unlike Equ.49), but which diverges $\propto t^{p-1}$ with $p < 1$ in the case of primary creep with power-law time dependence, i.e. for $\varepsilon^c = At^p\sigma^n$. This is a less strong divergence than Equ.50.

(C) Widespread Plasticity $(\varepsilon^p_{ref} > 0)$

The estimation formulae are again just Equs.43-45. The only difference from case (B) is that ε^p_{ref} is found by substituting the reference stress into the hardening curve, as usual. The short time behaviour is essentially the same as for case (B), ie. finite at time zero for secondary creep, but diverging as t^{p-1} with $p < 1$ in the case of primary creep with power-law time dependence, i.e. for $\varepsilon^c = At^p\sigma^n$.

Hence, we conclude that the t = 0 singularity in C(t) persists in the case of primary creep, despite the effects of plasticity in ameliorating its strength.

If the plastic strains are much larger than the elastic strains, $\varepsilon^p_{ref} >> \varepsilon^e_{ref}$, then $\phi \to \left(K_{TOT} / K_P \right)^2$ and hence the short term behaviour becomes,

$$C(t) \to \left(\frac{K_{TOT}}{K_P}\right)^2 C^*_{ref} = \frac{\dot{\varepsilon}^c_{ref}}{\sigma_{ref}} K^2_{TOT} \qquad \textit{for} \quad \tau << \frac{1}{n+1}\left(\frac{K_P}{K_{TOT}}\right)^2 \qquad \text{(Equ.52)}$$

6. CONCLUSIONS / IMPLICATIONS / WARNINGS

1) Equs.43-45 together with Equ.47 provide the most general estimation formula for $C(t)/C_{ref}^*$ **,** where C_{ref}^* is found from the R5 reference stress formula and is time dependent in primary creep. It is remarkable that Equs.43-45 can be derived for arbitrary primary creep (arbitrary B(t) in Equs.1).

2) The plasticity correction term in Equs.43-45 can be derived only under the assumption that the plasticity and creep hardening indices m and n are equal. FE studies (eg. Ref.6) suggest that Equs.43-45 continue to be a reasonable approximation for $m > n$, but that for $m < n$ the ameliorating effect of plasticity is much less marked.

3) As far as I am aware, Equs.43-45 represent the first time that secondary loading and plasticity have been considered in combination as regards their influence on C(t). Note, however, that only the plastic strains due to the primary loads have been included in this estimation procedure. Since plasticity reduces C(t), the procedure is likely to be conservative, especially for secondary stresses beyond yield. (Added by RAWB, March 2008).

4) When there are no secondary loadings, and when the plastic strains are large compared with the elastic strains, Equs. 43-45 show that the plasticity correction term ϕ becomes unity and hence that $C(t) = C_{ref}^*$. The physical reason for this is that, with $m = n$, the plasticity pre-establishes crack tip fields which are essentially the same as those due to creep, and hence the subsequent creep "does nothing" as regards the amplitude of the crack tip singularity, i.e. C(t). We can also understand the FE results of Ref.6, which imply the same holds for $m > n$, because, in this case, the plastic strain singularity is more severe than that of the creep. (Added by RAWB, March 2008).

5) It should be noted that Equ.44 shows that the correct definition of τ involves a denominator of ε_{ref}^{ep} , i.e. the elastic plus plastic strains, rather than just the elastic strains as is sometimes assumed. This is potentially important because it implies that τ is smaller, and hence C(t) larger, than would be the case if the elastic strain were used. However, the ϕ term is likely to be dominant, resulting in reduced C(t) due to plasticity.

6) For primary creep it is inappropriate to refer to τ as a dimensionless time since Equ.44 shows that τ is non-linearly related to time. For primary creep it is therefore incorrect to approximate τ as t/t_{red}.

7) The behaviour of C(t) for vanishingly short times is:-

Constitutive Law	**Secondary creep**	**Primary creep**
Elastic-creep	$K_{TOT}^2 / (n+1)Et$	$pK_{TOT}^2 / (n+1)Et$
Elastic-plastic-creep	Finite = C^*/ϕ	Diverges as $1/t^{1-p}$ $(= C_{ref}^* / \phi)$

7. References

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3) R5: Assessment Procedure for the High Temperature Response of Structures, Issue 2, British Energy, 1998. I’ve left this reference as it was when I wrote the Note, but you should now (i.e. at March 2008) be using R5 Issue 3 (June 2003).

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