Appendix E
The Weak Nuclear Force: Fermi's Constant, G_F

In defining the universal constants associated with the nuclear forces we face a particular difficulty which did not occur in the case of gravity and electromagnetism. Whilst gravity and electromagnetism can be described in classical terms, the nuclear "forces" are essentially quantum mechanical in nature. There is no strictly correct classical description of the nuclear forces comparable with that given in Appendices C and D for gravity and electromagnetism. The very concept of "force" does not actually apply, despite the continued use of the term. On the other hand, we have no wish to discuss quantum field theory. As a compromise, we shall take an approach based on an effective energy density. As a partial recognition of the quantum mechanical nature of these things, the 'particles' will be described in terms of a distribution, i.e. a probability density. Otherwise we adhere to the fiction that we are dealing with classical fields.

Consider firstly the electrostatic energy density which results when a proton, described by a probability density distribution ρ, is in an electrostatic field of potential ϕ:-

$$L = \tilde{e}\rho(\bar{r})\phi(\bar{r}) \qquad \text{(E.1)}$$

Note that both the proton density and the electric potential might vary with position, $\bar{r}$. This is a generalisation of Equ.(D.7) for a 'smeared out' proton as represented by the probability density ρ.

[Aside: Although it is not necessary for our present argument, some readers may like to note that (E.1) is the non-relativistic approximation to,

$$L = j_\mu(\bar{r})A^\mu(\bar{r}) \qquad \text{(E.2)}$$

which applies for a moving charge, with current density $j_\mu = \tilde{e}\rho v_\mu$ where v_μ is its 4-velocity, and A^μ is the electromagnetic 4-vector potential, which includes the magnetic field effects.]

We now attempt to describe the weak nuclear "force" by analogy with (D.1). In the oldest version of the theory (due to Fermi) there is no analogue of the electrostatic potential, ϕ. Instead, the energy density is supposed to be proportional to the product of the densities of the two particles involved. Thus, the weak nuclear energy between a proton and an electron is given, in our crude heuristic approximation, by,

$$L \approx G_F\rho_p(\bar{r})\rho_e(\bar{r}) \qquad \text{(E.3)}$$

where ρ_p and ρ_e are the proton and electron probability densities, and will vary with position, $\bar{r}$. G_F is our new universal constant (Fermi's constant) which defines the strength of the weak nuclear force. Since L is, by definition, an energy density, and because ρ_p and ρ_e have units of 1/volume, it follows that G_F has units: Energy x Volume. It is given by,

$$G_F = 1.43 \times 10^{-62}\ \text{Jm}^3 \qquad \text{(E.4a)}$$

In texts this is often written as,

$$G_F = \frac{1.02 \times 10^{-5}}{M_p^2} \cdot \frac{\hbar^3}{c} \qquad \text{(E.4b)}$$

where M_p is the proton mass (1.67 x 10^{-27} kg = 0.93827 GeV). Most texts will also confuse the unwary by working in units for which $\hbar = c = 1$, so that this is written,

$$G_F = \frac{1.02 \times 10^{-5}}{M_p^2} \qquad \text{(E.4c)}$$

Inserting the mass of the proton gives a numerical value G_F = 1.16 x 10^{-5} GeV^{-2}. [Note that the true dimensions of G_F, energy x volume, are equivalent to 1/energy^2 when using units with $\hbar = c = 1$]. Substituting 1 GeV = 1.6 x 10^{-10} J yields an alternative numerical value for G_F as 4.53 x 10^{14} J^{-2}. Yet another numerical value results from noting that, in units with $\hbar = c = 1$, J^{-1} = 3.16 x 10^{-26} m, hence G_F = 1.43 x 10^{-11} m/J. This latter value is reached directly from,

$$G_F = \frac{1.02 \times 10^{-5}}{M_p^2} \cdot \frac{\hbar}{c^3} = 1.43 \times 10^{-11} \text{ m/J} \qquad \text{(E.4d)}$$

In summary, the true MKS value for G_F is given by (E.4a), but, when using units with $\hbar = c = 1$, there are many different numerical values for G_F depending upon the unit chosen.

A particular problem arises in describing the nuclear forces which did not occur with the classical forces. When the weak force acts it changes the nature of the particles involved. For example, we have seen in Appendix B of the "Cosmic Coincidences" that the weak nuclear force can cause the following inter-conversions,

$$p^+ + e^- \Leftrightarrow n + \nu_e \qquad \text{(E.5)}$$

Hence, (E.3) can also be regarded as the weak nuclear energy density due to the interaction of a neutron and a neutrino. In quantum field theory, this energy density (the interaction Lagrangian) can be used to calculate the transition rate for reactions like (E.5).

Again we note in passing, though it is not crucial to our argument, that the Fermi interaction Lagrangian is properly written as,

$$L = \frac{G_F}{\sqrt{2}} J_{N\mu} J_L^{\mu} + \text{Hermetian conjugate} \qquad \text{(E.6)}$$

where the nucleon and lepton currents, J_N and J_L are given by,

$$J_{N\mu} = \overline{\psi}_n \gamma_\mu (1 + \gamma_5) \psi_p \quad \text{and} \quad J_{L\mu} = \overline{\psi}_e \gamma_\mu (1 + \gamma_5) \psi_{\nu_e} \qquad \text{(E.7)}$$

Interpreted as quantum fields, involving creation and annihilation operators, the interaction represented by (E.6, 7) 'explains' how reactions like (E.5) can occur.

We now want to compare and contrast (D.1) for the electrostatic energy density and (E.3) for the weak nuclear energy density. Note that if a second charge is responsible for the electric potential then it can be written,

$$\phi(\bar{r}) = \int \frac{\tilde{e}\rho_2(\bar{r}')}{4\pi|\bar{r}-\bar{r}'|} dV' \qquad \text{(E.8)}$$

Hence, (C.1) becomes,

$$L = \tilde{e}^2 \rho(\bar{r}) \int \frac{\rho_2(\bar{r}')}{4\pi|\bar{r}-\bar{r}'|} dV' \qquad \text{(E.9)}$$

If both particles are localised at a point, so that the probability densities are Dirac delta-functions, $\delta(\bar{r}-\bar{r}_1)$ and $\delta(\bar{r}-\bar{r}_2)$ respectively, then the total electrostatic energy is,

$$\text{Electrostatic energy} = \int L dV = e^2 \int \rho(\bar{r}) \left[\int \frac{\rho_2(\bar{r}')}{4\pi|\bar{r}-\bar{r}'|} dV' \right] dV = \frac{\tilde{e}^2}{4\pi|\bar{r}_1 - \bar{r}_2|} \qquad \text{(E.10)}$$

in agreement with (C.6).

The point of this rather heavy-handed derivation is that if we now insert the same delta-function distributions into (E.3) for the weak nuclear potential energy between an electron and a proton (say), the result is zero! The reason is that these two distributions are not both non-zero at any point. So, because the interaction Lagrangian involves their product, it is zero everywhere. The physical reason for this is that, in the case of the electrostatic interaction, the electric field mediates between the two separated particles. The physical manifestation of this mediation is that photons travel from one particle to the other, thus forming a connection between the two. In contrast, in the Fermi theory of the weak nuclear force, there *is* no weak nuclear field – so there is nothing to mediate between separated particles. Hence, in the (over simplified) Fermi theory of the weak interaction, particles can only interact "by contact", as it were.

Consequently, it is not possible to compare the strengths of the weak and electrostatic forces on a strictly equitable basis. The weak force is only non-zero at a vanishing particle separation, whereas the electrostatic force is divergent at vanishing separation. However, as a rough indication we initially use a size scale typical of atomic dimensions, i.e. 1 Angstrom = 10^{-10} m. The electrostatic energy between two charge quanta with this separation is 2.3 x 10^{-18} J. If we now assume that our proton and electron are both confined to a region of this size, namely a cube of side 1 Angstrom, then each has a probability density of 10^{30} m^{-3}. From (E.3), the total weak nuclear potential energy (remembering to multiply by the volume to covert energy density to total energy) is thus,

$$E_{weak} = G_F \times 10^{30} \text{ m}^{-3} \times 10^{30} \text{ m}^{-3} \times 10^{-30} \text{ m}^3 = 1.43 \times 10^{-32} \text{ J} \qquad \text{(E.11)}$$

So the electrostatic energy is some 14 orders of magnitude larger. However, if we had chosen to confine our particles within a cube of side X, the weak nuclear energy would have been,

$$E_{weak} = G_F \text{ x } X^{-3} \text{ m}^{-3} \text{ x } X^{-3} \text{ m}^{-3} \text{ x } X^3 \text{ m}^3 = 1.44 \text{ x } 10^{-32} \left(\frac{10^{-10}\,\text{m}}{X}\right)^3 \text{ J} \quad \text{(E.12)}$$

Thus, the weak nuclear energy increases (in this simple theory) as the reciprocal of the linear dimension cubed. In contrast, the electrostatic energy is inversely proportion to the particle separation ($\propto 1/X$). However, even if we used a linear dimension (X) typical of nuclear sizes, i.e. 1 Fermi = 10^{-15} m, the electrostatic energy (which is then 2.3 x 10^{-13} J) is still ~16,000 times larger than the weak nuclear energy. Hence the weak nuclear force does indeed deserve its name.

The electrostatic and weak nuclear energies do not become comparable until X ~ 8 x 10^{-18} m = 0.008 fm, which is two orders of magnitude smaller than nuclear sizes. This is significant. In quantum mechanics a length scale can be related to a mass, namely $\hbar / Xc$. A size scale roughly two orders of magnitude smaller than nuclear sizes therefore implies a mass about two orders of magnitude greater. It turns out that the weak nuclear force is actually mediated by a weak nuclear *field*, whose quanta (known as W and Z bosons, see Appendix B of the Cosmic Coincidences) do indeed have a mass roughly 100 times that of nucleons. Thus, for energies which are comparable with the masses of the weak force field quanta (i.e. of order 90 GeV), the 'weak' force is no longer weak, but comparable in strength to the electrostatic field. Thus, the weak force appears to be weak at lower energies, not because it is intrinsically weak, but because it is very short range as a consequence of the high mass of the weak field quanta.

In modern gauge field theories, the electromagnetic and weak nuclear forces are actually manifestations of a single underlying interaction. The electromagnetic forces are mediated by the zero-mass photon, whereas the weak force is mediated by the heavy W and Z bosons. At energies which are comparable with, or larger than, the masses of the W and Z bosons (approximately 80 GeV and 91 GeV respectively), the electromagnetic and 'weak' forces are of comparable strength. However, it can be shown that the strength of a force mediated by particles with mass reduces exponentially with the separation of the particles. Specifically the force is given by a Yukawa potential of the form,

$$V_0 \propto \frac{e^{-r/X}}{r/X} \quad \text{where, } X = \frac{\hbar}{Mc} \qquad \text{(E.13)}$$

where M is the mass of the mediating particle (the W and Z bosons in this case). Whilst V_0 measures the intrinsic strength of the potential (or force), it becomes rapidly many orders of magnitude smaller when r >> X. Hence, at lower energies, which is equivalent to larger spatial scales, the strength of the weak nuclear force decreases dramatically.

Conversely, at sufficiently high energies, such that the corresponding length scale $\hbar c / E$ is less than X, that is for $E > Mc^2$, the exponential term becomes approximately unity and the weak and electromagnetic forces become comparable. The detailed formulation of the unified theory of the so-called electro-weak field earned Salam and Weinberg the Nobel prize in 1979. It was the first successful example of a spontaneously broken gauge symmetric quantum field theory.

The W and Z particles were discovered, in 1983, following the formulation of the electroweak theory and in dramatic confirmation of it. It is worth pointing out that the existence of the weak field quanta, W and Z, underwrites the view of the basic "forces" described in Appendix B. Specifically it reaffirms the physical reality of the weak field in its own right. The force fields are not mere mathematical artifacts, but have a physical reality comparable with that of matter. So much so that, in this case, the field becomes manifest in the form of quanta that have mass. Observe that the original Fermi theory contained no weak field. Rather the effects of the weak interaction were accomplished by the "direct action" of one particle on another. But increased knowledge of the way that nature actually behaves, as revealed through experiments, has lead inevitably back to a physically real weak field.