The Limit State

Statement and Proof of the Upper and Lower Bound Theorems of Plasticity
With Examples of Their Application

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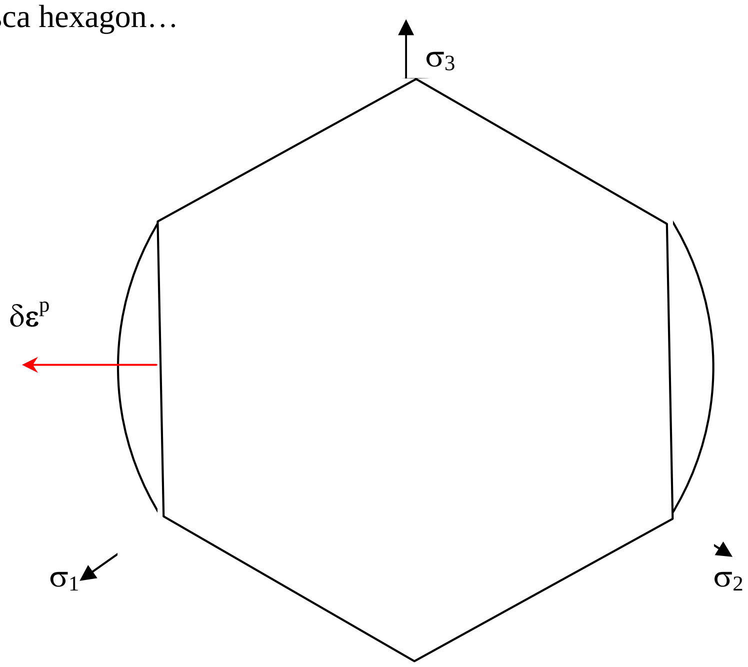
Contents:-

Preliminaries: The Formulation of Incremental Plasticity (Isotropic)

- Call the principal stresses $\sigma_1, \sigma_2, \sigma_3$
- The hydrostatic stress is $\sigma_h = (\sigma_1 + \sigma_2 + \sigma_3)/3$
- The deviatoric stresses (in principal coordinates) are $\hat{\sigma}_i = \sigma_i - \sigma_h$
- The deviatoric stresses obviously obey $\hat{\sigma}_1 + \hat{\sigma}_2 + \hat{\sigma}_3 = 0$
- Hydrostatic stress does not cause any plastic strain increment
- (plastic deformation = shear = dislocation movement = planes of atoms sliding)
- Hence, we need only consider the 2D deviatoric stress plane.
- In general, the yield surface is **convex** – for example, the Mises circle or the Tresca hexagon…



- Plastic deformation is incompressible (sliding again!), so the principal plastic strain increments obey $d\varepsilon_1^p + d\varepsilon_2^p + d\varepsilon_3^p = 0$.
- Hence, the plastic strain increments also lie in the “deviatoric plane”.
- Example: Pure shear with $\sigma_3 = 0,\ \sigma_1 = -\sigma_2 > 0$ clearly results in $d\varepsilon_3^p = 0,\ d\varepsilon_1^p = -d\varepsilon_2^p > 0$ and hence the plastic strain increment is perpendicular to the yield surface (Tresca or Mises or any other isotropic yield surface).
- **Flow Rule:** The plastic strain increment is always **normal** to the yield surface.

We shall see that it is the **convexity** of the yield surface, and the fact that the plastic strain increment is **normal** to the yield surface that results in the Upper and Lower Bound Theorems.

Statement of the Upper and Lower Bound Limit Load Theorems

The Lower Bound Theorem: If some postulated distribution of stresses within a body is;

(a) in equilibrium everywhere, and,
(b) in equilibrium with some applied loads, P_i, and,
(c) the equivalent stress does not exceed the yield stress anywhere,

then the loads P_i are a lower bound estimate of the loads required to cause plastic collapse.

Commentary:-

1) The displacements and strains are irrelevant and play no part in the lower bound theorem.
2) Being "in equilibrium everywhere" means that the stress distribution obeys $\frac{\partial \sigma_{ij}}{\partial x_j} = b_i$ at every point in the body (where b_i is the body force per unit volume, most often zero).
3) Different equivalent stresses can be chosen according to which theory of plasticity one favours. More generally the condition that "the equivalent stress does not exceed yield anywhere" can be replaced by "the yield condition is not violated anywhere".
4) The loads P_i may be any combination of point loads, pressures, tractions, body forces etc, as long as they are all *load controlled* loadings.
5) For a poor guess regarding the stresses, the loads P_i may be a uselessly low lower bound – but always safe.

Corollaries:-

- Adding material* to a body cannot reduce its plastic collapse load.
- Removing material* from a body cannot increase its plastic collapse load.

**assuming the added weight is negligible.*

Note that these obvious sounding theorems are not true for failure by fracture, since,

- Adding a block of material containing a crack can reduce the load carrying capacity of the structure, and,
- Removing material from around a crack tip can increase the load carrying capacity of the structure.

Limitations: The Lower Bound Theorem applies rigorously for,

a) Elastic-perfectly plastic behaviour;
b) Small strains (body geometry virtually unchanged by deformation);
c) Arbitrarily large ductility.

Item (c) is not normally a problem provided that all the loads are being treated as primary. The (plastic) ductility need only be large enough to permit the limit state to be attained. Often <1% will suffice – but there may be occasions when this becomes restricting. Restriction (c) becomes significant for long range displacement controlled loads. Whilst such loads could in theory be regarded as secondary, the strains required may be prohibitive.

Item (b) is not normally a problem either, for the same reasons – i.e. if the strains are modest the deformation will generally be small. However, there are cases when the assumption can be non-conservative, e.g. an upward slanted cantilever. There are also cases where it can be grossly over-conservative, e.g. an initially out-of-round internally pressurised pipe. Good practice when using FEA for collapse is to obtain results both with and without updated geometry. There is no general rule as to which is the more onerous or the more accurate.

Item (a) is the most significant simplification. Real structural steels exhibit large amounts of strain hardening. Specific assessment procedures will advise on the material property to employ in lieu of a perfectly plastic yield stress. For R6 this is the lower bound 0.2% proof stress (for Lr). R6 also uses a "flow" stress, defined as the lower bound average of the 0.2% proof stress and the UTS, for the Lr cut-off, i.e. for pure plastic collapse.

Proof of the Lower Bound Theorem

- Suppose that scaling all the applied loads, P_i, by some factor, λ, (just) results in collapse. Then the true collapse loads are $\tilde{P}_i = \lambda P_i$. We wish to demonstrate that $\lambda \geq 1$ so that our applied load is a lower bound to the true collapse load.
- Denote the actual plastic strain increments as collapse is approached by $d\tilde{\varepsilon}_K^p$, where 'K' denotes the 2 directions of principal deviatoric space, i.e. the plastic strain increment is effectively a 2D vector as shown in the above diagram.
- Call the velocity fields as collapse is approached $d\tilde{u}_i$, where 'i' denotes the usual Cartesian directions in 3D space. Hence $d\tilde{u}_i$ are kinematically (i.e. geometrically) compatible with the strains $d\tilde{\varepsilon}_K^p$. They must be because they are both the true values!
- The true stress distribution at collapse is written $\tilde{\sigma}_K$, where again 'K' denotes the 2 directions of principal deviatoric space. Hence, $\tilde{\sigma}_K$ are in equilibrium with the true collapse loads, $\tilde{P}_i$.
- Now consider some stress distribution σ_K, which is arbitrary apart from being in equilibrium with the loads, P_i, and not breaking through the yield surface.
- The principle of virtual work states that the work done by the hypothetical loads P_i must equal the integral of the work done on each element of the body, i.e.,

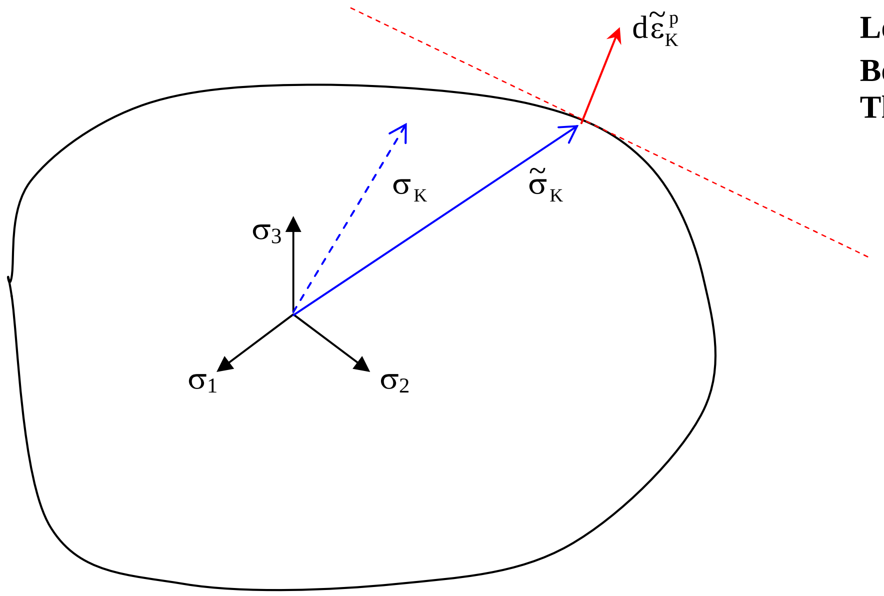
$$\sum_i P_i d\tilde{u}_i = \int \sigma_K d\tilde{\varepsilon}_K^p \qquad (1)$$

NB: The principle of virtual work applies because σ_K and Pi are in equilibrium and $d\tilde{\varepsilon}_K^p$ and $d\tilde{u}_i$ are kinematically compatible. Of course, the same relationship holds also for the true collapse load and stresses, i.e.,

$$\sum_i \tilde{P}_i d\tilde{u}_i = \int \tilde{\sigma}_K d\tilde{\varepsilon}_K^p \qquad (2)$$

Now, whenever the plastic strain increment is non-zero, the true stress $\tilde{\sigma}$ must lie on the yield surface. On the other hand, the hypothetical stress σ lies on or within the yield surface, by hypothesis. Hence, in the 2D deviatoric stress plane we have,

Lower Bound Theorem



It is clear from the above diagram that,

$$\sigma_K d\tilde{\varepsilon}_K^p \le \tilde{\sigma}_K d\tilde{\varepsilon}_K^p \tag{3}$$

at all points in the body. From (1) and (2) it thus follows that,

$$\sum_i P_i d\tilde{u}_i \le \sum_i \tilde{P}_i d\tilde{u}_i = \sum_i \lambda P_i d\tilde{u}_i \tag{4}$$

and hence that $\lambda \ge 1$. Thus, our postulated load is a lower bound to the true collapse load. QED.

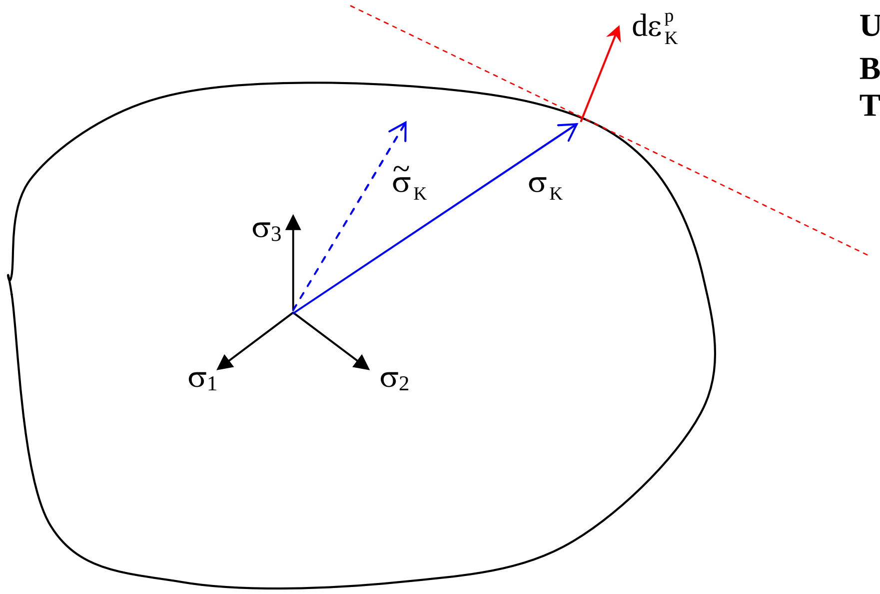
[The alert will have noticed that the proof assumes that the increments of elastic strain, $d\varepsilon_{ij}^{el}$, are zero at collapse. That is, that the elastic regions behave rigidly at collapse. This is true although we have not proved it here. This is the bit where the assumption of perfect plasticity, i.e. no hardening, is important].

<u>The Upper Bound Theorem</u>: An estimate, P_i, of the collapse load of a structure is made by,

(a) Postulating a 'mechanism' (i.e. a kinematically consistent distribution of displacement increments, du_i, and plastic strain increments, $d\varepsilon_K^p$);
(b) Evaluating the work done within the structure by the postulated plastic strain increments together with the corresponding stress on the yield surface;
(c) Equating the work done by the loads P_i to the above internal work.

Such an estimate is an upper bound to the true collapse load.

NB: The “corresponding stress on the yield surface” means that deviatoric stress at which the postulated plastic strain increment is normal to the yield surface. The situation is illustrated by a diagram very similar to that for the lower bound theorem:-



The differences are that, (a)the stresses with and without the tilda have swapped places, and, (b)the plastic strain increment is postulated rather than the true one. The stress without the tilda is defined as the stress corresponding to the postulated plastic strain increment. The stress with the tilda is the true collapse stress.

In the upper bound theorem we *define* the loads, Pi, by their work rate, i.e., by,

$$\sum_i P_i du_i = \int \sigma_K d\varepsilon_K^p \tag{4}$$

(*all* the quantities in Equ.(4) being *postulated* rather than true). Applying the virtual work argument to the true stress distribution, $\tilde{\sigma}_K$, in equilibrium with the true collapse loads, $\tilde{P}_i$, together with the postulated mechanism gives,

$$\sum_i \tilde{P}_i du_i = \int \tilde{\sigma}_K d\varepsilon_K^p \tag{5}$$

But from the diagram we have that,

$$\tilde{\sigma}_K d\varepsilon_K^p \leq \sigma_K d\varepsilon_K^p \tag{6}$$

at all points in the body. From (4) and (5) it thus follows that,

$$\sum_i \tilde{P}_i du_i = \sum_i \lambda P_i du_i \leq \sum_i P_i du_i \tag{7}$$

and hence that $\lambda \leq 1$. Thus, our postulated load is an upper bound to the true collapse load. QED.

Physical Interpretation of the Bound Theorems (My Words)

<u>The Lower Bound theorem</u>: As the load is increased, plasticity spreads through the structure causing more and more redistribution of the stresses until the only thing that can bring the process to an end is the inability to both satisfy equilibrium and the yield condition. *Probably, at the limit load*…the equivalent stress reaches the yield stress along the whole of a continuous surface which divides the structure into two parts…*though I've never seen this proved.*

<u>The Upper Bound Theorem</u>: Any mechanism which permits unbounded displacements whilst being consistent with the yield surface and the conservation of energy must already be beyond the collapse load.

Example Applications of the Bound Theorems

Rectangular Section Bar In Bending (Lower Bound)

The bar width is B and its thickness is t. The maximum magnitude of axial stress consistent with the yield condition is σ_y (because the stressing is uniaxial). Assume $+\sigma_y$ on one side of the neutral axis and $-\sigma_y$ on the other side. The load carried by one half of the section is thus $\sigma_y Bt/2$, and the centre of this load is at a distance of t/4 from the centre of the bar. It therefore contributes a moment of $\sigma_y Bt^2/8$. The other half contributes an equal moment, making $\sigma_y Bt^2/4$ in all. This results in a lower bound estimate of the collapse moment of,

$$M_y = \frac{Bt^2\sigma_y}{4} = \frac{3}{2} M^{first}_{yield} \tag{8}$$

where the "first yield" moment relates to the onset of yielding on the outer fibre, i.e. the maximum elastic stress reaching yield. As always, the characteristic stress that is used as "σ_y" is debatable.

The ratio of the collapse moment to the moment to cause first yield will always be greater than 1. For the rectangular section bar, Equ.(8) shows that it is 3/2. This is the origin of the factor of 1.5 by which design codes often permit membrane-plus-bending stresses to exceed the membrane design limit. However, for global bending of other sections this ratio will generally be different.

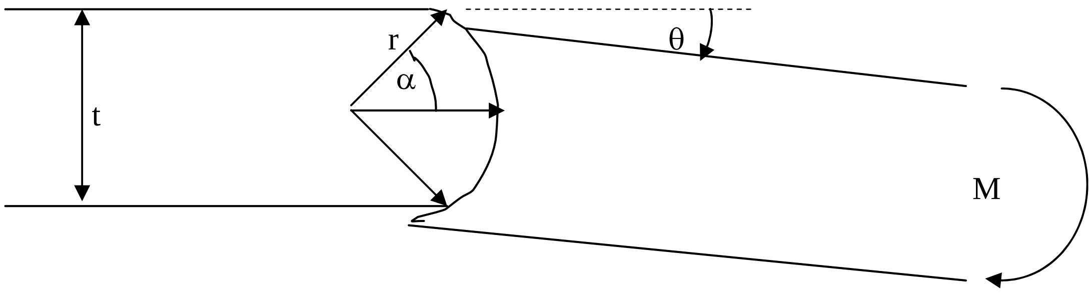
Thin Cylinder Under Global Bending (Lower Bound)

The axial stress is again assumed to be $+\sigma_y$ on one side of the neutral axis and $-\sigma_y$ on the other side. If y is the distance from the neutral axis the moment contributed by one quarter of the shell is $\int_0^{\pi/2} t\sigma_y y ds = \int_0^{\pi/2} t\sigma_y r \sin\theta . r d\theta = t\sigma_y r^2$. So the total moment is $4t\sigma_y r^2$. For a thin shell the section modulus is $I = \pi t r^3$ and so the moment at first

yield of the outer fibre is given by $M_{yield}^{first} = I\sigma_y / r = \pi t \sigma_y r^2$. Hence, $M_y = \frac{4}{\pi} M_{yield}^{first}$.
The ratio in this case is only $4/\pi$, significantly less than 1.5.

Rectangular Section Bar in Bending (Upper Bound)
The trick to applying the upper bound approach is to assume rigid (elastic) regions moving with respect to each other simply by sliding along slip lines. This represents a region of intense (in fact divergent) shear strain, but of small (in fact zero) volume. The work done is simply the area of the slip line times the shear yield stress (which gives the tangential force) times the distance slipped.

Inspired by our knowledge of the true plastic hinge, we postulate that slip occurs along a circular arc of some unknown radius 'r',



Geometry: $2r\sin\alpha = t$, Slip distance = $r\theta$. Slip line area = $2Br\alpha$

$$\text{Hence, work done} = M_y\theta = 2Br\alpha\tau_y.r\theta = \frac{Bt^2\theta\tau_y}{2} \cdot \frac{\alpha}{\sin^2\alpha} \qquad (9)$$

We are free to choose the angle α (or equivalently, the radius r) as we wish. Since Equ.(9) gives an upper bound for the collapse moment, we wish to minimise it to get the most accurate result (i.e. closest to reality). The minimum of the function $\frac{\alpha}{\sin^2\alpha}$ is easily shown to be 1.38 at an angle of $\alpha = 67^o$. Hence our "least upper bound" collapse solution is,

$$M_y = \frac{1.38Bt^2\tau_y}{2} = \frac{1.38Bt^2\sigma_y}{4}\text{ (Tresca)} = \frac{1.38Bt^2\sigma_y}{2\sqrt{3}}\text{ (Mises)} \qquad (10)$$

Thus, this upper bound result is between 38% and 59% larger than the lower bound result, Equ.(8), depending upon the yield theory used (i.e. a factor of 2.07 to 2.39 times the moment to first yield, compared with 1.5 for the lower bound result). <u>BUT it is an upper bound, so it would be non-conservative to use in an assessment</u>. There are better (i.e. smaller) upper bound solutions.

The upper bound result 'predicts' a reasonable shape for the plastic hinge.

NB: It makes no difference if we put the other side of the plastic hinge into the picture. Each side ('arc') of the plastic hinge is merely subject to half the angular displacement, and hence the result is the same.

Thick Pipe With Internal Pressure (Lower Bound, Tresca)
The equilibrium equation in polar coordinates is,

$$r\frac{\partial \sigma_r}{\partial r} + (\sigma_r - \sigma_h) = 0 \qquad (11)$$

where the subscripts denote the radial and hoop stresses. Assuming the yield stress is reached everywhere (the limit condition) gives $\sigma_h - \sigma_r = \sigma_y$ assuming the Tresca yield theory (and noting that the radial stress is compressive and the hoop stress tensile). Hence, Equ.(11) becomes simply,

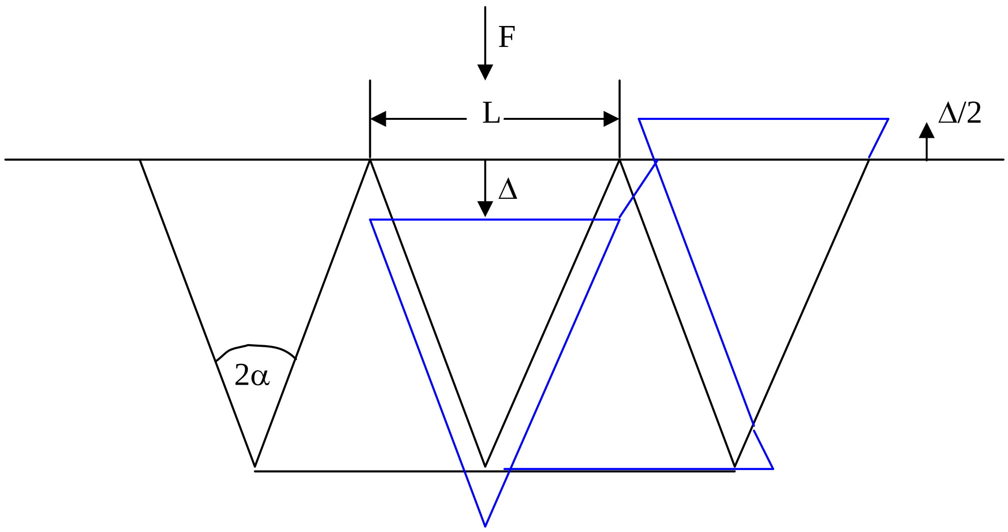
$$r\frac{\partial \sigma_r}{\partial r} = \sigma_y \qquad (12)$$

The boundary conditions are that the radial stress is $-P_y$ (the collapse pressure) on the inner radius (r = a) and zero on the outer radius (r = b). Hence, integrating gives,

$$\int_a^b d\sigma_r = \sigma_r(b) - \sigma_r(a) = P_y = \int_a^b \frac{\sigma_y}{r} dr = \sigma_y \log\left(\frac{b}{a}\right) \quad \text{(Tresca)} \qquad (13)$$

which is the usual 'log' solution for a thick cylinder.

Indentation of a Semi-Infinite Slab (Upper Bound)
"Metal forming" type processes usually use the upper bound approach since the issue is ensuring the machine has sufficient load capacity to do its job. A flat indenter of width L is pressed into the surface of a semi-infinite slab of material of shear yield strength τ_y. The indenter is assumed to be sufficiently hard that it is undeformed. A mechanism is postulated consisting of five sliding isosceles triangles, each of whose apex angles is 2α, i.e.,



The blue lines show the displaced triangles – on one side only – due to a downwards displacement of the indenter by Δ. *(Apologies for the distortions – either my skill with WORD is poor, or the software is not up to the job).*

Note: Do not be perturbed by the fact that the mechanism looks impossible due to the corners of some triangles penetrating into solid material. The volumes of these regions are of second order in the small displacement Δ. It can be shown that if the slipping zones are given a finite width the regions of overlap disappear.

Elementary geometry gives the horizontal sliding of four of the five triangles to be $\Delta \tan\alpha$. Similarly it is easily seen that the upward displacement of the outer triangles is $\Delta/2$. The slip distance along the first interface is $\Delta/\cos\alpha$. The slip distance along the other 'long' edges is half this. Also, the long side of the triangles is $L/2\sin\alpha$.

For a width B into the plane of the paper, the work done by the slipping zones (i.e. the sum of the products of the areas of the slipping lines, $\times\tau_y$, times their slip distance) is,

$$\begin{aligned} \text{Work Done} &= 2B\tau_y\left\{\frac{\Delta}{\cos\alpha}\cdot\frac{L}{2\sin\alpha}+2\cdot\frac{\Delta}{2\cos\alpha}\cdot\frac{L}{2\sin\alpha}+\Delta\tan\alpha\cdot L\right\} \\ &= 2B\Delta L\tau_y\left\{\frac{1}{\sin\alpha\cos\alpha}+\tan\alpha\right\} \end{aligned} \tag{14}$$

We must equate this to the work done by the indenter, i.e. $F\Delta$. Hence, the upper bound yield pressure is,

$$\frac{F}{BL} = 2\tau_y\left\{\frac{1}{\sin\alpha\cos\alpha}+\tan\alpha\right\} \tag{15}$$

Choosing equilateral triangles ($\alpha = 30^o$) this becomes,

$$\frac{F}{BL} = \frac{10\tau_y}{\sqrt{3}} = 5.77\tau_y = 3.33\sigma_y \text{ (Mises)} \tag{16}$$

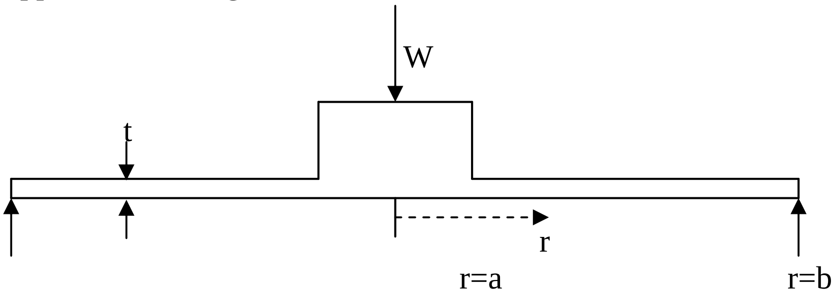
The best solution for our assumed mechanism is given by finding the *minimum* of the above function of angle, {…} in Equ.(15), which is 2.828 for $\alpha = 35.2^o$. This is actually very little different from the equilateral triangle case,

$$\frac{F}{BL} = 5.66\tau_y = 3.27\sigma_y \text{ (Mises)} \tag{17}$$

Assuming that this is not a wildly poor estimate, this implies that the "yield" stress of a material could be estimated (crudely) from a hardness test by multiplying the average indenter pressure by 0.3. Of course, this would be a strain hardened "yield" stress, corresponding to some strain representative of the indentation depth. I don't suggest that it's accurate. Also, hardness test indenters are not flat faced but pointed.

Transverse Load On A Circular Plate

A circular plate of radius b is assumed to be loaded via a central boss of radius a, and is simply supported at its edge:-



The shear load resultant at radius r is clearly $F = \frac{W}{2\pi r}$. (18)

The equilibrium equation for the moment resultant in polar coordinates is,

$$\frac{dM}{dr} + \frac{M}{r} + F = 0 \tag{19}$$

The boundary condition for simple support at the edge is that M(b) = 0. Thus, substituting (18) into (19) and solving with this boundary condition gives,

$$M = \frac{W}{2\pi}\left(\frac{b}{r} - 1\right) \qquad \text{and hence, at r = a:} \qquad M(a) = \frac{W}{2\pi}\left(\frac{b}{a} - 1\right) \tag{20}$$

The moment is largest at r = a. Equating the stress at r = a, i.e. $6M(a)/t^2$, to the yield stress thus gives a lower bound collapse load of,

$$W_y = \frac{2\pi}{6} \cdot \frac{\sigma_y a t^2}{b - a} \tag{21}$$

Note that this solution assumes the shear stress to be negligible compared with the bending stress. Hence (21) will become inaccurate when (b – a) is sufficiently small and shear becomes significant.

Rectangular Section Bar Under End Load and Bending

This provides an example of an "interaction curve" between two independent load resultants, F and M. Define a stress distribution with $+\sigma_y$ in the region $-X < x < X$, where x = 0 is the centre of the section. Let the stress also be $+\sigma_y$ in the region $X < x < t/2$, but $-\sigma_y$ in the region $-t/2 < x < -X$. Hence the net force is $F = 2BX\,\sigma_y$ and the total moment is $M = 2\left(\frac{t}{2} - X\right)\sigma_y\left(\frac{t}{4} + \frac{X}{2}\right) = \sigma_y\left(\frac{t^2}{4} - X^2\right)$. Define the collapse force at zero moment as $F_{y0} = tB\sigma_y$ and the collapse moment at zero force as

$M_{y0} = \frac{\sigma_y Bt^2}{4}$. Hence $\frac{F}{F_{y0}} = \frac{2X}{t}$ and $\frac{M}{M_{y0}} = 1 - \left(\frac{2X}{t}\right)^2 = 1 - \left(\frac{F}{F_{y0}}\right)^2$. Or, re-arranging, $\frac{M}{M_{y0}} + \left(\frac{F}{F_{y0}}\right)^2 = 1$. This is a "parabolic" interaction curve.

Often collapse formulations assume a circular interaction: $\left(\frac{M}{M_{y0}}\right)^2 + \left(\frac{F}{F_{y0}}\right)^2 = 1$.

Clearly the parabolic interaction is more onerous (it is not necessarily the best lower bound).

The definition of reference stress is $\sigma_{ref} = \frac{F}{F_{collapse}}\sigma_y$ or $\sigma_{ref} = \frac{M}{M_{collapse}}\sigma_y$. In the case of more than one applied load, these expressions beg the question as to what value the other load resultant takes. The sensible definition is to assume both loads are scaled in proportion. Defining ξ as the factor which, when applied to both loads, just results in collapse, i.e. $\frac{\xi M}{M_{y0}} + \left(\frac{\xi F}{F_{y0}}\right)^2 = 1$ for our parabolic solution, then the reference stress is given by $\sigma_{ref} = \frac{\sigma_y}{\xi}$.

Thin Pipe Under End Load and Bending (Lower Bound)

The end load is taken to be caused by pressure acting on the end caps (but not on the internal curved surface, i.e. no hoop stress). The mean axial stress is thus Pr/2t.

Inspired by the limit state for a bar in bending alone, we assume the axial stress due to bending (only) to be a constant $\sigma_t > 0$ for angles above $\theta = \theta_o$, and a constant $-\sigma_c < 0$ for angles below $\theta = \theta_o$. Since the bending moment gives rise to no net axial load we have,

$$F = 2\left\{\int_{\theta_o}^{\pi/2} \sigma_t rtd\theta + \int_{-\pi/2}^{\theta_0} (-\sigma_c) rtd\theta\right\} = 2tr\left\{\sigma_t\left(\frac{\pi}{2} - \theta_0\right) - \sigma_c\left(\frac{\pi}{2} + \theta_0\right)\right\} = 0 \qquad (22)$$

Hence,
$$\frac{\sigma_t}{\sigma_c} = \frac{\pi/2 + \theta_0}{\pi/2 - \theta_0} \qquad (23)$$

The bending moment is,

$$M = 2\left\{\int_{\theta_o}^{\pi/2}\sigma_t rt.r\sin\theta.d\theta + \int_{-\pi/2}^{\theta_0}(-\sigma_c)rt.r\sin\theta.d\theta\right\} \tag{24}$$
$$= 2tr^2(\sigma_t + \sigma_c)\cos\theta_0$$

Hence, for bending about the central axis ($\theta_o = 0$) at the limit condition ($\sigma_t = \sigma_c = \sigma_y$) we have a limit moment of $M_y = 4tr^2\sigma_y$ in the absence of any end load.

The axial stress due to the combined end load and moment is,

$$\sigma_z = \frac{Pr}{2t} + \sigma_t = \sigma_y \qquad \text{for} \qquad \theta > \theta_o \tag{25a}$$
$$\sigma_z = \frac{Pr}{2t} - \sigma_c = -\sigma_y \qquad \text{for} \qquad \theta < \theta_o \tag{25b}$$

In the limit state we equate the above axial stresses to $+\sigma_y$ and $-\sigma_y$ respectively, as indicated above. Substituting Equs.(25) into Equ.(23) results in,

$$\theta_0 = -\frac{\pi}{2}\cdot\tilde{p}_a \quad \text{where,} \quad \tilde{p}_a = \frac{Pr}{2t\sigma_y} \tag{26}$$

[NB: $\tilde{p}_a = 1$ represents the limit pressure, in the absence of bending, if pressure acts on the end caps only].

Also, by subtracting Equs.(25a) and (25b) we have $\sigma_t + \sigma_c = 2\sigma_y$. Substituting this together with Equ.(26) into Equ.(24) gives,

$$m = \frac{M}{M_y} = \cos\theta_0 = \cos\left(\frac{\pi}{2}\tilde{p}_a\right) \tag{27}$$

in agreement with Miller and other sources.

Thin Pipe Under Pressure and Bending (Lower Bound, Mises)

To develop a lower bound theorem 'solution' to this problem strictly requires consideration of the radial stress and how all stress components vary with radius, as was done for the case of pressure alone (above). This full solution has been presented by Ainsworth, E/REP/GEN/0027/00, including the case with an additional (non-pressure) end load and covering a thick pipe also.

In this example we consider a simpler solution with the same end result. It applies for the thin pipe case and for pressure plus bending. For a thin pipe we can ignore the radial stress and the radial variation of the other stresses. Using the same assumption for the axial stress as in the previous example, that is,

$$\sigma_z = \frac{\sigma_h}{2} + \sigma_t \ \text{ for } \theta > \theta_0 \ \text{ and } \ \sigma_z = \frac{\sigma_h}{2} - \sigma_c \ \text{ for } \theta < \theta_0 \tag{28}$$

where $\sigma_h = \frac{Pr}{t}$, we find that the Mises stress is given by,

$$2\bar{\sigma}^2 = \sigma_h^2 + \sigma_z^2 + (\sigma_h - \sigma_z)^2 = \frac{3}{2}\sigma_h^2 + 2\sigma_t^2 \quad \text{for } \theta > \theta_0 \tag{29}$$

Surprisingly an identical expression holds for $\theta < \theta_0$ with σ_t replaced by σ_c. Hence, we can make the Mises stress reach yield on both sides of bending by choosing $\sigma_t = \sigma_c$. From Equ.(23) this requires $\theta_o = 0$ to ensure that the net axial load is that due to pressure only.

Putting $\tilde{p}_h^M = \frac{\sqrt{3}}{2} \cdot \frac{\sigma_h}{\sigma_y}$ we see that $\tilde{p}_h^M$ reaches 1 when the Mises limit state is achieved under pressure alone (for a thin pipe). Equ.(29) gives the limit state as,

$$\left(\tilde{p}_h^M\right)^2 + \left(\frac{\sigma_t}{\sigma_y}\right)^2 = 1 \tag{30}$$

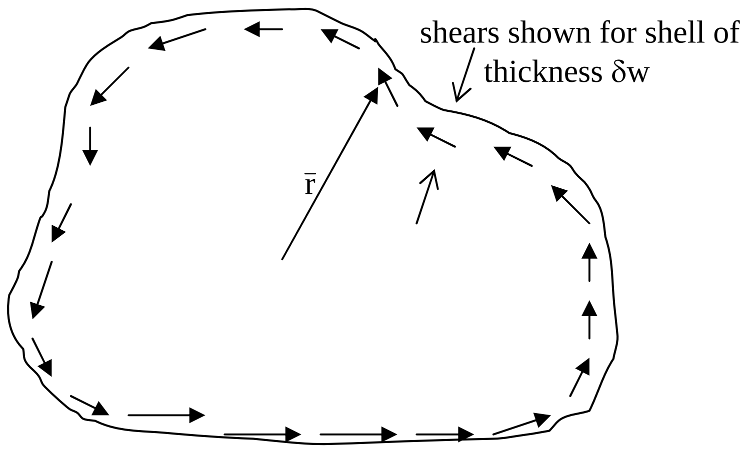
But from Equ.(24) we have $M = 4tr^2\sigma_t$ when $\sigma_t = \sigma_c$ and $\theta_o = 0$. This gives,

$$\left(\tilde{p}_h^M\right)^2 + m^2 = 1 \tag{31}$$

i.e. circular interaction applies in this case, as per Ainsworth Equ.(15). This is as per IMAN#4 and as demonstrated to be accurate by comparison with FEA solutions by Steve Booth.

Torsion of a Shaft With and Without An Axial Split (Lower Bound)

Imagine a shaft with an arbitrary cross section:-



The arrows represent the shears acting over the cross section. Only those on a shell of thickness δw are shown. The magnitude of the shear stress is taken to be at yield, τ_y, everywhere. For any shell defined by a closed path of constant width δw it is clear that this corresponds to zero net force (because the vectorial sum of the force elements is zero by virtue of being a closed path). Dividing the whole cross section into a set of nested "onion skin" shells like this therefore ensures no net force.

To evaluate the torque, consider a vector element of force $\delta\bar{F} = \tau_y \delta w \delta\bar{s}$, where $\delta\bar{s}$ is a vector element of the shell's perimeter. The contribution of this to the torque about some arbitrary origin with respect to which the element under consideration has a position vector $\bar{r}$ is $\delta\bar{T} = \bar{r} \times \delta\bar{F}$. The torque is necessarily in the axial direction since $\bar{r}$ and $\delta\bar{F}$ are both in the plane of the cross section. However we note that $|\bar{r} \times \delta\bar{s}|$ is just twice the area of the triangle formed by the radius r and the element of length δs. Hence the contribution of *one shell* (one onion skin) to the torque is,

$$\delta T = 2\tau_y A \delta w \tag{32}$$

where A is the area *enclosed* by the shell – NOT the area of the shell itself, i.e. A is a 'large' quantity not an infinitesimal quantity.

<u>Example (1): Hollow Circular Shaft</u>: inner & outer radii a and b: At any radius r we have $A = \pi r^2$ and we can identify δw with δr. Thus, the limit torque is,

$$T_y = \int \delta T = \int_a^b 2\tau_y \cdot \pi r^2 \cdot dr = \frac{2\pi}{3}\tau_y \left(b^3 - a^3\right) \tag{33}$$

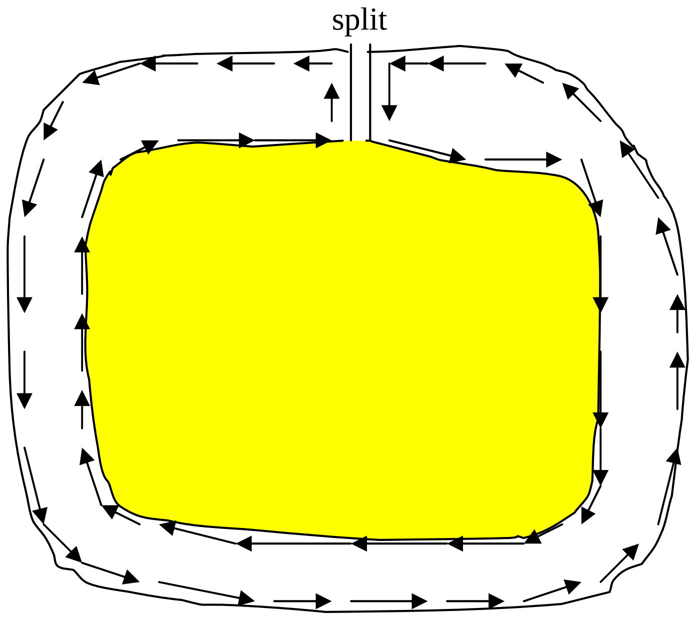
For future use note that for a thin shaft, i.e. for t = b – a << a, we have approximately, $T_y^{thin} \approx 2\pi\tau_y r^2 t$.

Example (2): Hollow Square Shaft: Let the half-side be B on the outside and A on the inside. If the half-side of some intermediate square shell is x, we can identify δw with δx, and the area A with $(2x)^2$. Hence, the limit torque is,

$$T_y = \int_A^B 2\tau_y \cdot 4x^2 \cdot dx = \frac{8}{3}\tau_y\left(B^3 - A^3\right) \qquad (34)$$

Comparing with (33) we see that the square section has a limit torque which is a factor $4/\pi$ larger than that of the in-scribed circular shaft of the same thickness. (I wonder whether this is realistic given the distortions that I would anticipate for a square section).

Example (3): A Shaft With An Axial Through-Thickness Split
The purpose of this example is to demonstrate the radical reduction in torsion strength that results from an axial split. The reason is that the pattern of shears shown in the above figure has to be modified in the presence of the split – so as to ensure that the new free surface represented by the split has no traction across it. In the general case the situation pictorially is…



The major difference is the reversal of the direction of the shears caused by the new free surface at the split. This means that the area within the "onion skin" does *not* include the area within the 'bore' - shown shaded yellow above – whereas previously it did.

For a hollow section of thickness 't', consider an "onion skin" at a depth 'w' below the surface. Consider a thin shaft for simplicity. If the perimeter of the shell (in one direction only) is 's', the area within the "onion skin" is $A \cong s(t-2w)$, in the thin approximation. The limit torque is thus,

$$T_y \approx 2\tau_y s \int_0^{t/2} (t - 2w)dw = \frac{\tau_y s t^2}{2} \qquad (35)$$

where s is the perimeter of the shaft (unsplit).

For example, in the case of a split, thin, hollow, circular shaft, we have s = 2πr and hence the limit torque is,

$$T_y^{split} \approx \pi \tau_y r t^2 \tag{36}$$

Comparing with the result for a thin unsplit circular shaft we find that the split has caused a strength reduction by a factor of,

$$\frac{T_y^{split}}{T_y^{unsplit}} = \frac{t}{2r} \tag{37}$$

i.e. to a very small fraction of the unsplit strength for a thin shaft.