Appendix B1:
A Cloud Of Gas Collapsing Under Gravity: Entropy Considerations

1. Introduction

The major issue in the formation of a star from a cloud of gas is the requirement for a cooling mechanism. In this Appendix we shall show that the requirement for a cooling mechanism follows from a most basic necessity, namely satisfying the second law of thermodynamics.

2. The Gas

Consider a system of N particles of mass m in a gravitationally bound state. We shall assume thermodynamic equilibrium prevails *within* the gas cloud (though not necessarily between the cloud and its surroundings – see below). Thus, the gas has some temperature, T. The average kinetic energy per particle is, for non-relativistic motion in three spatial dimensions, 3kT/2. The virial theorem tells us that the average total kinetic energy, K = 3NkT/2, is –½P, where P is the average total potential energy. Since, for a large gas cloud, the total potential energy will not vary significantly in time, we can drop the requirement for averaging. We thus have,

$$P = -3NkT \tag{1}$$

But the gravitational potential energy of a spherically symmetric mass distribution of total mass M = Nm and radius R is,

$$P = -\eta \frac{GM^2}{R} \tag{2}$$

where η is a dimensionless constant of order unity depending on the distribution (e.g. η = 3/5 for uniform density). In terms of the volume of the cloud,

$$V = \frac{4\pi}{3} R^3 \tag{3}$$

we thus have,

$$kT = \frac{\eta}{3}\left(\frac{4\pi}{3}\right)^{1/3} Gm^2 \cdot \frac{N}{V^{1/3}} \tag{4}$$

Now the entropy of an ideal gas is given by the Sackur-Tetrode equation…

$$S_{gas} = Nk\left\{\frac{5}{2} + \log\left[\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \cdot \frac{V}{N}\right]\right\} \tag{5}$$

This gives the absolute total entropy. For our purposes we only need know the change in entropy between two states of different volume and temperature. Thus we can simplify (5) to,

$$S_{gas} = Nk \log\left[VT^{3/2}\right] + \text{constant} \tag{6}$$

Using (4) we can substitute for T in terms of V, and hence (6) this simplifies further to,

$$S_{gas} = Nk\log\left[V^{1/2}\right] + \text{constant} = \frac{Nk}{2}\log[V] + \text{constant} \qquad (7)$$

In passing we note that the total energy of the gas cloud is,

$$E_{gas} = K + P = \frac{P}{2} = -\frac{3}{2}NkT = -\frac{\eta}{2}\cdot\frac{GM^2}{R} \qquad (8)$$

From (8) we note that,

(a) The total energy of the gas is negative (inevitably because it is a gravitational bound system – we could not have employed the Virial Theorem otherwise);

(b) As the cloud collapses (R reduces) the magnitude of the energy increases, i.e. the absolute energy E_{gas} decreases;

(c) As the cloud collapses (R reduces) the temperature increases.

(d) From (b) and (c) it follows that the gas cloud has a negative specific heat (the temperature increases as energy is removed from it);

Whilst from (7) it follows that,

(e) As the cloud collapses (R and V reduce) the entropy of the gas <u>decreases</u>.

This latter observation is initially disconcerting due to the apparent[1] violation of the second law of thermodynamics. It is a consequence of the reduction in volume winning over the increase in temperature, due to (4) and (6). If the temperature increased with decreasing volume faster than $T \propto V^{-2/3}$ then the entropy of the gas would increase. However, the relatively modest rate of increase in temperature given by (4), i.e. $T \propto V^{-1/3}$, yields a decreasing gas entropy.

Note that the assumption underlying our derivations is that the gas cloud remains gravitationally bound as it collapses [otherwise (1) would not be true, since it is derived from the Virial Theorem, 2K + P = 0]. This assumption leads to the requirement that the gas cloud loose energy as it collapses.

If, alternatively, we required that the gas cloud collapse without loosing energy, then by *reductio ad absurdum* it follows that 2K + P is not zero, and hence that the cloud ceases to be gravitationally bound.

3. The Radiation

[1] Apparent, not real, as we see in the next Section.

It is clear that the problem posed in Section 2 is incomplete in the sense that the conditions imposed lead to the conclusion that the gas looses energy but how it does so and where the energy goes has not been specified. If we imagine our gas cloud to be surrounded by the vacuum of interstellar space, and given that we are assuming the cloud is gravitationally bound throughout (and hence cannot eject matter), it follows that the only means of loosing energy is via electromagnetic radiation. The second law of thermodynamics is preserved if this radiation carries off more entropy than the gas cloud looses. This is what we shall check in the following Section. In this Section we derive the entropy of the radiation.

Since the gas is in internal thermodynamic equilibrium (by assumption – this being the condition for its temperature to be defined) it follows that the radiation emitted is black body radiation. In black body radiation the energy density is proportion to T^4, i.e.,

$$\text{radiant energy density} = U = AT^4 \qquad (9)$$

The entropy of a given volume V_r of radiation is found from the definition,

$$dS_{radiation} = \frac{dQ}{T} \qquad (10)$$

where $dQ = V_r dU$. Actually, (10) is all that we shall require, but for completeness we note that,

$$dQ = V_r.d(AT^4) = 4AV_r T^3 dT \qquad (11)$$

so (10) gives,

$$S_{radiation} = \int \frac{4AV_r T^3 dT}{T} = \int 4AV_r T^2 dT = \frac{4}{3} AV_r T^3 = \frac{4}{3} \cdot \frac{Q}{T} \qquad (12)$$

4. The Second Law of Thermodynamics

We can now examine if the overall entropy increases or decreases. From (7) we find the change of the entropy of the gas for a given small change in volume,

$$dS_{gas} = \frac{Nk}{2V} dV \qquad (13)$$

The radiation energy increase, dQ, is minus the change in the total energy of the gas, which from (8) is,

$$dQ = -dE_{gas} = \frac{3}{2} NkdT \qquad (14)$$

Using (4) we can write,

$$kT = \frac{B}{V^{1/3}} \Rightarrow kdT = -\frac{1}{3} \cdot \frac{B}{V^{4/3}} = -\frac{1}{3} \cdot \frac{kT}{V} dV \qquad (15)$$

Hence, substituting (15) into (14) gives,

$$dQ = -\frac{1}{2} \cdot \frac{NkT}{V} dV \qquad (16)$$

and hence, using (10) for the entropy change of the radiation we find,

$$dS_{radiation} = -\frac{1}{2} \cdot \frac{Nk}{V} dV \qquad (17)$$

Adding (13) and (17) we find the change in the total entropy of the gas plus the radiation for a change in gas volume of dV to be zero, i.e.,

$$dS_{TOTAL} = dS_{gas} + dS_{radiation} = \frac{1}{2} \cdot \frac{Nk}{V} dV + \left(-\frac{1}{2} \cdot \frac{Nk}{V} dV \right) \equiv 0 \qquad (18)$$

Thus, the overall change in entropy is zero, consistent with the second law of thermodynamics in the limiting case of a reversible change. Q.E.D.

[Aside: This implies that we could return to the original gas volume and temperature by applying radiant heat from outside].

[NB: There may be some confusion regarding the possible use of (12) rather than (10) for the entropy change of the radiation. The former may seem to imply that $dS_{radiation}$ = (4/3)dQ/T, rather than simply dQ/T. The paradox is resolved by noting that the temperature also changes when the volume changes. Taking this into account using (9) shows that both approaches result in dQ/T].

5. Implications For Star Formation

It follows from the above discussion that if the gas cloud were opaque, so that radiation could not escape from it, it could not collapse since there would be no mechanism for obeying the second law of thermodynamics. In practice, it will always be possible to loose heat at some rate, if only from the surface of the gas cloud. The lesson we learn from the preceding analysis is that the rate at which the gas cloud collapses will be controlled by the rate at which heat can be radiated away from it. The latter will depend upon the opacity of the gas and upon whatever other heat transport mechanisms might be active (e.g. convection). In these circumstances, i.e. when the gas is not perfectly transparent to radiation, there will inevitably be variations in the temperature of the gas, and hence also its pressure and density, at different depths. Here we have the beginnings of star formation.