# Appendix A0
# Black Body Radiation of Photons and Relativistic Fermions

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**1. Overview**: In this opening Appendix we derive from first principles formulae for the black body spectrum, energy density, particle density, entropy, radiation pressure and radiation flux. Initially this is done for photons (i.e. for electromagnetic radiation). This derivation is expressed in such a way that it applies to any boson field in the limit that virtually all particles have extreme relativisitic energies ($kT >> mc^2$), so the particles may be approximated as massless. The derivation is then repeated for fermions. Again the formulae are developed for the extreme relativistic (or massless) limit. It is shown that the energy density and entropy of a black body fermion field is a factor 7/8 less than that for a boson field (per particle species and per spin state). The particle density for relativistic fermions is a factor ¾ smaller than that for relativistic bosons (per particle species and per spin state). The derivations are carried out so that the generalisation to non-relativistic energies, when the formulae become dependent upon particle mass, is clear. Closed form expressions are not derived for this general case only because the integrals involved are not conducive to this (*at least I think not…perhaps I should try harder?*).

## 2. Density of Modes

Fields obeying the wave equation, or the Klein-Gordon equation, are confined to a rectangular box with Dirichlet boundary conditions on the walls, have modes with spatial variations given by,

$$\phi = \phi_0 \sin\left(\frac{n\pi}{L_x}\right)\sin\left(\frac{m\pi}{L_y}\right)\sin\left(\frac{p\pi}{L_z}\right) \tag{1}$$

i.e. with $$\bar{k} = \left(\frac{n\pi}{L_x}, \frac{m\pi}{L_y}, \frac{p\pi}{L_z}\right) \quad \text{with } n, m, p = 1, 2, 3, 4, \ldots. \tag{2}$$

Hence, each mode occupies a volume $\pi^3 / V$ in k-space, where $V = L_x L_y L_z$ is the volume of the box in real space.

The derivation of black body formulae is correct only for large volumes, V – or more accurately, when all three dimensions L are large. In this case the spacing between modes in k-space, $\pi/L$, can be taken to be arbitrarily small compared with the typical wavenumbers of the field, i.e. $\pi / L << k \approx k_B T / \hbar c$. The density of modes in k-space is thus 1 per k-volume $\pi^3 / V$.

Noting that it is only the +,+,+ octant of k-space that contributes possible modes (i.e. reversing the sign of any component of k reproduces the same mode), the total number of modes with $k = |\bar{k}|$ between k and k + dk is,

$$dN = \frac{1}{8}.4\pi k^2 dk.\frac{1}{\frac{\pi^3}{V}} = \frac{V}{2\pi^2} k^2 dk \qquad (3)$$

Now Eq.(3) assumes that fields are uniquely defined for a given particle type by (1). Where particles come in $N_s$ spin states, and $N_a$ antiparticle states, then (3) becomes,

$$dN = N_s N_a \frac{V}{2\pi^2} k^2 dk \qquad (4)$$

Thus, $N_a = 2$ if a particle is distinct from its anti-particle, otherwise $N_a = 1$. The number of spin states for non-zero mass particles of spin L is generally $N_s = 2L+1$. Thus for electrons, protons and neutrons we have $N_a = 2$ and $N_s = 2$. For photons, their masslessness means that they have only two spin states despite being spin 1, so $N_a = 1$, $N_s = 2$. Similarly, massless neutrinos of spin ½ do not have two spin states, but only one[1], so the conventional prescription is $N_a = 2$, $N_s = 1$[2]. For the scalar pions the neutral pion is its own antiparticle, whereas the antiparticle of the + pion is the – pion, so the total number covering all pions is 3.

Note that (4) remains the correct density of modes in k-space even if the particle in question is massive ($M \neq 0$). What changes in this case is the expression for the energy, or the frequency, in terms of k, i.e.,

$$E^2 = (\hbar\omega)^2 = (\hbar kc)^2 + (Mc^2)^2 \qquad (5)$$

Hence, if we wish to express the density of modes per unit energy range, dE, or per unit frequency range, $d\omega$, then Equ.(5) needs to be used.

## 3. Expected Mode Occupancy Number for Bosons

The defining characteristic of bosons is that each mode may be occupied by any integral number of bosons, i.e. the occupancy may be 0, 1, 2, 3,… Consider any specified mode which has a frequency $\omega$. [NB: This means specifying n, m, p in Equ(1,2)]. If there are j bosons in this mode, the total energy associated with this single mode is $j\hbar\omega$.

Note that the occupancy integer, j, has nothing at all to do with the mode defining integers n,m,p. Nor are we making any assumptions about the dependence of the energy (or frequency, $\omega$) on the mode numbers n,m,p. Our derivation of the occupancy number is true for the general expression, Equ.(5) – or indeed for any other relation between frequency and wavefunction.

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[1] This is not certain, but is consistent with all experimental observations to date (as far as I am aware). In gauge theories of the electroweak force, only one helicity state of the neutrino couples to the other fields. However, this does not mean that neutrinos of the opposite helicity do not exist. However, such a particle would interact only via gravity. Such a particle is not logically ruled out.

[2] Neutrinos are normally taken to be distinct from their antiparticles. However I am not sure that this is established. If neutrinos were identical with their antiparticles (Majorana neutrinos) then lepton number would not be conserved.

The probability that the occupancy of any state of frequency ω is j at temperature T is proportional to the Boltzmann factor, i.e.,

$$P_j(\omega, T) \propto e^{-j\hbar\omega/k_B T} = r^j, \quad \textit{where} \quad r = e^{-\hbar\omega/k_B T} \tag{6}$$

Normalisation is achieved by dividing by,

$$\sum_j P_j = \sum_0^\infty r^j = \frac{1}{1-r} \tag{7}$$

so that,

$$P_j(\omega, T) = (1-r)e^{-j\hbar\omega/k_B T} = (1-r)r^j \tag{8}$$

The expectation value (mean) of the occupation number of any mode of frequency ω is thus,

$$\langle j \rangle = \sum_j jP_j = \sum_0^\infty (1-r)jr^j \tag{9}$$

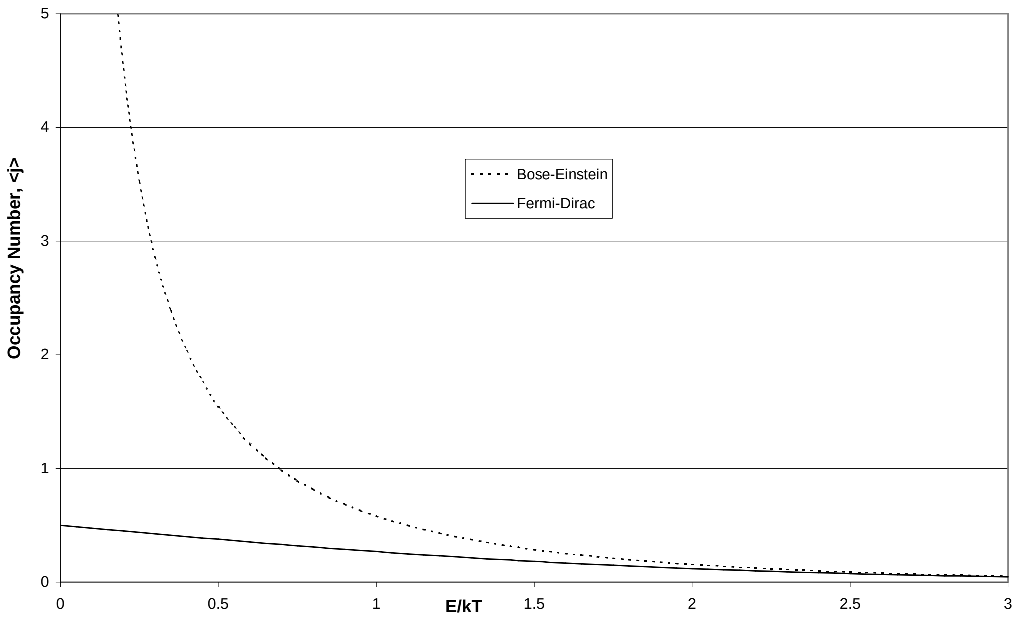
But,

$$\begin{aligned} \sum_0^\infty jr^j &= -\frac{\partial}{\partial\alpha}\sum_0^\infty e^{-\alpha j} = -\frac{\partial}{\partial\alpha}\left(\frac{1}{1-e^{-\alpha}}\right) \\ &= \frac{e^{-\alpha}}{\left(1-e^{-\alpha}\right)^2} = \frac{r}{(1-r)^2} \end{aligned} \tag{10}$$

where, $\alpha = \hbar\omega / k_B T$. Hence, (9) gives the expectation value of the mode occupancy number to be,

$$\langle j \rangle = \frac{r}{1-r} = \frac{e^{-\alpha}}{1-e^{-\alpha}} = \frac{1}{e^{\alpha}-1} = \frac{1}{e^{\hbar\omega/k_B T}-1} \tag{11}$$

which is the Bose-Einstein distribution. It is plotted below, in comparison with the Fermi-Dirac distribution (derived later).



The qualitative features of the two distributions are,

- The Bose-Einstein occupancy of modes with energy much less than $k_B T$ is divergent;
- The Fermi-Dirac occupancy at these low energies is 0.5;
- For $E = k_B T$ the Bose-Einstein occupancy is 0.582, more than double that for fermions, which is 0.269.
- The two distributions become the same at energies much higher than $k_B T$, when the occupancies are small (<0.05 or so).

**4. The Black Body Spectrum (Bosons)**

The energy associated with all modes with wavenumbers in the range k to k+dk is the product of the number of modes in this range, the occupancy of these modes, and the energy of each mode, i.e., multiplying (4) by (11) and by the energy $\hbar kc$,

$$d(\textit{Energy in volume V and range dk}) = N_s N_a \frac{V}{2\pi^2} \cdot \frac{\hbar c k^3 dk}{e^{\hbar kc / k_B T} - 1} \qquad (12)$$

In general (12) is not the energy spectrum because the distribution of energy has been parameterised against the mode descriptor k, rather than the frequency, ω. For massless particles, such as photons, or in the extreme relativistic limit when the rest mass energy is negligible compared with $k_B T$, the general dispersion relation (5) can be simplified to $E = \hbar\omega = \hbar kc$, and hence the black body energy distribution (12) can be re-written as the spectrum,

$$d\xi = \frac{N_s N_a}{2\pi^2 c^3} \cdot \frac{\hbar \omega^3 d\omega}{e^{\hbar\omega / k_B T} - 1} \qquad (13)$$

where we have divide by the volume V to give the energy density, ξ, on the LHS. Equ(13) applies to any bosons in the far relativistic limit, $k_B T >> Mc^2$. For the particular case of the electromagnetic field (photons) we get,

$$d\xi = \frac{1}{\pi^2 c^3} \cdot \frac{\hbar \omega^3 d\omega}{e^{\hbar\omega / k_B T} - 1} \qquad (14)$$

**5. Total Black Body Energy Density and Stefan's Constant**

The total energy density, integrated over all frequencies, is (for photons),

$$U(T) = \int d\xi = \frac{1}{\pi^2 c^3} \cdot \int_0^\infty \frac{\hbar \omega^3 d\omega}{e^{\hbar\omega / k_B T} - 1} = \frac{\hbar}{\pi^2 c^3} \cdot \left(\frac{k_B T}{\hbar}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} \qquad (15)$$

The following integrals are useful in what follows;

$$\int_0^\infty \frac{x^{n-1} dx}{e^x - 1} = \Gamma(n)\varsigma(n) \quad \textit{and} \quad \int_0^\infty \frac{x^{n-1} dx}{e^x + 1} = \left(1 - 2^{1-n}\right)\Gamma(n)\varsigma(n) \qquad (16)$$

where the gamma function is given for integer arguments by $\Gamma(n) = (n-1)!$ and the zeta function for the first few integer arguments >1 is,

| **n** | **ζ(n)** |
|---|---|
| 2 | $\pi^2/6$ = 1.645… |
| 3 | 1.202056… |
| 4 | $\pi^4/90$ = 1.0823… |
| 5 | 1.0369…… |
| 7 | 1.0083492773… |

Hence we have in particular,

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \tag{17}$$

Substitution of (17) into (15) gives the total black body photon energy as,

$$U(T) = \frac{\pi^2 k_B^4}{15 c^3 \hbar^3} T^4 \tag{18}$$

We derive below that the energy flux, J (or power per unit area) is related to the total energy density U by J = ¼ Uc. Now Stefan's law for black body radiation is,

$$J = \sigma T^4 \tag{19}$$

Hence, (18) and (19) allow Stefan's constant to be found in terms of the fundamental constants,

$$\sigma = \frac{\pi^2 k_B^4}{60 c^2 \hbar^3} \tag{20}$$

Substitution of the numerical values,

$k_B = 1.38 \times 10^{-23}$ J/$^o$K; $\hbar = 1.054 \times 10^{-34}$ Jsec $c = 3 \times 10^8$ m/sec

give, $\sigma = 5.66 \times 10^{-8}$ Wm$^{-2o}$K$^{-4}$, which is the correct value.

Comparing (13) and (14) it is clear that the total energy density for any boson field in the far relativistic limit ($k_B T >> Mc^2$) is,

$$U(T) = \left(\frac{N_s N_a}{2}\right) \frac{\pi^2 k_B^4}{15 c^3 \hbar^3} T^4 \tag{21}$$

## 6. Entropy of a Black Body Distribution

The general relation between entropy, total energy and temperature is,

$$S = \int \frac{dU}{T} \tag{22}$$

Hence, from (21), forming dU will produce a term in $4T^3dT$, which when divided by T gives $4T^2dT$, and this integrates to 4T3/3. Taking the entropy to be zero at absolute zero, (22) with (21) thus give,

$$S = \left(\frac{N_s N_a}{2}\right)\frac{4\pi^2 k_B^4 V}{45c^3\hbar^3}T^3 = \frac{4}{3}\cdot\frac{U}{T}V \tag{23}$$

as the entropy of a volume V of bosons at temperature T. In terms of Stefan's constant this is,

$$S = \frac{8N_s N_a}{3c}\sigma V T^3 \tag{24}$$

Equ(24) applies for any massless boson, or for bosons in the far relativistic limit, ($k_BT >> Mc^2$).

## 7. Number of Particles Per Unit Volume

The number of particles in modes with wavenumbers in the range k to k+dk is the product of the number of modes in this range and the occupancy of these modes, i.e., multiplying (4) by (11) gives, in the zero-mass or far relativistic regime,

$$dN_p = N_s N_a \frac{V}{2\pi^2}\cdot\frac{k^2 dk}{e^{\hbar kc/k_BT} - 1} \tag{25}$$

Where the subscript $_p$ is to distinguish the number of particles, $N_p$, from the number of modes, N. Integrating gives the total number of particles,

$$N_p = N_s N_a \frac{V}{2\pi^2}\cdot\int\frac{k^2 dk}{e^{\hbar kc/k_BT} - 1} = N_s N_a \frac{V}{2\pi^2}\cdot\left(\frac{k_BT}{\hbar c}\right)^3\int_0^\infty \frac{x^2dx}{e^x - 1} \tag{26}$$

and from (16) we have,

$$\int_0^\infty \frac{x^2dx}{e^x - 1} = 2.4041... \tag{27}$$

hence,

$$N_p = 2.404 N_s N_a \frac{V}{2\pi^2}\cdot\left(\frac{k_BT}{\hbar c}\right)^3 = 0.1218 N_s N_a V.\left(\frac{k_BT}{\hbar c}\right)^3 \tag{28}$$

Note that we could have made a rough estimate of the number of particles from the total energy divided by a typical energy per particle, i.e. $k_B T$. From (21) this gives,

*crude estimate* $$N_p \approx \left(\frac{N_s N_a}{2}\right)\frac{\pi^2 k_B^3 V}{15 c^3 \hbar^3} T^3$$

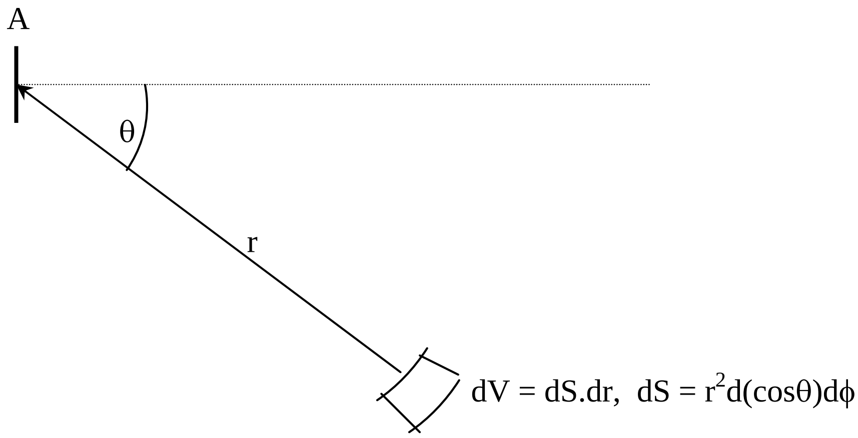
This crude estimate is an over-estimate by x2.7 compared with the correct integration. To put this another way, the true average energy per particle is not $k_B T$ but $2.7 k_B T$ in the case of relativistic bosons, e.g. photons.

**8. Derivation of Radiation Flux and Radiation Pressure Formulae**

Geometrical factors of ¼ or $^1/_3$ occur in formulae for radiation flux and pressure respectively. These are derived here.

Flux of Particles is defined as the number of particles which pass through unit area per unit time from one side only. Thus, the particle flux is also the number of particles hitting the walls of the container of the radiation per unit time per unit area. (If we had defined flux by counting particles from both sides of the element of area, the wall flux would have been half this flux). We consider particles of a specified energy, E, (within dE) and denote the particle flux by $J_E^P$, where the subscript identifies the energy and the superscript identified a particle flux (as opposed from an energy flux which we will consider below).

Consider an element of area A whose normal defines a polar coordinate z direction. Consider all the particles of energy E (within dE) which pass through A from (say) the RHS between times t and t + dt of some arbitrary datum time. Since we are considering only particles of energy E it follows that they all have the same speed, v. hence they all originate at time zero from the same distance r = vt from A (in the limit that A is small compared with r).




Consider a small volume element dV at orientation θ from A. The proportion of the particles within this volume at time 0 which hit A at time t is given by the solid angle subtended by A at dV, divided by 4π, i.e. the fraction is $A\cos\theta / 4\pi r^2$. If we denote the density of particles with energy E (within dE) by $\rho_E^P$ it follows that the number of particles hitting A from the RHS in time dt is,

$$J_E^P A dt = \rho_E^P \int dV . \frac{A\cos\theta}{4\pi r^2} = \rho_E^P \int_{\theta=0}^{\theta=\pi/2} \frac{A\cos\theta}{4\pi r^2} dr.r^2 d(\cos\theta) d\phi \tag{29}$$

Note that dr = vdt. The integral over $\phi$ divided by $4\pi$ gives a factor of ½ (which is simply because we are including particles from the right-hand side only). Hence (29) becomes,

$$J_E^P = \rho_E^P v \int_0^1 \frac{c}{2} dc = \frac{1}{4} \rho_E^P v \tag{30}$$

i.e. we get another factor of ½ from averaging over the resolved direction of the particles, i.e. the average velocity component normal to A is v/2.

If we now want the total particle flux, irrespective of energy, we would, in general, have to integrate (30), i.e.,

$$J^P = \frac{1}{4} \int_{Mc^2}^{\infty} \rho_E^P v dE \tag{31}$$

where v is the velocity at energy E. For zero mass particles, or massive particles in the extreme relativistic regime, this is simplified because the particles all have speeds close to c. Hence we have simply,

$$J^P = \frac{1}{4} \rho^P c \tag{32}$$

i.e. the total particle flux is obtained from the total particle density (counding particles of all energies) by multiplying by c/4.

<u>The Energy Flux</u> due to particles with energies around E is clearly,

$$J_E = \frac{1}{4} E \rho_E^P v = \frac{1}{4} \rho_E v = \frac{1}{4} \xi v \tag{33}$$

the absence of the superscript P indicating that this J is an energy flux, and the presence of the subscript E meaning that it is the flux due to particles with individual energies in the range E to E + dE. Equ(3) relates this flux to both the particle density and the energy density spectrum, i.e.,

$$E \rho_E^P = \rho_E \equiv \xi \tag{34}$$

In the general case we would need to integrate (33) over energies to get the total energy flux from particles of all individual energies (i.e. mode frequencies), thus,

$$J = \frac{1}{4} \int_0^{\infty} \xi v d\omega \tag{35}$$

Once again this simplifies in the case of massless particles, or in the extreme relativistic limit, when all particles are moving at close to (or exactly) c:-

$$J = \frac{1}{4} Uc \tag{36}$$

which is the result used above to derive (20) for Stefan's constant.

The Radiation Pressure is derived in a similar fashion. Once again we consider only particles (radiation) approaching from one side. (If both sides were considered, the pressure would obviously be zero for isotropic radiation). We also assume spectral reflection, so each particle produces an impulse on the barrier twice that which would apply if the particle were absorbed. At first sight this seems arbitrary, and probably incorrect. However, if we are dealing with a radiation field in equilibrium with the container walls, it is perfectly reasonable. This is because every particle which is absorbed must be re-emitted later.

Considering again the particles with energy E and speed v hitting area A from an angle of θ, they each produce an impulse on A of 2pcos(θ), where p is the particle momentum. For a massive and non-relativistic particle, the moment is given by,

$$p = \sqrt{(E/c)^2 - (Mc)^2} \tag{37}$$

The impulse is thus,

$$\begin{aligned} d(\text{impulse}) &= \rho_E^P \int_{\theta=0}^{\theta=\pi/2} 2p\cos(\theta) \frac{A\cos\theta}{4\pi r^2} vdt.r^2 d(\cos\theta) d\phi \\ &= \rho_E^P A dt \int_{\theta=0}^{\theta=\pi/2} p\cos^2(\theta) v d(\cos\theta) \\ &= \frac{1}{3}\rho_E^P A p v dt \end{aligned} \tag{38}$$

Hence we see that the expected geometrical factor of 1/3 arises from averaging over a product of two cos factors; one arises from the proportion of particles hitting A and originating at an angle to the normal to A; the other arises due to having to resolve the component of the impulse normal to A. Because force is impulse/time and pressure is force/area we deduce that the pressure due to particles of energy E is,

$$P_E = \frac{1}{3}\rho_E^P pv = \frac{4}{3} p J_E^P \tag{39}$$

Once again, below the extreme relativistic limit, an integral is required to find the total pressure. That is, both p and v vary with E in a manner which is not simply proportional. However, for massless particles, or in the extreme relativistic limit, we have v = c and p = E/c, so that we get the total radiation pressure to be,

$$P = \frac{1}{3}\rho_E^P E = \frac{\xi}{3} \tag{40}$$

i.e. the radiation pressure is just the energy density / 3.

## 9. Black Body Field of Fermions

Mode Occupancy: This is (even) simpler than for bosons because the defining characteristic of fermions is that the possible occupancies are 0 or 1 only. The probability that a mode of energy E is occupied is proportional to $e^{-E/k_B T}$, compared to a relative probability of unity for not being occupied. Hence, the normalised probabilities are,

Occupied: $$P(1) = \frac{e^{-E/k_B T}}{1 + e^{-E/k_B T}} \tag{41}$$

Unoccupied: $$P(0) = \frac{1}{1 + e^{-E/k_B T}} \tag{42}$$

And the expectation value of the occupancy is simply 0 x P(0) + 1 x P(1) = P(1), i.e.,

$$< j >= \frac{e^{-E/k_B T}}{1 + e^{-E/k_B T}} = \frac{1}{e^{E/k_B T} + 1} \tag{43}$$

which is the Fermi-Dirac distribution (plotted on Page 3).

Density of Modes: There is no difference between fermions and bosons in respect of the density of modes in k-space, i.e. Equ.(4) also applies. As was the case with bosons, translating the density in k-space into a density in frequency space (i.e. to derive a spectrum) becomes messy when the particle is massive and not extremely relativistic.

Black Body Spectrum for Massless or Extremely Relativistic Fermions
Equ.(12) is just the same except the Fermi-Dirac distribution is substituted for the Bose-Einstein distribution. Hence, the energy due to fermion particles of individual energy E to E + dE is,

$$d(\textit{Energy in volume V and range dE}) = N_s N_a \frac{V}{2\pi^2 c^3 \hbar^3} \cdot \frac{E^3 dE}{e^{E/k_B T} + 1} \tag{44}$$

or, in terms of the energy density spectrum,

$$d\xi = \frac{N_s N_a}{2\pi^2 c^3 \hbar^3} \cdot \frac{E^3 dE}{e^{E/k_B T} + 1} \tag{45}$$

Integrating to find the total energy density gives,

$$d\xi = \frac{N_s N_a}{2\pi^2 c^3 \hbar^3} \cdot \int \frac{E^3 dE}{e^{E/k_B T} + 1} = \frac{N_s N_a}{2\pi^2 c^3 \hbar^3} (k_B T)^4 \int_0^\infty \frac{x^3 dx}{e^x + 1} \tag{46}$$

From (16) we have that $\int_0^\infty \frac{x^3 dx}{e^x + 1} = \left[1 - 2^{-3}\right]\Gamma(4)\varsigma(4) = \frac{7}{8} \cdot \frac{\pi^4}{15}$ (47)

hence the Fermion black body total energy density is,

$$\xi = \frac{7}{8} \cdot \frac{\pi^2 N_s N_a}{30 c^3 \hbar^3} (k_B T)^4 \tag{48}$$

and comparing with (21) we see that the Fermion energy density spectrum is just a factor of 7/8 times that of a boson field at the same temperature. The origin of this factor is the ratio of the integrals,

$$\frac{\int_0^\infty \frac{x^3 dx}{e^x + 1}}{\int_0^\infty \frac{x^3 dx}{e^x - 1}} = \frac{7}{8} \tag{49}$$

We would expect the Fermion energy density to be less because the Fermion occupancy <j> is less than the boson occupancy at all energies, as illustrated by the graph on Page 3.

The same factor of 7/8 works through to the total entropy which may be evaluated in an identical fashion to that for bosons above, i.e., for fermions,

$$S = \frac{7}{8}\left(\frac{N_s N_a}{2}\right)\frac{4\pi^2 k_B^4 V}{45 c^3 \hbar^3} T^3 = \left(\frac{N_s N_a}{2}\right)\frac{7\pi^2 k_B^4 V}{90 c^3 \hbar^3} T^3 \tag{50}$$

The number of particles involves a different integral, however, and in this case is a factor of ¾ times that for bosons,

$$dN_p = \frac{N_s N_a V}{2\pi^2 c^3 \hbar^3} \cdot \int \frac{E^2 dE}{e^{E/k_B T} + 1} = \frac{N_s N_a V}{2\pi^2 c^3 \hbar^3} (k_B T)^3 \int_0^\infty \frac{x^2 dx}{e^x + 1} \tag{51}$$

From (16) $\int_0^\infty \frac{x^2 dx}{e^x + 1} = \left[1 - 2^{-2}\right]\Gamma(3)\varsigma(3) = \frac{3}{4} x 2.4041 = 1.803$ (52)

Giving,

$$dN_p = 1.803 \frac{N_s N_a V}{2\pi^2 c^3 \hbar^3} (k_B T)^3 = 0.0913 \frac{N_s N_a V}{c^3 \hbar^3} (k_B T)^3 \tag{53}$$

which is just ¾ times (28) for bosons. This originates from the ratio,

$$\frac{\int_0^\infty \frac{x^2 dx}{e^x + 1}}{\int_0^\infty \frac{x^2 dx}{e^x - 1}} = \frac{3}{4} \qquad (54)$$