

All Finite (Anti)Hermitian Irreducible Representations of the de Sitter and Anti-de Sitter Lie Algebras and Their Lorentz Structure

R.A.W. Bradford

University of Bristol, Faculty of Engineering

Abstract

Because of the importance of unitarity in quantum physics, work on the representations of the de Sitter group has focussed on the unitary case, which necessarily means infinite dimensional matrices for this non-compact group. Here we address the finite dimensional representations resulting from the requirement that the Lie algebra generators are either Hermitian or anti-Hermitian. The complete classification of all such irreducible representations is found and their matrix elements specified. These irreducible representations (irreps) are based on “backbones” defined as the homogeneous Lorentz sub-algebra and consisting of direct sums of the finite irreps of the homogeneous Lorentz algebra (HLA). Only two types of such “backbones” arise (see 5.1a,b herein). Consequently, only certain dimensions of representation are possible, namely 4, 5, 10, 14, 20, 30, 35, 55, 56, 91... or generally either $\frac{1}{6}N(N+1)(N+2)$ or $\frac{1}{6}N(N+1)(2N+1)$ where $N = 2,3,4 \dots$ is the number of HLA irreps in the “backbone” (minimum 2). The two Casimir invariants can be specified in terms of a single integral or half-integral parameter, p . For irreps based on (5.1a), $-C1 = p(p+1) - 2$ and $C2 = 0$ with $p \in \{2, 3, 4 \dots\}$. For irreps based on (5.1b), $-C1 = 2(p^2 - 1)$ and $-C2 = p^2(p^2 - 1)$ with $p \in \{\frac{3}{2}, 2, \frac{5}{2}, 3 \dots\}$. These correspond to the same expressions found for the unitary representations, $-C1 = p(p+1) + (q+1)(q-2)$ and $-C2 = p(p+1)q(q-1)$ with $q = 0$ and $q = p$ respectively for the two types of irrep. There is thus a far more restricted set of finite irreps with Hermitian or anti-Hermitian generators than for the discrete infinite dimensional unitary irreps. The corresponding irreps of the anti-de Sitter group follow immediately from the replacement of the 4-momentum operators thus $V_\mu \rightarrow iV_\mu$.

Keywords: de Sitter group, representation, anti-de Sitter group, Hermitian, anti-Hermitian

MSC (2020) class: 22E45

Contents

1. Purpose.....	3
2. The Lie Algebra	3
3. The Anti-de Sitter Case	5
4. The Irreps of the HLA.....	5
5. All the Irreps of the dSA	6
6. Proof of the Assertion of §5	7
6.1 There is No Representation of the dSA based on a One-Block Backbone	8
7. The Representations of $V\mu$ and Compatibility between Blocks	8
7.1 The Poincaré versus the dS/AdS Groups	14
7.2 Imposing the Hermitian/Anti-Hermitian Condition.....	14
7.3 Terminology: Compatibility, Connectedness, Monotonic	15
8. Proof that Representations with Backbones Given by (5.1a,b) Exist	15
8.1 Backbone Type (5.1a)	16
8.2 Backbone Type (5.1b)	19
9. There are No Irreps other than Those Based on the Backbones of (5.1a,b), Assuming No Duplicates	21
9.1 Expressions for the On-BD Components of Vx, Vy	21
9.2 The Minimum A and B in Any Representation is Zero.....	22
9.3 Unique Non-Monotonic Sequences	23
9.4 There are No Other Irreps than (5.1a,b): The Proof Without Duplicates.....	24
10. The Defeat of the Proof of §9 by Duplicates.	29
11. Uniqueness: Any Backbone has at Most One Representation (up to Equivalence).....	29
12. The Structure of Reducible Representations and the Role of Duplicates	31
13. The Casimir Matrices.....	37
14. Conclusion	38
15. References.....	39

1. Purpose

The representations of the de Sitter group (dSG) and the anti-de Sitter group (AdSG) have received considerable attention in the past, initially by Thomas, Ref.[1], which was subject to some later corrections by Newton, Ref.[2], and consolidated by Dixmier, Ref.[3]. Most recently Enayati et al have presented the topic in comprehensive book form, Ref.[4]. However, because of the importance of unitarity in quantum mechanics, the focus has been on unitary representations. But the dSG and AdSG are non-compact and hence admit only infinite dimensional unitary representations. Consequently, existing work has focussed on the infinite dimensional unitary representations. From the perspective of the Lie algebra, unitarity of the group arises from all the generators of the Lie algebra being Hermitian.

In this paper we present the complete set of irreducible finite dimensional representations of the dSA and AdSA subject to the less restrictive condition that the generators are either Hermitian (H) or anti-Hermitian (AH). (Note that with this requirement to be H or AH, irreducibility and indecomposability are the same). Expressions are provided for the matrix components of all the generators.

The homogeneous Lorentz algebra (HLA) is a sub-algebra of the dSA and the AdSA. We are interested in the Lorentz structure of the irreducible finite representations of the dSA and AdSA; in other words, to what set of representations of the HLA does an irreducible representation (irrep) of dSA/AdSA reduce? This structure is automatically provided if the finite irreps of the dSA/AdSA are built from a “backbone” consisting of direct sums of the known finite irreps of the HLA (see (4.5), below, for details).

Existing work on the infinite dimensional unitary representations of the dSG has allowed all such representations to be parametrised in terms of an integral, or half-integral, parameter p , and another parameter, q . The latter may take integral or half-integral values, or, for some classes of representation, values from a continuous range. The two Casimir invariants, which characterise the irreps, are given by simple expressions in terms of p and q . The same expressions are found to apply also in the finite dimensional case, restricted to H or AH generators, but occur only for the cases $q = 0$ or $q = p$. Thus, the complete set of all irreps in the finite, H or AH, case is severely restricted compared with the set of irreps available in the infinite dimensional unitary case – even if attention is confined to the discrete series of the latter.

Throughout we use the term “half integral” to mean an odd integer / 2 (hence, if referring to spin, “half-integral” would mean fermionic).

2. The Lie Algebra

It will be convenient to express the dSA/AdSA algebra in terms of the familiar rotation generators, J_i , the generators of boosts, K_i , and the four generators of displacements, V_μ . Latin subscripts take the values x, y, z whilst Greek subscripts take values t, x, y, z . The J_i, K_i, V_μ generate the group of isometries of a 4-sphere in (4,1) space in the dSA case, or in (3,2) space in the AdSA case, i.e., group elements are transformations that preserve the 5-dimensional metric $t^2 \mp w^2 - (x^2 + y^2 + z^2)$ where the upper sign refers to the dSA and the lower sign to the AdSA.

The HLA is a sub-algebra whose commutation relations (CRs) are,

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad [K_i, K_j] = -i\epsilon_{ijk}J_k \quad [J_i, K_j] = i\epsilon_{ijk}K_k \quad (2.1)$$

The remaining CRs all involve the V_μ , the generalisation of the usual 4-momentum operators. They may be derived from the stated isometry condition. Importantly, the CRs between V_μ and J_i or K_i are equivalent to requiring that the four quantities V_μ transform as a Lorentz 4-vector (thus justifying the notation). Alternatively, the CRs imbued by the isometry condition imply that V_μ is a Lorentz 4-vector. These CRs are,

$$[J_i, V_j] = i\epsilon_{ijk}V_k \quad [K_i, V_j] = -iV_t\delta_{ij} \quad [J_i, V_t] = 0 \quad [K_i, V_t] = -iV_i \quad (2.2)$$

So far, (2.1) and (2.2) are identical to the equivalents in the Poincaré Lie algebra. However, in flat Minkowski space the 4-momentum operators, i.e., the generators of displacements, are mutually commuting. The hallmark of intrinsic curvature is that the generators of displacements do not commute. For the dSA/AdSA geometries we find,

$$[V_i, V_j] = \pm i\epsilon_{ijk}J_k \quad [V_t, V_i] = \pm iK_i \quad (+ = \text{dS}, - = \text{AdS}) \quad (2.3)$$

The first CR in (2.1) is not consistent with the J_i being anti-Hermitian (AH). So the J_i must be Hermitian (H) – for our purposes, i.e., because we are interested only in the case that all generators are H or AH.

All the CRs in (2.1), (2.2) and (2.3) are consistent with all the other generators, K_i and V_μ , being H or all AH (or certain mixtures). However, no finite representations will exist with all generators H; some must be AH. This is familiar from the well-known classification of all the finite representations of the HLA sub-algebra, i.e., (2.1). The corresponding group is also non-compact so the finite representations must involve some generators which are AH. But the J_i must be H, so the K_i must be AH. This is explicitly manifest by considering the complexification of the algebra and defining,

$$A_i = (J_i + iK_i)/2 \quad B_i = (J_i - iK_i)/2 \quad (2.4a)$$

$$\text{i.e.,} \quad J_i = A_i + B_i \quad K_i = i(B_i - A_i) \quad (2.4b)$$

In terms of which the CRs (2.1) can be written in decoupled form as,

$$[A_i, A_j] = i\epsilon_{ijk}A_k \quad [B_i, B_j] = i\epsilon_{ijk}B_k \quad [A_i, B_j] = 0 \quad (2.5)$$

In other words, the HLA is isomorphic to the direct sum of two copies of the (complexified) $\mathfrak{su}(2)$. But CRs (2.5) imply that both of A_i and B_i must be H, so that (2.4b) implies that the K_i must be AH, as we have already concluded.

It can thus be seen that the CRs in (2.1), (2.2) and (2.3) are consistent with V_i being H if V_t is AH, or vice-versa. These cases align with dS and AdS respectively. This can be seen by noting that, as well as the 3-rotations, $\{J_i\}$ and the HLA, $\{J_i, K_i\}$ being sub-algebras, the $\{J_i, V_i\}$ also provide a sub-algebra. If we choose the dSA sign in (2.3), i.e., $[V_i, V_j] = i\epsilon_{ijk}J_k$, then $\{J_i, V_i\}$ generates $\text{SO}(4)$, i.e., rotations in E^4 , which requires the V_i to be H to generate these finite unitary transformations. Conversely, if we choose the AdS sign in (2.3), i.e., $[V_i, V_j] = -i\epsilon_{ijk}J_k$, then the $\{J_i, V_i\}$ provide an algebra identical to the HLA with K_i replaced by V_i and hence requiring the K_i to be AH.

Note that in both cases the $\{J_i, V_i\}$ provide a sub-algebra which is either isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ or its complexification is isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. This has formed the

basis of some approaches to the representations of the dSA, using as the “backbone” direct sums of the known irreps of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, i.e., the $SO(4)$ subgroup. This alternative means of building the irreps will be important in a later part of this paper (§11).

To summarise we have,

- A_i and B_i are Hermitian.
- J_i are Hermitian
- K_i are Anti-Hermitian.
- dS: V_i are Hermitian, V_t is Anti-Hermitian.
- AdS: V_i are Anti-Hermitian, V_t is Hermitian.

3. The Anti-de Sitter Case

All irreps of the AdSA can be formed from those of the dSA by the replacement $V_j \rightarrow iV_j$ and $V_t \rightarrow iV_t$, noting that this leaves all CRs (2.1) and (2.2) invariant, whereas the RHS of (2.3) change sign, as required. We therefore confine attention to the dS case, the AdS case being understood.

4. The Irreps of the HLA

These are well known, following from those of $\mathfrak{su}(2)$, and we simply state the result without elaboration. Exactly one irrep of $\mathfrak{su}(2)$ exists, up to equivalence, for every positive integral dimension, 1, 2, 3... We write this dimension as $(2A + 1)$ where A takes all positive indefinite integral or half-integral values. (Throughout we use the term “half integral” to mean an odd integer / 2). Consider the following combinations,

$$J^\pm = J_x \pm iJ_y \quad K^\pm = K_x \pm iK_y \quad A^\pm = A_x \pm iA_y \quad B^\pm = B_x \pm iB_y \quad (4.1)$$

Indexing the rows/columns of the representation by indices $a \in \{-A, -A + 1, \dots, A - 1, A\}$ the representations are,

$$(A^{+(A)})_{a_1 a_2} = r_{a_2}^{(A)} \delta_{a_1 a_2 + 1} \quad (4.2a)$$

$$(A^{-(A)})_{a_1 a_2} = s_{a_2}^{(A)} \delta_{a_1 a_2 - 1} \quad (4.2b)$$

$$(A_z^{(A)})_{a_1 a_2} = a_1 \delta_{a_1 a_2} \quad (4.2c)$$

where,
$$r_{a_2}^{(A)} = \sqrt{(A - a_2)(A + a_2 + 1)} \quad (4.2d)$$

$$s_{a_2}^{(A)} = \sqrt{(A + a_2)(A - a_2 + 1)} \quad (4.2e)$$

The same expressions apply, of course, to B_i . Note that although the indices a_1 , etc., may be integral or half-integral, adjacent indices differ by 1.

When the abstract generators are realised as matrices, the summations in (2.4a,b) must be understood in terms of implied direct products, that is, for example,

$$J_i = A_i \otimes \mathbb{I}_{2B+1} + \mathbb{I}_{2A+1} \otimes B_i \quad (4.3)$$

where \mathbb{I}_{2A+1} and \mathbb{I}_{2B+1} are unit matrices of dimension equal to that of the A_i and B_i matrices respectively. It is convenient when dealing with direct products of matrices to use a double index notation. Hence the J_i and K_i are represented by the following matrices,

$$(J^{+(A,B)})_{a_1 b_1, a_2 b_2} = r_{a_2}^{(A)} \delta_{a_1 a_2 + 1} \delta_{b_1 b_2} + r_{b_2}^{(B)} \delta_{a_1 a_2} \delta_{b_1 b_2 + 1} \quad (4.4a)$$

$$(J^{-(A,B)})_{a_1 b_1, a_2 b_2} = s_{a_2}^{(A)} \delta_{a_1 a_2 - 1} \delta_{b_1 b_2} + s_{b_2}^{(B)} \delta_{a_1 a_2} \delta_{b_1 b_2 - 1} \quad (4.4b)$$

$$i(K^{+(A,B)})_{a_1 b_1, a_2 b_2} = r_{a_2}^{(A)} \delta_{a_1 a_2 + 1} \delta_{b_1 b_2} - r_{b_2}^{(B)} \delta_{a_1 a_2} \delta_{b_1 b_2 + 1} \quad (4.4c)$$

$$i(K^{-(A,B)})_{a_1 b_1, a_2 b_2} = s_{a_2}^{(A)} \delta_{a_1 a_2 - 1} \delta_{b_1 b_2} - s_{b_2}^{(B)} \delta_{a_1 a_2} \delta_{b_1 b_2 - 1} \quad (4.4d)$$

$$(J_z^{(A,B)})_{a_1 b_1, a_2 b_2} = (a_1 + b_1) \delta_{a_1 a_2} \delta_{b_1 b_2} \quad (4.4e)$$

$$i(K_z^{(A,B)})_{a_1 b_1, a_2 b_2} = (a_1 - b_1) \delta_{a_1 a_2} \delta_{b_1 b_2} \quad (4.4f)$$

Hence, the irreps of the HLA are uniquely labelled by the pair of (half) integers, A, B . Every finite irrep of the dSA must reduce to a finite direct sum of such HLA irreps which can be denoted,

$$(A_1, B_1) \oplus (A_2, B_2) \oplus (A_3, B_3) \oplus \dots \oplus (A_N, B_N) \quad (4.5)$$

for some finite positive integer, N , which is the number of HLA irreps involved in the dSA irrep. N will also be referred to as the number of “blocks” because in matrix notation each of the HLA irreps, (A_i, B_i) , is a block in a block-diagonal matrix representation of (4.5).

The direct sum, (4.5), will be called the “backbone” of the irrep of the dSA. The order of the terms (blocks) in the direct sum makes no difference, of course. Also reversing $A_i \leftrightarrow B_i$ for all i merely changes the sign of K_i and an equivalent representation then results if the sign of V_t is also reversed (all CRs 2.1-3 being invariant under this transformation).

5. All the Irreps of the dSA

We state the result of this paper, the purpose of the rest of which is to prove this assertion and to provide explicit expressions for the complete set of representations of the irreps.

All irreps of the dSA have a backbone of one of the following two forms,

$$(A, A) \oplus \left(A - \frac{1}{2}, A - \frac{1}{2}\right) \oplus (A - 1, A - 1) \oplus \dots \oplus (0, 0) \quad (5.1a)$$

Or
$$(A, 0) \oplus \left(A - \frac{1}{2}, \frac{1}{2}\right) \oplus (A - 1, 1) \oplus \dots \oplus (0, A) \quad (5.1b)$$

in which there are at least two blocks.

Necessarily, $A = \frac{N-1}{2}$ where N is the number of blocks in the backbone ($N \geq 2$). Hence, in order of ascending dimension, the first few irreps’ backbones are as given in Table 1. Most dimensions have no representation. The possible dimensions are $\frac{1}{6}N(N+1)(2N+1)$ for Type 5.1a (the even reference numbers in Table 1) or $\frac{1}{6}N(N+1)(N+2)$ for Type 5.1b (the odd reference numbers), where $N \in \{2, 3, 4, \dots\}$ is the number of blocks in the backbone.

Table 1: The Backbones of the First Ten Finite Irreps of the dSA. The first figure is a label for future convenience. The second figure is the dimension.

Rep	Dim	Backbone
1	4	$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$
2	5	$\left(\frac{1}{2}, \frac{1}{2}\right) \oplus (0,0)$
3	10	$(1,0) \oplus \left(\frac{1}{2}, \frac{1}{2}\right) \oplus (0,1)$
4	14	$(1,1) \oplus \left(\frac{1}{2}, \frac{1}{2}\right) \oplus (0,0)$
5	20	$\left(\frac{3}{2}, 0\right) \oplus \left(1, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 1\right) \oplus \left(0, \frac{3}{2}\right)$
6	30	$\left(\frac{3}{2}, \frac{3}{2}\right) \oplus (1,1) \oplus \left(\frac{1}{2}, \frac{1}{2}\right) \oplus (0,0)$
7	35	$(2,0) \oplus \left(\frac{3}{2}, \frac{1}{2}\right) \oplus (1,1) \oplus \left(\frac{1}{2}, \frac{3}{2}\right) \oplus (0,2)$
8	55	$(2,2) \oplus \left(\frac{3}{2}, \frac{3}{2}\right) \oplus (1,1) \oplus \left(\frac{1}{2}, \frac{1}{2}\right) \oplus (0,0)$
9	56	$\left(\frac{5}{2}, 0\right) \oplus \left(2, \frac{1}{2}\right) \oplus \left(\frac{3}{2}, 1\right) \oplus \left(1, \frac{3}{2}\right) \oplus \left(\frac{1}{2}, 2\right) \oplus \left(0, \frac{5}{2}\right)$
10	91	$\left(\frac{5}{2}, \frac{5}{2}\right) \oplus (2,2) \oplus \left(\frac{3}{2}, \frac{3}{2}\right) \oplus (1,1) \oplus \left(\frac{1}{2}, \frac{1}{2}\right) \oplus (0,0)$

6. Proof of the Assertion of §5

The proof of the assertion consists of the following steps.

- 1) Proof that there is no representation of the dSA based on a backbone with a single block. A proof has been presented by Shurleff in Ref.[5] but will be reiterated in this section, below, for completeness.
- 2) Proof that some non-zero off-block-diagonal (off-BD) terms are essential in an irrep of the dSA and that non-zero off-BD terms between blocks P and Q can occur only if $|A_P - A_Q| = \frac{1}{2}$ and $|B_P - B_Q| = \frac{1}{2}$. This has also been proved in Ref.[5] but will be reiterated here for completeness, a simple extension of Ref.[5] being to show that the result applies to an arbitrary number of blocks in the backbone. Ref.[5] also derived explicit expressions for the off-BD terms which we will summarise and use extensively in the rest of this paper. These issues are dealt with in §7.
- 3) §8 will show by explicit construction that irreps of the dSA exist based on the backbones of (5.1a,b).

- 4) On the assumption that the backbone contains no duplicate blocks, §9 will prove that any backbones other than those of (5.1a,b) cannot provide an irrep of the dSA. Here by “duplicate” is meant two blocks with the same A and B .
- 5) §10 points out why the proof of §9 fails if duplicates are present.
- 6) §11 shows that for any proposed backbone there can be at most one representation, up to equivalence, i.e., that given a possible backbone the associate representation is unique, up to equivalence.
- 7) §12 discusses the significance of duplicates and shows how valid representations may be constructed in the presence of duplicates – but in a manner which is explicitly reduceable. Appeal to the uniqueness result of §11 is then used to conclude that any representation based on the same backbone (with duplicates) is equivalent and hence reduceable. Hence there are no irreps with duplicates. It is pointed out that the most general (reducible) representation will involve many duplicates.
- 8) This completes the proof, showing that the irreps given explicit form in §8 and based on the backbones of (5.1a,b) are the complete set of finite irreps of the HLA subject to all generators being H or AH .

The rest of this section will complete Step 1: that there is no representation of the dSA based on a one-block backbone.

6.1 There is No Representation of the dSA based on a One-Block Backbone

From CR's (2.2) it follows that $[J_z, [J_z, V_x]] = J_z^2 V_x - 2J_z V_x J_z + V_x J_z^2 = V_x$. Using the explicit expression, (4.4e), for J_z which applies for the HLA irrep (A, B) this implies,

$$(V_x)_{ab, a'b'} = [(a + b) - (a' + b')]^2 (V_x)_{ab, a'b'} \quad (6.1)$$

But the CRs (2.2) also require $[K_z, V_x] = 0$ and using (4.4f) for K_z this gives,

$$[(a - b) - (a' - b')] (V_x)_{ab, a'b'} = 0 \quad (6.2)$$

The only way we can avoid concluding from (6.1) and (6.2) that $(V_x)_{ab, a'b'} = 0$ is if both,

$$[(a + b) - (a' + b')] = \pm 1 \quad \text{and} \quad [(a - b) - (a' - b')] = 0 \quad (6.3)$$

But adding these implies $(a - a') = \pm 1/2$, which is impossible because the indices can differ only by an integer. Hence we conclude that $V_x = 0$. The other CRs in (2.2) then immediately imply that all the V_μ are zero, but the CRs (2.3) cannot then hold and so there is no such representation. QED.

7. The Representations of V_μ and Compatibility between Blocks

This section generalises to an arbitrary number of blocks the derivation by Shurtleff, Ref.[5], of the conditions under which block-components of V_μ can be non-zero. Note that Ref.[5] addressed the Poincaré group, not the dSG, but the derivations summarised in this section require only the CRs (2.1) and (2.2), which are common between the two algebras and hence Ref.[5] is applicable. Satisfaction of CRs (2.3) will come in later sections.

Considering for illustration the case of a backbone with four blocks, the representation (rep) of the J_i and the K_i can be written in block-matrix form as,

$$\begin{pmatrix} J_i^{(1)} & 0 & 0 & 0 \\ 0 & J_i^{(2)} & 0 & 0 \\ 0 & 0 & J_i^{(3)} & 0 \\ 0 & 0 & 0 & J_i^{(4)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} K_i^{(1)} & 0 & 0 & 0 \\ 0 & K_i^{(2)} & 0 & 0 \\ 0 & 0 & K_i^{(3)} & 0 \\ 0 & 0 & 0 & K_i^{(4)} \end{pmatrix} \quad (7.1)$$

where $J_i^{(1)}$ is shorthand for $J_i^{(A_1, B_1)}$, $J_i^{(2)}$ is shorthand for $J_i^{(A_2, B_2)}$, etc., and similarly for the $K_i^{(P)}$. (We shall use capital Latin letters to denote block numbers). Note that $J_i^{(A_1, B_1)}$ is a matrix of dimension $(2A_1 + 1)(2B_1 + 1)$, etc. We initially write the rep of V_μ as a full matrix whose PQ block is $V_{\mu PQ}$, each one of which is, in general, a rectangular matrix with $(2A_P + 1)(2B_P + 1)$ rows and $(2A_Q + 1)(2B_Q + 1)$ columns. The PQ block of the CR $[J_i, V_\mu]$ is thus,

$$[J_i, V_\mu]_{PQ} = J_i^{(P)} V_{\mu PQ} - V_{\mu PQ} J_i^{(Q)} \quad (7.2)$$

with a similar expression for $[K_i, V_\mu]_{PQ}$. But on-block-diagonal (on-BD) blocks then do not mix blocks, i.e., $[J_i, V_\mu]_{PP} = J_i^{(P)} V_{\mu PP} - V_{\mu PP} J_i^{(Q)}$ involves only the on-BD block PP . But we have already shown in §6 that the only solution for the $V_{\mu PP}$ is that they are zero. Hence we conclude that the V_μ matrices have zero blocks on the BD.

We now present the proof that the off-BD block $V_{\mu PQ}$ between blocks P and Q can be non-zero only if $|A_P - A_Q| = \frac{1}{2}$ and $|B_P - B_Q| = \frac{1}{2}$.

Using the CRs (2.2) we have $[J_z, [J_z, V_x]] = J_z^2 V_x - 2J_z V_x J_z + V_x J_z^2 = V_x$. Considering the off-BD blocks 12 and 21, this decouples into two parts,

$$V_{x12} = (J_z^{A,B})^2 V_{x12} - 2J_z^{A,B} V_{x12} J_z^{C,D} + V_{x12} (J_z^{C,D})^2 \quad (7.3a)$$

$$V_{x21} = (J_z^{C,D})^2 V_{x21} - 2J_z^{C,D} V_{x21} J_z^{A,B} + V_{x21} (J_z^{A,B})^2 \quad (7.3b)$$

where we have taken blocks 1 and 2 to have HLA irreps (A, B) and (C, D) respectively. Using the explicit expression, (4.4e), for J_z for both the (A, B) and (C, D) parts we find, from (7.3a),

$$(V_{x12})_{ab,cd} = [(a + b) - (c + d)]^2 (V_{x12})_{ab,cd} \quad (7.4)$$

where c takes the integral or half-integral values $-C, -C + 1 \dots C$, and similarly for d in terms of D . From (2.2), the CR $[K_z, V_x] = 0$ gives in similar fashion,

$$[(a - b) - (c - d)] (V_{x12})_{ab,cd} = 0 \quad (7.5)$$

To avoid (7.4) and (7.5) implying $(V_{x12})_{ab,cd} = 0$ (and by immediate implication of the CRs, (2.2), that all the $V_\mu = 0$) we require,

$$[(a + b) - (c + d)] = \pm 1 \quad \text{and} \quad [(a - b) - (c - d)] = 0 \quad (7.6)$$

Adding the two eqs (C8) implies $a - c = b - d = \pm 1/2$. This is possible only if A is an integer and C is a half-integer, or vice-versa, and similarly for B and D . The conditions (7.6) establish the only components of $(V_{x12})_{ab,cd}$ which can be non-zero, namely those with either $a = c + \frac{1}{2}$ and $b = d + \frac{1}{2}$ or $a = c - \frac{1}{2}$ and $b = d - \frac{1}{2}$, and hence we can write,

$$(V_{x12})_{ab,cd} = t_{ab}^{12} \delta_{a,c+\frac{1}{2}} \delta_{b,d+\frac{1}{2}} + u_{ab}^{12} \delta_{a,c-\frac{1}{2}} \delta_{b,d-\frac{1}{2}} \quad (7.7)$$

By using the CR $[J_z, V_x] = iV_y$ we find that $i(V_{y12})_{ab,cd} = [a + b - c - d](V_{x12})_{ab,cd}$ and hence,

$$i(V_{y12})_{ab,cd} = t_{ab}^{12} \delta_{a,c+\frac{1}{2}} \delta_{b,d+\frac{1}{2}} - u_{ab}^{12} \delta_{a,c-\frac{1}{2}} \delta_{b,d-\frac{1}{2}} \quad (7.8)$$

Defining $V_{\pm 12} = (V_{x12} \pm iV_{y12})/2$, (7.7,8) simplify to,

$$(V_{+12})_{ab,cd} = t_{ab}^{12} \delta_{a,c+\frac{1}{2}} \delta_{b,d+\frac{1}{2}} \quad (V_{-12})_{ab,cd} = u_{ab}^{12} \delta_{a,c-\frac{1}{2}} \delta_{b,d-\frac{1}{2}} \quad (7.9a)$$

Symmetry means we can obtain the corresponding expressions for $V_{\pm 21}$ by the replacements $a \leftrightarrow c, b \leftrightarrow d$, giving,

$$(V_{+21})_{cd,ab} = t_{cd}^{21} \delta_{c,a+\frac{1}{2}} \delta_{d,b+\frac{1}{2}} \quad (V_{-21})_{cd,ab} = u_{cd}^{21} \delta_{c,a-\frac{1}{2}} \delta_{d,b-\frac{1}{2}} \quad (7.9b)$$

So far we have used only the CRs involving J_z . Defining $J_{\pm} = J_x \pm iJ_y$ the CRs, (2.2), involving the x and y components can be simplified to,

$$[J_{\pm}, V_{\pm}] = 0 \quad \text{and} \quad [J_{\pm}, V_{\mp}] = \pm V_z \quad (7.10)$$

In terms of block components we can write,

$$([J_+, V_+]_{12})_{ab,cd} = (J_+^{A,B})_{ab,a'b'} (V_{+12})_{a'b',cd} - (V_{+12})_{ab,c'd'} (J_+^{C,D})_{c'd',cd} \quad (7.11)$$

where the dashed indices are summed. Using the explicit expressions (4.4a-f) together with (7.9a) this zero commutator gives,

$$(r_{a-1}^A t_{a-1,b}^{12} - r_c^C t_{ab}^{12}) \delta_{a,c+\frac{3}{2}} \delta_{b,d+\frac{1}{2}} = 0 \quad (7.12a)$$

$$\text{and,} \quad (r_{b-1}^B t_{a,b-1}^{12} - r_d^D t_{a,b}^{12}) \delta_{a,c+\frac{1}{2}} \delta_{b,d+\frac{3}{2}} = 0 \quad (7.12b)$$

In the same manner $([J_-, V_-]_{12})_{ab,cd} = 0$, $([J_+, V_+]_{21})_{cd,ab} = 0$ and $([J_-, V_-]_{21})_{cd,ab} = 0$ give respectively,

$$(s_{a+1}^A u_{a+1,b}^{12} - s_c^C u_{ab}^{12}) \delta_{a,c-\frac{3}{2}} \delta_{b,d-\frac{1}{2}} = 0 \quad (7.12c)$$

$$\text{and,} \quad (s_{b+1}^B u_{a,b+1}^{12} - s_d^D u_{a,b}^{12}) \delta_{a,c-\frac{1}{2}} \delta_{b,d-\frac{3}{2}} = 0 \quad (7.12d)$$

$$(r_{c-1}^C t_{c-1,d}^{21} - r_a^A t_{cd}^{21}) \delta_{c,a+\frac{3}{2}} \delta_{d,b+\frac{1}{2}} = 0 \quad (7.12e)$$

$$\text{and,} \quad (r_{d-1}^D t_{c,d-1}^{21} - r_b^B t_{cd}^{21}) \delta_{c,a+\frac{1}{2}} \delta_{d,b+\frac{3}{2}} = 0 \quad (7.12f)$$

$$(s_{c+1}^C u_{c+1,d}^{21} - s_a^A u_{cd}^{21}) \delta_{c,a-\frac{3}{2}} \delta_{d,b-\frac{1}{2}} = 0 \quad (7.12g)$$

$$\text{and,} \quad (s_{d+1}^D u_{c,d+1}^{21} - s_b^B u_{cd}^{21}) \delta_{c,a-\frac{1}{2}} \delta_{d,b-\frac{3}{2}} = 0 \quad (7.12h)$$

Consider (7.12a). So long as we choose b and d such that $b = d + \frac{1}{2}$ then (7.12a) implies, for any a and b compatible with this and with $a = c + \frac{3}{2}$,

$$r_c^C t_{c+\frac{3}{2},b}^{12} = r_{c+\frac{1}{2}}^A t_{c+\frac{1}{2},b}^{12} \quad (7.13)$$

For any given b , (7.13) is a recursion formula for t_{ab}^{12} as it gives each t_{ab}^{12} in terms of the one before, i.e., $t_{a-1,b}^{12}$. Providing that $C \geq A + \frac{3}{2}$ we can choose $a = -A$ in (7.13) and still have a c compatible with $a = c + \frac{3}{2}$. With $a = -A$, (7.13) gives,

$$r_{-A-\frac{3}{2}}^C t_{-A,b}^{12} = r_{-A-1}^A t_{-A-1,b}^{12} = 0 \quad (7.14)$$

because (4.2d) gives $r_{-A-1}^A = 0$. If we assume this implies $t_{-A,b}^{12} = 0$ then with $a = -A + 1$, (7.13) gives,

$$r_{-A-\frac{1}{2}}^C t_{-A+1,b}^{12} = r_{-A}^A t_{-A,b}^{12} = 0 \quad (7.15)$$

So $t_{-A+1,b}^{12} = 0$ also, and so on, finding that all the t_{ab}^{12} are zero. The only way of avoiding this conclusion is if, in (7.14), $r_{-A-\frac{3}{2}}^C = \sqrt{\left(C + A + \frac{3}{2}\right)\left(C - A - \frac{1}{2}\right)} = 0$, i.e., $C = A + \frac{1}{2}$. But this contradicts the initial assumption that $C \geq A + \frac{3}{2}$. So we conclude that this initial assumption leads to all the t_{ab}^{12} being zero.

The same assumption, $C \geq A + \frac{3}{2}$, can be used in the same way to deduce from (7.12e) that all the t_{cd}^{21} are zero. And again, the assumption, $C \geq A + \frac{3}{2}$ can be used in the same way to deduce from (7.12c) and (7.12g) that all the u_{ab}^{12} and all the u_{ab}^{21} are zero. Hence, $C \geq A + \frac{3}{2}$ necessarily leads to $V_{\mu 12} = V_{\mu 21} = 0$.

The symmetry amongst the Eqs.(7.12a-h) immediately implies that this conclusion will follow also for $A \geq C + \frac{3}{2}$. To derive this explicitly, note that we can now assume $c = C$ in (7.13) whilst being compatible with $a = c + \frac{3}{2}$. But this gives,

$$r_{C+\frac{1}{2}}^A t_{C+\frac{1}{2},b}^{12} = r_C^C t_{C+\frac{3}{2},b}^{12} = 0 \quad (7.16)$$

because (4.2d) gives $r_C^C = 0$. And for $c = C - 1$ we get,

$$r_{C-\frac{1}{2}}^A t_{C-\frac{1}{2},b}^{12} = r_{C-1}^C t_{C+\frac{1}{2},b}^{12} = 0 \quad (7.17)$$

and so on. If (C22) is taken to imply $t_{C+\frac{1}{2},b}^{12} = 0$ then this sequence gives all the $t_{ab}^{12} = 0$. To

avoid this we require $r_{C+\frac{1}{2}}^A = 0 = \sqrt{\left(A - C - \frac{1}{2}\right)\left(A + C + \frac{3}{2}\right)}$ and hence $A = C + \frac{1}{2}$ which contradicts the initial assumption $A \geq C + \frac{3}{2}$. The other Eqs.(7.12a-h) similarly imply that $A \geq C + \frac{3}{2}$ would give $t_{cd}^{21} = u_{ab}^{12} = u_{ab}^{21} = 0$, i.e., the $V_{\mu 12} = V_{\mu 21} = 0$.

Hence we conclude that both $C \geq A + \frac{3}{2}$ and $A \geq C + \frac{3}{2}$ are ruled out if $V_{\mu 12}, V_{\mu 21}$ are to be non-zero. In other words, given that they are integral or half-integral, $C - 1 \leq A \leq C + 1$.

But we have already established that, for non-zero $V_{\mu 12}, V_{\mu 21}$ either A is an integer and C is a half-integer, or vice-versa. Consequently, the only remaining possible cases which avoid all the V_{μ} being zero are $A = C \pm \frac{1}{2}$.

The symmetry of Eqs.(7.12a-h) under the exchanges $a \leftrightarrow c, b \leftrightarrow d, A \leftrightarrow C, B \leftrightarrow D$ means that $B = D \pm \frac{1}{2}$ is also required for non-zero $V_{\mu 12}, V_{\mu 21}$. There is no requirement for the sign in $B = D \pm \frac{1}{2}$ to be related to the sign in $A = C \pm \frac{1}{2}$. Pairs of blocks respecting the conditions $A = C \pm \frac{1}{2}$ and $B = D \pm \frac{1}{2}$ will be described as ‘‘compatible’’.

This almost completes the proof of Step 2 except that we need to show that when $A = C \pm \frac{1}{2}$ and $B = D \pm \frac{1}{2}$ a non-zero solution for the $V_{\mu 12}, V_{\mu 21}$ exists, and what it is. The fact that the four cases $A = C \pm \frac{1}{2}$ and $B = D \pm \frac{1}{2}$ (independent signs) do indeed produce non-trivial solutions can be seen as follows. Assume $C = A + \frac{1}{2}$. Since we require $a = c + \frac{3}{2}$ for (7.13) to hold we cannot assume $a = -A$ because this would imply an impossible c . Instead the smallest a with which we can start the recursion is $a = -A + 1$. This gives,

$$r_{-A-\frac{1}{2}}^C t_{-A+1,b}^{12} = r_{-A}^A t_{-A,b}^{12} \quad (7.18)$$

But (4.4d) gives $r_{-A}^A = \sqrt{2A}$ and $r_{-A-\frac{1}{2}}^C = \sqrt{\left(C + A + \frac{1}{2}\right)\left(C - A + \frac{1}{2}\right)}$ which is non-zero for our assumed $C = A + \frac{1}{2}$. Hence we can take some non-zero $t_{-A,b}^{12}$ to start the recursion, in terms of which all the t_{ab}^{12} are found for the assumed b . Similar recursion applies, of course, to the b index, and this leads to all the t_{ab}^{12} being found as factors times a single parameter, t_{AB}^{12} .

Similar remarks apply for t_{cd}^{21}, u_{ab}^{12} and u_{cd}^{21} .

However, Shurtleff, Ref.[5], has also shown that the full set of recursion relations like (7.18) lead to $u_{-A,-B}^{12} = \pm t_{AB}^{12}$, etc., so there are in fact only two free parameters, t_{AB}^{12} and t_{CD}^{21} . Moreover, in the notation, the AB in t_{AB}^{12} is implied by the 12, and the CD in t_{CD}^{21} is implied by the 21, so the two undetermined constants can be written more simply as t_{12} and t_{21} . $V_{\mu 12}$ is proportional to t_{12} whilst $V_{\mu 21}$ is proportional to t_{21} .

The derivation so far has focussed on V_x and V_y (or equivalently V_{\pm}) and their CRs with J_i . Bringing in the CRs with K_i a similar derivation may be conducted for V_t and V_z (or the equivalent quantity $W_{\pm} = (V_z \pm V_t)/2$).

Shurtleff, Ref.[5], has thus derived the explicit expressions for all the $V_{\mu 12}$ and $V_{\mu 21}$ matrices, for all four combinations of cases $A = C \pm \frac{1}{2}$ and $B = D \pm \frac{1}{2}$. These are reproduced in Table 2 but in a compact notation in which all four cases can be written as a single expression.

We have also adopted a different normalisation for the constants t_{12} and t_{21} which simplifies the expressions (and also simplifies their numerical values as will be seen later). The relationship between the t_{12}, t_{21} used here and those of Ref.[5] are given in Table 3 for completeness.

The matrices given in Table 2 exclude the factors of t_{12}, t_{21} and so are denoted as $U_{\mu 12}$ and $U_{\mu 21}$. The actual $V_{\mu 12}$ and $V_{\mu 21}$ matrices in any given case will therefore be given by,

$$V_{\mu 12} = t_{12} U_{\mu 12} \quad \text{and} \quad V_{\mu 21} = t_{21} U_{\mu 21} \quad (7.19)$$

And we have also defined $V_{\pm} = (V_x \pm iV_y)/2$ and $W_{\pm} = (V_z \pm V_t)/2$. Table 2 gives expressions for the combinations $U_{\pm 12} = (U_{x12} \pm iU_{y12})/2$ and $\tilde{U}_{\pm 12} = (U_{z12} \pm U_{t12})/2$, and similarly for $U_{\pm 21}, \tilde{U}_{\pm 21}$.

Finally, all the above analysis applies between any pair of (different) blocks, so 12 can be replaced throughout by arbitrary blocks PQ , and the signs in Table 2, S_A and S_B , align with,

$$A_P = A_Q + \frac{S_A}{2} \text{ and } B_P = B_Q + \frac{S_B}{2} \quad (7.20)$$

Table 2: Explicit Expressions for the $U_{\mu 12}$ and $U_{\mu 21}$ Matrices (based on Ref.[5])

Quantity	Matrix Elements
$(U_{\pm 12})_{a_1 b_1, a_2 b_2}$	$S_{\pm} \sqrt{(A_{12} \pm S_A a_{12})(B_{12} \pm S_B b_{12})} \cdot \delta_{a_1, a_2 \pm \frac{1}{2}} \delta_{b_1, b_2 \pm \frac{1}{2}}$
$(U_{\pm 21})_{a_2 b_2, a_1 b_1}$	$S_{\pm} \sqrt{(A_{12} \mp S_A a_{12})(B_{12} \mp S_B b_{12})} \cdot \delta_{a_1, a_2 \mp \frac{1}{2}} \delta_{b_1, b_2 \mp \frac{1}{2}}$
$(\tilde{U}_{\pm 12})_{a_1 b_1, a_2 b_2}$	$-S_{AB}^{\pm} \sqrt{(A_{12} \pm S_A a_{12})(B_{12} \mp S_B b_{12})} \cdot \delta_{a_1, a_2 \pm \frac{1}{2}} \delta_{b_1, b_2 \mp \frac{1}{2}}$
$(\tilde{U}_{\pm 21})_{a_2 b_2, a_1 b_1}$	$S_{AB}^{\pm} \sqrt{(A_{12} \mp S_A a_{12})(B_{12} \pm S_B b_{12})} \cdot \delta_{a_1, a_2 \mp \frac{1}{2}} \delta_{b_1, b_2 \pm \frac{1}{2}}$

Conventions/Definitions:

- Writing $A_1 = A, B_1 = B, A_2 = C, B_2 = D$ the signs S_A, S_B are defined by $A = C + \frac{S_A}{2}$ and $B = D + \frac{S_B}{2}$. There are four cases which we may write $++$, $+-$, $-+$, $--$.
- $S_+ = +, S_- = -S_A S_B$.
- $S_{AB}^+ = S_B, S_{AB}^- = S_A$
- A_{12} is the larger of A or C (or A_1 and A_2); B_{12} is the larger of B or D (or B_1 and B_2).
- a_{12} is a_1 or a_2 according to A_{12} being A_1 or A_2 . Similarly, b_{12} is b_1 or b_2 according to B_{12} being B_1 or B_2 .
- $U_{\pm 12} = (U_{x12} \pm iU_{y12})/2$ and $\tilde{U}_{\pm 12} = (U_{z12} \pm U_{t12})/2$, similarly for $U_{\pm 21}, \tilde{U}_{\pm 21}$.
- $J_{\pm} = J_x \pm iJ_y$ and $K_{\pm} = K_x \pm iK_y$

Table 3: Relationship between the t_{12} , t_{21} used here and those of Ref.[5]

This work	Case ($S_A S_B$)	Equivalent in Ref.[5]
t_{12}	++	$\frac{t_{12}}{2\sqrt{A_1 B_1}}$
	--	$\frac{t_{12}}{2\sqrt{A_1 B_1}}$
	+-	$\frac{t_{12}}{\sqrt{2A_1}}$
	-+	$\frac{t_{12}}{\sqrt{2B_1}}$
t_{21}	++	$\frac{t_{21}}{2\sqrt{A_2 B_2}}$
	--	$\frac{t_{21}}{2\sqrt{A_2 B_2}}$
	+-	$\frac{t_{21}}{\sqrt{2B_2}}$
	-+	$\frac{t_{21}}{\sqrt{2A_2}}$

7.1 The Poincaré versus the dS/AdS Groups

It may now be checked directly that the explicit expressions of Table 2, together with those for J_i, K_i , (4.2a-e), obey all the CRs of (2.1) and (2.2). These CRs are common between the Poincaré and dS/AdS groups and so the representations of Table 2 and (4.2a-e) apply to both. Where the Poincaré and dS/AdS groups differ is in respect of CRs (2.3), the commutation relations of the V_μ amongst themselves. The only freedom left in order to constrain the representations of Table 2 to obey these further CRs is the freedom to choose the parameters t_{PQ} . In the case of the Poincaré group, the corresponding CRs are $[V_\mu, V_\nu] = 0$, i.e., that the 4-momenta commute in flat Minkowski spacetime. This can be achieved only by requiring either $t_{PQ} = 0$ and, for compatible blocks, $t_{QP} \neq 0$, or vice-versa, so that the V_μ are neither Hermitian nor anti-Hermitian. In this work that case is ruled out because we are interested only in representations for which all generators are Hermitian or anti-Hermitian.

It follows that in the rest of this paper, provided V_μ of the form given by Table 2 are used, the CRs of (2.1) and (2.2) will automatically be respected, and we need concentrate upon satisfying the CRs (2.3) only.

7.2 Imposing the Hermitian/Anti-Hermitian Condition

Recall that the dSA requires $V_k^+ = V_k$ and $V_t^+ = -V_t$ and these require $V_+^+ = V_-$ as well as $W_+^+ = W_-$. But, considering the 12 blocks only, $V_+ = \begin{pmatrix} 0 & t_{12}U_{+12} \\ t_{21}U_{+21} & 0 \end{pmatrix}$ and $V_- = \begin{pmatrix} 0 & t_{12}U_{-12} \\ t_{21}U_{-21} & 0 \end{pmatrix}$. So, for real t_{ij} , $V_+^+ = V_-$ requires $t_{12}U_{+12}^+ = t_{21}U_{-21}$. But for cases ++ and -- Table 2 tells us that $U_{+12}^+ = -U_{-21}$ so for these cases $t_{12} = -t_{21}$. Alternatively, for cases +- and -+ Table 2 tells us that $U_{+12}^+ = U_{-21}$ so for these cases $t_{12} = t_{21}$. These conditions are consistent with the Hermitian/anti-Hermitian requirements upon W_\pm for the same reason. In summary,

$$\text{For cases } ++ \text{ and } --: \quad t_{PQ} = -t_{QP} \quad (7.21a)$$

$$\text{For cases } +- \text{ and } -+: \quad t_{PQ} = t_{QP} \quad (7.21b)$$

The simplicity of these requirements is an additional reason for adopting our present convention for the definitions of the t_{PQ} , differing from that of Ref.[5], see Table 3.

7.3 Terminology: Compatibility, Connectedness, Monotonic

A convenient terminology will be that two blocks P and Q which meet the requirement that the off-DB PQ *could* be non-zero, namely that $A_P = A_Q \pm \frac{1}{2}$ and $B_P = B_Q \pm \frac{1}{2}$ (signs unrelated) is to say that the blocks P and Q are “compatible”.

If the off-DB blocks $V_{\mu PQ}$ and $V_{\mu QP}$ are actually non-zero, the blocks P and Q are said to be “connected”.

An ordered sequence of three or more backbone blocks is said to be “monotonic” if both the A and the B parameters vary monotonically over the sequence. Hence the possible monotonic sequences of three blocks are $++$, $+-$ and $-+$, $+-$, $-+$ and $+-$, $-+$. The backbones of (5.1a,b) are fully monotonic. Up to reversal, they are the only fully monotonic backbones for which the minimum A and the minimum B are both zero (and we will see in §9.2 that any representation must meet the latter conditions).

8. Proof that Representations with Backbones Given by (5.1a,b) Exist

We have proved already in §6 that there must be two or more blocks in the backbone. With J_i, K_i given in block-diagonal form, e.g., (7.1), we assert that V_μ consistent with CRs (2.3) and with backbones given by (5.1a,b) can be written in the following manner... exemplifying this by the four block case for illustration,

$$\begin{pmatrix} 0 & V_{\mu 12} & 0 & 0 \\ V_{\mu 21} & 0 & V_{\mu 23} & 0 \\ 0 & V_{\mu 32} & 0 & V_{\mu 34} \\ 0 & 0 & V_{\mu 43} & 0 \end{pmatrix} \quad (8.1)$$

In other words that possible V_μ can be written with off-BD blocks which are BD-adjacent, i.e., within one block row and column of the BD. In the four-block case illustrated this means we can take $V_{\mu 14} = V_{\mu 41} = 0$. Note that the structure of the backbones (5.1a,b), and the order in which the direct sums are written, ensures that adjacent blocks are compatible and hence can be taken as connected (unless it should turn out that the corresponding $V_{\mu PQ}$ is identically zero).

The claim that V_μ of this form meet all the requirements to ensure a representation exists is to be proved in this section. This section will not prove that the constructed representations are irreps, but this will follow from §13 which shows that the Casimir operators are multiples of the unit matrix in these representations. Nor will this section prove that no other *inequivalent* representation exists based upon the same backbone, but this will be shown in §11.

Consequently, the assumption that there are no off-BD blocks which are not BD adjacent will then be justified (i.e., any such representation must be equivalent to those of the type considered here, based on the backbones of (5.1a,b)).

To prove the assertion it will suffice to show that $t_{12}, t_{23}, t_{34}, \dots$ exist which respect the CRs (2.3), noting that $t_{21}, t_{32}, t_{43}, \dots$ follow from (7.21a,b). This is by no means obvious because, on the face of it, the situation is hugely over-constrained. For example, in the four irrep case there are just 3 variables to be determined in order to satisfy 6 matrix equations, (2.3), where each matrix equation consist of 30 component equations (case 5.1a) or 20 component equations (case 5.1b). Of course, these equations are not all independent.

Consider $[V_\alpha, V_\beta]$. We will illustrate the construction of the solution using the four block case, the extension to an arbitrary number of blocks will be clear. The on-BD blocks in $[V_\alpha, V_\beta]$ are,

$$[V_\alpha, V_\beta]_{11} = (V_{\alpha 12} V_{\beta 21} - V_{\beta 12} V_{\alpha 21}) \quad (8.2a)$$

$$[V_\alpha, V_\beta]_{22} = (V_{\alpha 21} V_{\beta 12} - V_{\beta 21} V_{\alpha 12}) + (V_{\alpha 23} V_{\beta 32} - V_{\beta 23} V_{\alpha 32}) \quad (8.2b)$$

$$[V_\alpha, V_\beta]_{33} = (V_{\alpha 32} V_{\beta 23} - V_{\beta 32} V_{\alpha 23}) + (V_{\alpha 34} V_{\beta 43} - V_{\beta 34} V_{\alpha 43}) \quad (8.2c)$$

$$[V_\alpha, V_\beta]_{44} = (V_{\alpha 43} V_{\beta 34} - V_{\beta 43} V_{\alpha 34}) \quad (8.2d)$$

As regards any larger number of blocks in the backbone, by assuming that only BD-adjacent blocks occur in the V_α there will only ever be one term like those in brackets in the first and last BD blocks, and only ever two such in the others. (In contrast, the inclusion of non-BD-adjacent blocks would bring in more such terms).

The (potentially) non-zero off-BD blocks are,

$$[V_\alpha, V_\beta]_{13} = (V_{\alpha 12} V_{\beta 23} - V_{\beta 12} V_{\alpha 23}) \quad (8.3a)$$

$$[V_\alpha, V_\beta]_{31} = (V_{\alpha 32} V_{\beta 21} - V_{\beta 32} V_{\alpha 21}) \quad (8.3b)$$

$$[V_\alpha, V_\beta]_{24} = (V_{\alpha 23} V_{\beta 34} - V_{\beta 23} V_{\alpha 34}) \quad (8.3c)$$

$$[V_\alpha, V_\beta]_{42} = (V_{\alpha 43} V_{\beta 32} - V_{\beta 43} V_{\alpha 32}) \quad (8.3d)$$

As regards any larger number of blocks in the backbone, by assuming that only BD-adjacent blocks occur in the V_α there will only ever be one term like those in brackets in all the off-BD blocks.

Because J_i, K_i are in block-diagonal form, e.g., (7.1), the CRs (2.3) require all the non-BD blocks of $[V_\alpha, V_\beta]$ to be zero, so we shall require all of (8.3a-d) to be identically zero.

8.1 Backbone Type (5.1a)

This is the case ++, ++, ++, (It can equally be considered as --, --, --, ..., by reversing the order of the blocks). Evaluating $[V_x, V_y]_{13}$ using (8.3a) and the representations of Table 2 shows it to be identically zero, and the same is found for all the $[V_\alpha, V_\beta]_{13}$. By symmetry the same is true for all the off-BD blocks, (8.3a-d). This confirms the CRs (2.3) are respected as regards the off-BD blocks. We will see in §9.3 that this is a particular example of monotonic sequences causing terms like the RHS of (8.3a-d) to be zero (see Table 5).

Turning now to the on-BD blocks, we find,

$$i \left([V_x, V_y]_{11} \right)_{a_1 b_1, a'_1 b'_1} = 4t_{12} t_{21} (A_1 b_1 + B_1 a_1) \delta_{a_1 a'_1} \delta_{b_1 b'_1} \quad (8.4)$$

This is required to be identically equal to $-(J_{z11})_{a_1 b_1, a'_1 b'_1} = -(a_1 + b_1)\delta_{a_1 a'_1} \delta_{b_1 b'_1}$, i.e.,

$$4t_{12}t_{21}(A_1 b_1 + B_1 a_1) \equiv -(a_1 + b_1) \quad (8.5)$$

We note that $A_1 > 0$ and $B_1 > 0$ because we have shown there must be at least two blocks in the backbone this case therefore requires $A_1 = B_1 \geq \frac{1}{2}$. Note that the minus sign in (8.5) is consistent with (7.21a) and we have,

$$t_{12} = -t_{21} = \frac{1}{2\sqrt{A}} = \sqrt{\frac{1}{2(N-1)}} \quad (8.6)$$

where $A_1 = B_1 = A = \frac{N-1}{2}$ and N is the number of blocks in the backbone. It can similarly be checked by direct substitution of the representations of Table 2 that,

$$[V_y, V_z]_{11} = iJ_x \quad \text{and} \quad [V_z, V_x]_{11} = iJ_y \quad (8.7)$$

But it is easier to simply note that both J_i and V_i transform as 3-vectors under rotations in E^3 and hence that (8.7) follows immediately from $[V_x, V_y]_{11} = iJ_z$.

Similarly, it can be checked directly from Table 2 that the CRs $[V_t, V_i] = iK_i$ are obeyed, but it is easier to simply observe that one of these CRs was used in the derivation of the expressions of Table 2 whilst the other two result from the 3-vector nature of K_i and V_i .

We note in passing that (8.5) would have no solution in the case $A_1 \neq B_1$.

Turning now to the next on-BD block, $[V_x, V_y]_{22}$ evaluation using (8.2b) and Table 2 gives,

$$i([V_x, V_y]_{22})_{a_2 b_2, a'_2 b'_2} = \quad (8.8)$$

$$[-4t_{12}t_{21}((A_2 + 1)b_2 + (B_2 + 1)a_2) + 4t_{23}t_{32}(A_2 b_2 + B_2 a_2)]\delta_{a_2 a'_2} \delta_{b_2 b'_2}$$

This must be identically equal to $-(J_{z22})_{a_2 b_2, a'_2 b'_2} = -(a_2 + b_2)\delta_{a_2 a'_2} \delta_{b_2 b'_2}$ hence,

$$-4t_{12}t_{21}((A_2 + 1)b_2 + (B_2 + 1)a_2) + 4t_{23}t_{32}(A_2 b_2 + B_2 a_2) = -(a_2 + b_2) \quad (8.9)$$

But $A_1 = B_1 = \frac{N-1}{2}$ and so $A_2 = B_2 = \frac{N-2}{2}$ and we have already found $t_{12}t_{21}$, (8.6), so we find that (8.9) is an identity provided we set,

$$t_{23} = -t_{32} = \sqrt{\frac{2N-1}{2(N-1)(N-2)}} \quad (8.10)$$

Block 22 of the other CRs of (2.3) will also be obeyed by the same rotational symmetry argument as before.

Turning now to the next on-BD block, $[V_x, V_y]_{33}$ it may be evaluated using (8.2c) and Table 2 but can be written down immediately from (8.8), giving,

$$i([V_x, V_y]_{33})_{a_3 b_3, a'_3 b'_3} = \quad (8.11)$$

$$[-4t_{23}t_{32}((A_3 + 1)b_3 + (B_3 + 1)a_3) + 4t_{34}t_{43}(A_3 b_3 + B_3 a_3)]\delta_{a_3 a'_3} \delta_{b_3 b'_3}$$

This must be identically equal to $-(J_{z33})_{a_3 b_3, a'_3 b'_3} = -(a_3 + b_3) \delta_{a_3 a'_3} \delta_{b_3 b'_3}$. Using $A_2 = B_2 = \frac{N-2}{2}$ and $A_3 = B_3 = \frac{N-3}{2}$ together with (8.10) we find that it is indeed an identity provided we set,

$$t_{34} = -t_{43} = \sqrt{\frac{3(N-1)}{2(N-2)(N-3)}} \quad (8.12)$$

The solution for the general $t_{n,n+1}$ is now clear, for $2 \leq n \leq N-1$, namely,

$$t_{n,n+1} = -t_{n+1,n} = \sqrt{\frac{(2N-n+1)n}{4(N-n)(N-n+1)}} \quad (8.13)$$

But finally we have the last on-BD block, $[V_x, V_y]_{NN}$, for which there is only one term as illustrated by (8.2d). Equating with J_{zNN} we find,

$$-4t_{N,N-1}t_{N-1,N}((A_N + 1)b_N + (B_N + 1)a_N) \equiv -(a_N + b_N) \quad (8.14)$$

But because $t_{N,N-1}t_{N-1,N} < 0$ due to (7.21a) the LHS, if non-zero, would be positive when neither of a_N or b_N are negative and at least one is non-zero, whereas the RHS would then be negative. Hence (8.14) can only be an identity if both a_N and b_N are constrained to be zero, i.e., for $A_N = B_N = 0$.

This establishes that, had we chosen the first block of the backbone, (A, A) , to have $A > \frac{N-1}{2}$, the final block of the backbone would not be $(0,0)$ and this would provide a conflict so that no solution would result. Similarly, if $A < \frac{N-1}{2}$ a conflict would again result. To see this, set $A = \frac{N'-1}{2}$ where $N' \leq N-1$ in which case (8.13) becomes, for $n = N-1$,

$$-t_{n,n+1}t_{n+1,n} = \frac{(2N' - N + 2)(N - 1)}{4(N' - N + 1)(N' - N + 2)}$$

But this is undefined due to the zero in the denominator when $N' = N-1$. For smaller N' this same problem will occur for smaller n , i.e., $n < N-1$. Hence we must have $A = \frac{N-1}{2}$, no other option is consistent.

The on-BD blocks of the other CRs (2.3) can be established for backbone (5.1a) in similar fashion, though, for the same argument noted above, there is no need to do so as this follows from the derivation of the representations of Table 2 together with the 3-vector nature of J_i, K_i and V_i . Table 4 gives explicitly the values of the coefficients t_{ij} for the first ten irreps (using the irrep reference numbers of Table 1).

Summary of §8.1

It has been established by explicit construction of the values of the $t_{n,n+1} = -t_{n+1,n}$, (8.13), that a representation of the dSA exists on the backbone of (5.1a). The solution was based on the assumption that the representations of the V_μ have only BD-adjacent blocks.

As a by-product we have also proved that there is no solution on a backbone of the following type,

$$(A, B) \oplus \left(A - \frac{1}{2}, B - \frac{1}{2}\right) \oplus (A - 1, B - 1) \oplus \dots \oplus \left(A - \frac{N-1}{2}, B - \frac{N-1}{2}\right)$$

unless it reduces to (5.1a), i.e., that $A = B = \frac{N-1}{2}$ so that the last block is (0,0).

8.2 Backbone Type (5.1b)

In this section we seek to carry out a similar derivation as in §8.1 for the backbone of (5.1b). This is the case $+-, +-, +-, \dots$ (which, reading the blocks in reverse order is equivalent to $-+, -+, -+, \dots$).

Evaluating $[V_x, V_y]_{13}$ using (8.3a) and the representations of Table 2 shows it to be identically zero, and the same is found for all the $[V_\alpha, V_\beta]_{13}$. To show this it is not necessary to assume that $B_1 = 0$, or any relation between A_1 and B_1 , and so the same must be true for all the off-BD blocks, (8.3a-d). This confirms the CRs (2.3) are respected as regards the off-BD blocks. Again we note that, as we will see in §9.3, this is a particular example of monotonic sequences causing terms like the RHS of (8.3a-d) to be zero (see also Table 5).

Turning now to the on-BD blocks, we find, using (8.2a) and Table 2,

$$i \left([V_x, V_y]_{11} \right)_{a_1 b_1, a'_1 b'_1} = 4t_{12}t_{21}(A_1 b_1 - (B_1 + 1)a_1) \delta_{a_1 a'_1} \delta_{b_1 b'_1} \quad (8.15)$$

and this is required to be identically equal to $-(J_{z11})_{a_1 b_1, a'_1 b'_1} = -(a_1 + b_1) \delta_{a_1 a'_1} \delta_{b_1 b'_1}$, i.e.,

$$4t_{12}t_{21}(A_1 b_1 - (B_1 + 1)a_1) = -(a_1 + b_1) \quad (8.16)$$

But $A_1 = A_2 + \frac{1}{2} > 0$ and so if $B_1 > 0$ (8.16) could not be identically true. Hence a backbone which starts $(A_1, B_1) \oplus \left(A_1 - \frac{1}{2}, B_1 + \frac{1}{2}\right)$ must have $B_1 = 0$. (8.16) then gives, recalling that (7.21b) requires $t_{21} = t_{12}$,

$$t_{21} = t_{12} = \frac{1}{2} \quad (8.17)$$

For the next on-BD block, we find, using (8.2b) and Table 2, (8.18)

$$i \left([V_x, V_y]_{22} \right)_{a_2 b_2, a'_2 b'_2} = [4t_{23}t_{32}(A_2 b_2 - (B_2 + 1)a_2) - ((A_2 + 1)b_2 - B_2 a_2)] \delta_{a_2 a'_2} \delta_{b_2 b'_2}$$

and this is required to be identically equal to $-(J_{z22})_{a_2 b_2, a'_2 b'_2} = -(a_2 + b_2) \delta_{a_2 a'_2} \delta_{b_2 b'_2}$, i.e.,

$$4t_{23}t_{32}(A_2 b_2 - (B_2 + 1)a_2) - ((A_2 + 1)b_2 - B_2 a_2) = -(a_2 + b_2) \quad (8.19)$$

We find that (8.19) is an identity iff,

$$t_{32} = t_{23} = \frac{1}{2} \quad (8.20)$$

For the next on-BD block, we find, in similar fashion, (8.21)

$$i \left([V_x, V_y]_{33} \right)_{a_3 b_3, a'_3 b'_3} = [4t_{34}t_{43}(A_3 b_3 - (B_3 + 1)a_3) - ((A_3 + 1)b_3 - B_3 a_3)] \delta_{a_3 a'_3} \delta_{b_3 b'_3}$$

and this is required to be identically equal to $-(J_{z22})_{a_3 b_3, a'_3 b'_3} = -(a_2 + b_2) \delta_{a_3 a'_3} \delta_{b_3 b'_3}$, i.e.,

$$4t_{34}t_{43}(A_3 b_3 - (B_3 + 1)a_3) - ((A_3 + 1)b_3 - B_3 a_3) = -(a_3 + b_3) \quad (8.22)$$

Again this is an identity iff,

$$t_{43} = t_{34} = \frac{1}{2} \quad (8.23)$$

For the case of a backbone with N blocks of type $(A, 0) \oplus \left(A - \frac{1}{2}, \frac{1}{2}\right) \oplus (A - 1, 1) \oplus \dots$ it is clear that from block $[V_x, V_y]_{nn}$ with $n \leq N - 1$ we will find,

$$t_{n+1, n} = t_{n, n+1} = \frac{1}{2} \quad (8.24)$$

That leaves only block $i[V_x, V_y]_{NN}$ which differs by containing only one term, see (8.2d).

Equating with $-J_{zNN}$ requires,

$$-4t_{N-1, N}t_{N, N-1}((A_N + 1)b_N - B_N a_N) = -(a_N + b_N) \quad (8.25)$$

But as $B_N > 0$ if $A_N > 0$ then (8.25) cannot be an identity. Hence it must be that $A_N = 0$, i.e., the backbone must terminate at $A_N = 0$. (8.25) is then an identity as long as we have,

$$t_{N-1, N} = t_{N, N-1} = \frac{1}{2} \quad (8.26)$$

which is consistent with (8.24), hence completing the proof that this representation exists, noting that the backbone terminates at $A_N = 0$ also means $A_1 = \frac{N-1}{2}$.

Summary of §8.2

It has been established by explicit demonstration that $t_{n+1, n} = t_{n, n+1} = \frac{1}{2}$ provides a representation of the dSA on the backbone of (5.1b). The solution was based on the assumption that the representations of the V_μ have only BD-adjacent blocks.

As a by-product we have also proved that there is no solution on a backbone of the following type,

$$(A, B) \oplus \left(A - \frac{1}{2}, B + \frac{1}{2}\right) \oplus (A - 1, B + 1) \oplus \dots \oplus \left(A - \frac{N-1}{2}, B + \frac{N-1}{2}\right)$$

unless it reduces to (5.1b), i.e., that $A = \frac{N-1}{2}$ and $B = 0$ so that the first block is $(A, 0)$ and the last block is $(0, A)$.

We shall refer to the representations based on the backbones of (5.1ab), whose existence has been proved in this section, as irreps as this will be demonstrated in §13.

Table 4: The Coefficients t_{ij} which Uniquely Specify the First Ten Irreps. The reference numbers of the irreps refers to Table 1. Only the even numbered irreps are listed and these are of Type 1a and have $t_{ji} = -t_{ij}$. For irreps of Type 1b (odd numbers in Table 1) $t_{ji} = t_{ij} = \frac{1}{2}$ for all $t_{n,n+1}$ for $n \in [1, N - 1]$.

Rep	t_{12}	t_{23}	t_{34}	t_{45}	t_{56}
2	$\frac{1}{\sqrt{2}}$	-	-	-	-
4	$\frac{1}{2}$	$\frac{\sqrt{5}}{2}$	-	-	-
6	$\frac{1}{\sqrt{6}}$	$\sqrt{\frac{7}{12}}$	$\frac{3}{2}$	-	-
8	$\frac{1}{\sqrt{8}}$	$\sqrt{\frac{3}{8}}$	1	$\sqrt{\frac{7}{2}}$	-
10	$\frac{1}{\sqrt{10}}$	$\sqrt{\frac{11}{40}}$	$\sqrt{\frac{5}{8}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{5}$

9. There are No Irreps other than Those Based on the Backbones of (5.1a,b), Assuming No Duplicates

A backbone contains a “duplicate” if there are two or more blocks on the backbone with the same (A, B) . This section proves there are no irreps other than those of (5.1a,b) assuming no duplicates. Sections 10 to 12 address the issue of duplicates.

9.1 Expressions for the On-BD Components of $[V_x, V_y]$

The on-BD blocks of $[V_x, V_y]$ can be written,

$$[V_x, V_y]_{II} = V_{xIJ}V_{yJI} + V_{xIK}V_{yKI} + \dots - (V_{yIJ}V_{xJI} + V_{yIK}V_{xKI} + \dots) \quad (9.1)$$

where J, K, \dots are all the blocks connected to block I . We can write the contribution of each connected block J, K, \dots to (9.1) as,

$$t_{IJ}t_{JI}Z_{IJ} = i(V_{xIJ}V_{yJI} \dots - V_{yIJ}V_{xJI}) \quad (9.2)$$

So that Z_{IJ} is real and we have,

$$i[V_x, V_y]_{II} = t_{IJ}t_{JI}Z_{IJ} + t_{IK}t_{KI}Z_{IK} + \dots \quad (9.3a)$$

Here we have factored out the dependence on $t_{IJ} = \pm t_{JI}$ so that Z_{IJ} has a unique expression independent of these unknown parameters. For the four possible cases of the connectivity between blocks I and J the expressions for Z_{IJ} can be calculated using Table 2. The results are given in Table 4 in terms of (A_I, B_I) and (a_I, b_I) . Note that due to the CRs (2.3), the on-BD block $[V_x, V_y]_{II}$ is required to be identically equal to $i(J_z)_{II}$, which is a diagonal matrix, given

by (4.4e). Care with the signs is essential, the real quantity $i[V_x, V_y]_{II} \equiv -(J_z)_{II} \equiv -(a_I + b_I)$. In the latter expression and throughout this section we shall generally drop the factors of Kronecker deltas when referring to diagonal matrices, e.g., $\delta_{a_I a'_I} \delta_{b_I b'_I}$, which should strictly be present. These should be understood.

Similarly we can define \tilde{Z}_{IJ} by,

$$[V_t, V_z]_{II} = t_{IJ} t_{JI} \tilde{Z}_{IJ} + t_{IK} t_{KI} \tilde{Z}_{IK} + \dots \quad (9.3b)$$

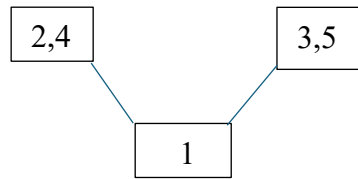
Again using Table 2 it is found that the \tilde{Z}_{IJ} are equal to the Z_{IJ} of Table 4 except with the sign of the terms in a_I reversed. As $[V_t, V_z]_{II} = i(K_z)_{II} = a_I - b_I$ is the same as $-(J_z)_{II}$ with the sign of a_I reversed, satisfaction of this CR also follows. Satisfaction of the other CRs (2.3) is automatic as Table 2 was derived on this basis.

Table 4: The Z_{IJ} in (9.2) and (9.3a).

Connectivity IJ	Z_{IJ}
++	$4[A_I b_I + B_I a_I]$
--	$-4[(A_I + 1)b_I + (B_I + 1)a_I]$
+-	$4[A_I b_I - (B_I + 1)a_I]$
-+	$-4[(A_I + 1)b_I - B_I a_I]$

9.2 The Minimum A and B in Any Representation is Zero

Figure 1



The irreps (5.1a,b) have minimum values of $A_{min} = 0$ and $B_{min} = 0$. In fact this must be the case for any proposed representation. A block at minimum $A = A_{min}$, say (A_{min}, B) , can be connected only to blocks at $(A_{min} + \frac{1}{2}, B - \frac{1}{2})$ and at $(A_{min} + \frac{1}{2}, B + \frac{1}{2})$. There may be multiple copies of such blocks, say blocks 2 and 4 at the former and blocks 3 and 5 at the latter. This connectivity is illustrated in Figure 1 which plots A vertically and B horizontally. This type of connectivity diagram will be used extensively from this point.

The connections 12 and 14 are of type $- +$ whereas connections 13 and 15 are $--$. Using (9.3a) and Table 4 gives, (9.4)

$$\frac{i}{4} [V_x, V_y]_{11} = -(t_{12} t_{21} + t_{14} t_{41}) [(A + 1)b_1 - B a_1] - (t_{13} t_{31} + t_{15} t_{51}) [(A + 1)b_1 + (B + 1)a_1]$$

But from (7.21a,b) we have $t_{12} = t_{21}$, $t_{14} = t_{41}$, $t_{13} = -t_{31}$, $t_{15} = -t_{51}$, so (9.4) becomes,

$$\begin{aligned} \frac{i}{4} [V_x, V_y]_{11} &= -(t_{12}^2 + t_{14}^2) [(A + 1)b_1 - B a_1] + (t_{13}^2 + t_{15}^2) [(A + 1)b_1 + (B + 1)a_1] \\ &= b_1 (A + 1) [(t_{13}^2 + t_{15}^2) - (t_{12}^2 + t_{14}^2)] + a_1 [B(t_{12}^2 + t_{14}^2) + (B + 1)(t_{13}^2 + t_{15}^2)] \end{aligned} \quad (9.5)$$

But CRs (2.3) give $\frac{i}{4}[V_x, V_y]_{11} \equiv -(J_z)_{11} = -(a_1 + b_1)$ and we see that this cannot be identical with (9.5), in which the coefficient of a_1 is non-negative, unless $A = A_{min} = 0$.

By symmetry the same must hold for B , i.e., $B_{min} = 0$.

A further observation which follows from (9.5) is that any block (A, B) which is connected only to a block (or blocks) “above” it, i.e., at $(A + \frac{1}{2}, B \pm \frac{1}{2})$, must have $A = 0$ if (9.5) is not to result in a contradiction.

By symmetry we see that any block (A, B) which is connected only to a block (or blocks) “to the right” of it, i.e., at $(A \pm \frac{1}{2}, B + \frac{1}{2})$, must have $B = 0$ if it is not to result in a contradiction.

Putting those two results together means that any block connected only to blocks above and to its right must be the origin, $(0,0)$. This result also follows from (9.5) when blocks 2 and 4 are omitted.

9.3 Unique Non-Monotonic Sequences

§7.3 defined a monotonic sequence. A sequence of connected blocks from block I to block J is a non-monotonic sequence if it is connected but not monotonic. Such a non-monotonic sequence is a unique non-monotonic sequence if there is no other non-monotonic sequence which connects block I to block J .

Considering a given sequence of three blocks, IKJ , whether monotonic or not, the corresponding off-BD block IJ of the CR $[V_x, V_y]$ will be denoted,

$$i[V_x, V_y]_{IJ} = t_{IK}t_{KJ}Z_{IKJ} \quad (9.4)$$

Here we have factored out the dependence on t_{IK} and t_{KJ} so that Z_{IKJ} has a unique expression independent of these unknown parameters. For a given block I there are 16 possible sequences of three blocks, IKJ , of which 4 are monotonic and 12 non-monotonic. Using Table 2 the resulting Z_{IKJ} can be evaluated in terms of (A_I, B_I) and (a_I, b_I) . The results are given in Table 5. Note that due to the CRs (2.3), and the fact that J_i are block-diagonal, the off-BD block $[V_x, V_y]_{IJ}$ is required to be identically zero.

Table 4 shows why monotonic sequences are special in this respect as they all have Z_{IKJ} which is identically zero.

In contrast, if in any proposed representation there is a unique non-monotonic sequence of three blocks between any pair of blocks I and J , the proposed representation cannot be valid because the corresponding Z_{IKJ} is not identically zero and so $[V_x, V_y]_{IJ}$ fails to be identically zero. (The cases $A_I = 0$ and/or $B_I = 0$ may seem to be an exception to this for some sequences in Table 5, but zero values for A_I and/or B_I are incompatible with the sequence in those cases, so there is no exception).

Note, however, that the stipulation that the non-monotonic sequence be unique is essential to this conclusion. If there were two or more non-monotonic sequences connecting blocks I and J , say IKJ, ILJ , etc., then (9.1) would become instead,

$$i[V_x, V_y]_{IJ} = t_{IK}t_{KJ}Z_{IKJ} + t_{IL}t_{LJ}Z_{ILJ} + \dots \quad (9.5a)$$

This does not now necessarily create a conflict with the requirement that $[V_x, V_y]_{IJ}$ be identically zero. This is because some of the expressions of Table 5 are linearly dependent upon others so that the RHS of (9.5a) could be identically zero if the values of $t_{IK}t_{KJ}$ and $t_{IL}t_{LJ}$ were chosen appropriately. This is why it is explicitly *unique* non-monotonic sequences which will be the focus of our attention, below, as these unambiguously indicate that a representation does not exist with the proposed connectivity.

Table 5: The Z_{IKJ} in (9.4) and (9.5a). The stated sequence relates to IK, KJ .

Sequence IK, KJ	Z_{IKJ}
++, ++	0
++, --	$4[A_I b_I + B_I a_I]$
++, +-	$-4B_I \sqrt{A_I^2 - a_I^2}$
++, -+	$-4A_I \sqrt{B_I^2 - b_I^2}$
--, ++	$-4[(A_I + 1)b_I + (B_I + 1)a_I]$
--, --	0
--, +-	$-4(A_I + 1) \sqrt{(B_I + 1)^2 - b_I^2}$
--, -+	$-4(B_I + 1) \sqrt{(A_I + 1)^2 - a_I^2}$
+-, ++	$4(B_I + 1) \sqrt{A_I^2 - a_I^2}$
+-, --	$4A_I \sqrt{(B_I + 1)^2 - b_I^2}$
+-, +-	0
+-, -+	$4[A_I b_I - (B_I + 1)a_I]$
-+, ++	$4(A_I + 1) \sqrt{B_I^2 - b_I^2}$
-+, --	$4B_I \sqrt{(A_I + 1)^2 - a_I^2}$
-+, +-	$4[B_I a_I - (A_I + 1)b_I]$
-+, -+	0

We can also define \tilde{Z}_{IKJ} via,

$$[V_t, V_z]_{IJ} = t_{IK}t_{KJ}\tilde{Z}_{IKJ} + t_{IL}t_{LJ}\tilde{Z}_{ILJ} + \dots \quad (9.5b)$$

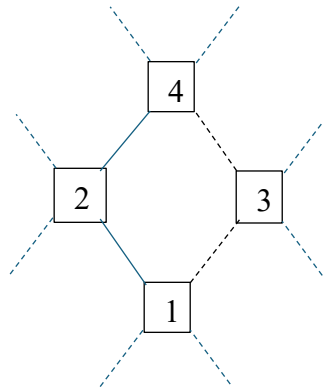
These \tilde{Z}_{IKJ} may be shown to equal the Z_{IKJ} of Table 5, but with the signs of all terms involving a_I reversed.

9.4 There are No Other Irreps than (5.1a,b): The Proof Without Duplicates

Consider some arbitrary block on the backbone, (A, B) , call it block number 1. Since there are no duplicates, any block can be connected to a maximum of four others. Consider the connectivity of block 1 with three other blocks, 2, 3, 4 as indicated in Figure 2. Here the A parameters are plotted on the vertical axis and the B parameters on the horizontal axis, and

connectivity is displayed by connecting lines. The blocks 2, 3, 4 are defined as $(A + \frac{1}{2}, B - \frac{1}{2})$, $(A + \frac{1}{2}, B + \frac{1}{2})$ and $(A + 1, B)$ respectively. The dashed lines between 3 and 4 and between 3 and 1 indicate that these blocks may or may not be connected, it makes no difference to the following argument.

Figure 2



All four blocks might be connected to further blocks, as indicated by the dashed lines.

But if block 2 is not connected to blocks other than 1 and 4 then the proposal does not provide a representation. This is shown by considering, from (9.3),

$$i[V_x, V_y]_{22} = t_{21}t_{12}Z_{12} + t_{24}t_{42}Z_{24}$$

But 21 has signature $+$ $-$ (the opposite of that for 12), and 24 has signature $-$ $-$, so Table 4 gives, using CR $i[V_x, V_y]_{22} = -(J_z)_{22}$ (9.6)

$$\frac{i}{4}[V_x, V_y]_{22} = t_{21}t_{12}[A_2b_2 - (B_2 + 1)a_2] - t_{24}t_{42}[(A_2 + 1)b_2 + (B_2 + 1)a_2] \equiv -(a_2 + b_2)/4$$

But from (7.21a,b) we have $t_{21} = t_{12}$ but $t_{24} = -t_{42}$, so that (9.6) becomes,

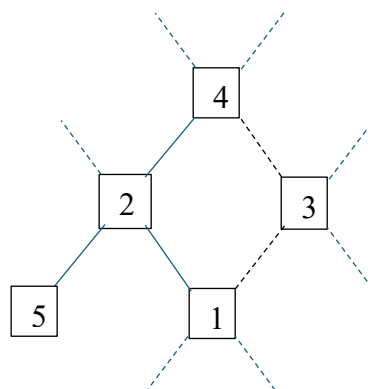
$$t_{12}^2[A_2b_2 - (B_2 + 1)a_2] + t_{24}^2[(A_2 + 1)b_2 + (B_2 + 1)a_2] \equiv -(a_2 + b_2)/4 \quad (9.10)$$

But, provided that $B_2 > 0$, the coefficient of b_2 on the LHS, which is $t_{12}^2A_2 + t_{24}^2(A_2 + 1)$, is positive definite and hence cannot equal that on the RHS, namely $-\frac{1}{4}$, so (9.10) is not possible.

Consequently block 2 must be connected to one or more further blocks unless $B_2 = 0$.

If we attempt to remedy that problem by adding a block 5 connected only to block 2, thus,

Figure 3



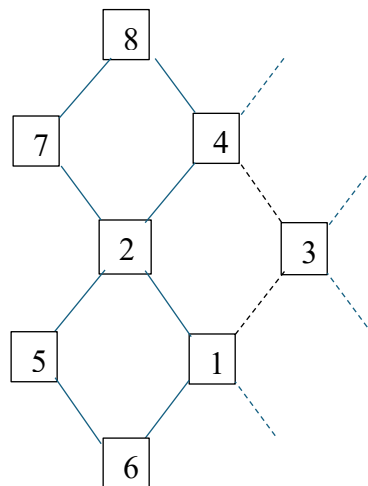
This creates a unique non-monotonic sequence 521, and hence cannot be a representation as shown in §9.3.

Adding a further block 6 connected to blocks 1 and 5 can potentially remedy that problem, but if block 5 is not connected to any further blocks, as in Figure 4, this just brings us back to the situation we encountered initially with block 2, namely that,

$$\frac{i}{4}[V_x, V_y]_{55} = t_{56}^2[A_5 b_5 - (B_5 + 1)a_5] + t_{25}^2[(A_5 + 1)b_5 + (B_5 + 1)a_5] \quad (9.11)$$

cannot be identical to $-(a_5 + b_5)/4$ unless $B_5 = 0$. Essentially the same situation is encountered if we add in block 7, and perhaps block 8, as also shown in Figure 4. These further blocks also result in inconsistencies with the requirements for a representation.

Figure 4:



By this process we are driven to add more blocks to the left until $B = 0$ is reached. (We have seen above that the smallest B must indeed be 0, and the observations of this section amount to another means of concluding this). For such blocks, with $B = 0$, the problem with $\frac{i}{4}[V_x, V_y]_{II}$ exemplified by (9.10) and (9.11) does not occur.

However, there is now another problem. The sequences of three blocks 652 and 527 and 278 in Figure 4 are all unique non-monotonic sequences – and the left boundary at $B = 0$ will always be driven to include such features if we start from Figure 2. So the connectivity we have been driven to is still not a possible representation (let alone an irrep). By symmetry the same problem would occur on the lower boundary at $A = 0$. Because we are addressing only finite representations the same problem will occur on the upper boundary at A_{max} and at the right boundary at B_{max} .

We conclude that the initial connectivity between blocks 1, 2 and 4 shown in Figure 1 leads unavoidably to a conflict and so there can be no such connectivity (i.e., non-monotonic) anywhere within the connectivity network. The only remaining possibilities are connection lines through the network which are monotonic everywhere, i.e., their gradient of ± 1 is constant, there being no changes of direction or branches.

The above derivation still drives the lines of connection leftwards or downwards to terminate at either $B = 0$ or $A = 0$ (or both). Indeed, we have already seen in §9.2 that this must be the case. In fact, the termination blocks of the monotonic sequences must be either $(0,0)$ or $(0, B_{max})$ or $(A_{max}, 0)$. This is established firstly by considering blocks 6, 1, 3, 7 and 8 in Figure 4 not to be present, so that we have a monotonic sequence of slope +1 of which 524 is a part. If block 4 is connected only to block 2, so that $B_4 = B_{max}$, then we require, noting that 42 is ++,

$$\frac{i}{4}[V_x, V_y]_{44} = t_{42}t_{24}[A_4b_4 + B_4a_4] = -t_{24}^2[A_4b_4 + B_4a_4] \equiv -(a_4 + b_4)/4 \quad (9.12)$$

and this requires $A_4 = A_{max} = B_4 = B_{max}$. But a connection line of slope 1 passing through $A_4 = B_4$ must start at (0,0) and ends in the other direction at (A_{max}, A_{max}) . This is precisely backbone (5.1a).

Alternatively we can consider the monotonic connection line 721 of slope -1 in Figure 3, so that blocks 8, 4, 3, 5 and 6 have now been dropped from the representation. We have already established that this line must terminate at $B = 0$ or $A = 0$, in fact it must terminate at $B = 0$ on the left and at $A = 0$ on the right, and, having a slope of -1, we again have $B_{max} = A_{max}$ and the termination points are $(0, A_{max})$ and $(A_{max}, 0)$. This is precisely backbone (5.1b).

Hence we have proved in this section that...

In the absence of duplicates, the only representations are those based on backbones (5.1a,b)

That these are irreps is shown in §13. Reducible representations can be defined using a backbone consisting of two or more of the backbones of (5.1a,b), for now as long as these do not intersect, as illustrated by Figures 5a,b. We will see shortly how intersecting backbones may be accommodated.

Figure 5a: An Example Backbone for a Reducible Representation. Only shaded boxes are present, others shown below are merely unoccupied place holders. Blocks which are connected are shown linked. Hence this is the backbone for the reducible representation based on the direct sum of the irrep based on $(2,0) \oplus \left(\frac{3}{2}, \frac{1}{2}\right) \oplus (1,1) \oplus \left(\frac{1}{2}, \frac{3}{2}\right) \oplus (0,2)$ and that based on $(0,0) \oplus \left(\frac{1}{2}, \frac{1}{2}\right)$. These are bosonic cases.

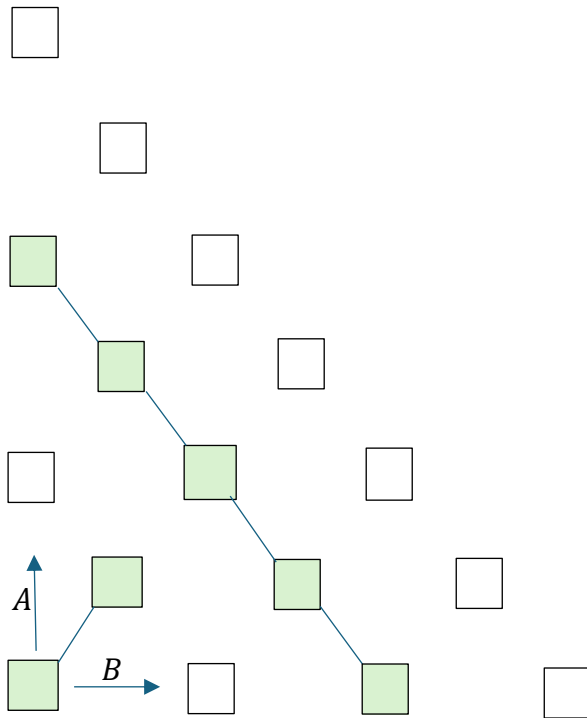
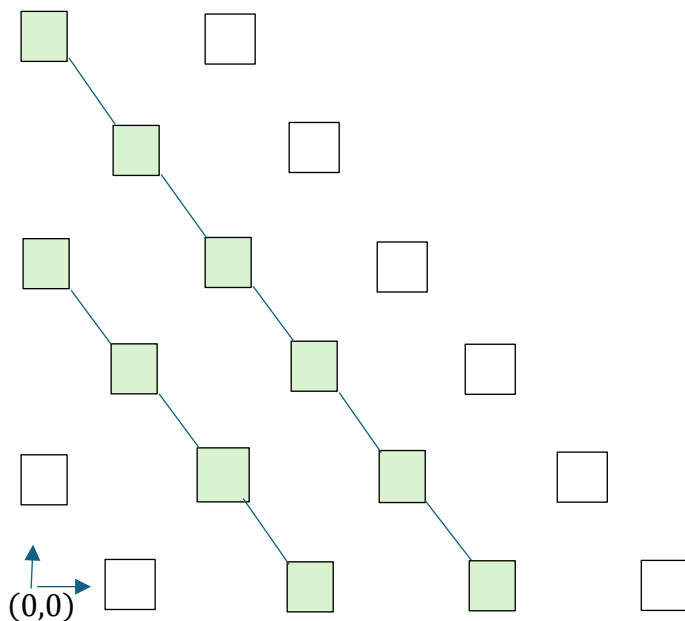


Figure 5b: A Further Example Backbone for a Reducible Representation. Only shaded boxes are present, others shown below are merely unoccupied place holders. Blocks which are connected are shown linked. Hence this is the backbone for the reduced representation based on the direct sum of the irrep based on $\left(\frac{3}{2}, 0\right) \oplus \left(1, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 1\right) \oplus \left(0, \frac{3}{2}\right)$ and $\left(\frac{5}{2}, 0\right) \oplus \left(2, \frac{1}{2}\right) \oplus \left(\frac{3}{2}, 1\right) \oplus \left(1, \frac{3}{2}\right) \oplus \left(\frac{1}{2}, 2\right) \oplus \left(0, \frac{5}{2}\right)$. These are fermionic cases.



10. The Defeat of the Proof of §9 by Duplicates.

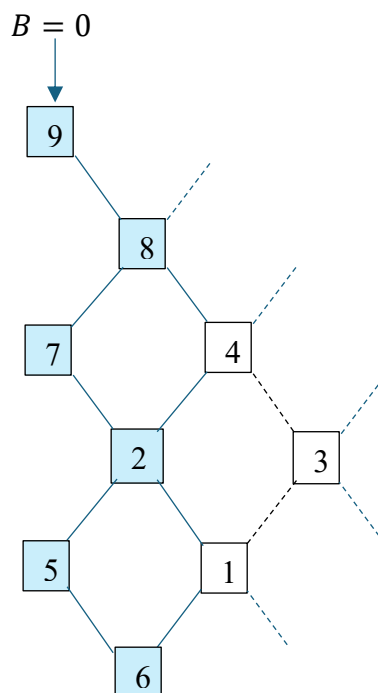
The possibility of duplicates undermines the proof of §9. To see why, suppose all the following backbone blocks were duplicated,

- All blocks with $A = 0$ or with $B = 0$;
- All blocks with $A = A_{max}$ or with $B = B_{max}$;
- All blocks with $A = \frac{1}{2}$ or with $B = \frac{1}{2}$;
- All blocks with $A = A_{max} - \frac{1}{2}$ or with $B = B_{max} - \frac{1}{2}$.

Figure 6 illustrates this near the $B = 0$ boundary. The other three boundaries are similar. The sequences of three blocks, 652, 527, 278 and 789, which would all be unique non-monotonic sequences if these blocks were singletons, have been rendered non-unique by duplication. Thus the argument of §9 fails.

Figure 6: Illustrating the Defeat of the Proof of §9 by Duplicates.

Blocks 5, 7, 9 have $B = 0$. All blocks have $0 < A < A_{max}$ and $B < B_{max}$. The shaded blocks, 5, 7, 9, 6, 2, 8, are duplicated so each of these in the diagram stand for two blocks with the same (A, B) .



However, we shall see that duplicates do not give rise to any further irreps. Instead the role of duplicates is to permit reducible representations consisting of the direct sum of an arbitrary number and arbitrary mix of irreps of type (5.1a,b). The presence of a duplicate in a valid representation will therefore indicate reducibility.

11. Uniqueness: Any Backbone has at Most One Representation (up to Equivalence)

Here we show that any proposed backbone, if it produces a representation at all, produces only one, up to equivalence. In other words, if a given backbone has two differently realised

representations for the V_μ then the two representations are equivalent (i.e., a unitary transformation exists to change one into the other).

The essence of the proof is simple and relies on the fact that irreps of the dSA can be constructed on a backbone of direct sums of the irreps of the algebra of the $SO(4)$ group, comprised of the generators J_i, V_i , instead of using, as we have here, backbones of direct sums of the irreps of the HLA comprised of the generators J_i, K_i . The key observation is that both algebras, up to complexification, are isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Consequently, a given representation expressed in terms of an HLA backbone can be unitarily transformed into a representation based on another $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ backbone, that for the J_i, V_i .

If the representation is reducible, and hence expressible in block diagonal form over irreps, a unitary transformation between the two irrep bases can be chosen so as to preserve the block diagonal structure, i.e., the number of irreps, and the dimensionality of each one, is independent of the irrep basis used. Suppose the backbone in the HLA basis is given by N blocks (A_i, B_i) and that in the $SO(4)$ basis by N blocks (A_i', B_i') . We thus have $(2A_i + 1)(2B_i + 1) = (2A_i' + 1)(2B_i' + 1)$ for all i . But the $SO(3)$ subgroup generated by the J_i is the same regardless of the basis, and so each block (A_i', B_i') must provide the same J eigenvalues as (A_i, B_i) , hence $A_i + B_i = A_i' + B_i'$ and $|A_i - B_i| = |A_i' - B_i'|$. This is sufficient to imply that, for every i independently, either $A_i' = A_i$ and $B_i' = B_i$ or alternatively that $A_i' = B_i$ and $B_i' = A_i$.

The case that $A_i' = B_i$ and $B_i' = A_i$ corresponds, via (2.4b), to the same J_i but a K_i which has changed sign. All the CRs, (2.1-3), are invariant under this change of sign, and hence the i^{th} irrep is equivalent whether $A_i' = A_i$ and $B_i' = B_i$ or $A_i' = B_i$ and $B_i' = A_i$. If $A_i' = B_i$ and $B_i' = A_i$ applied for all i then the overall representation would be equivalent to that using $A_i' = A_i$ and $B_i' = B_i$. However, if $A_i' = B_i$ and $B_i' = A_i$ applies for some but not all the N irreps, and hence $A_i' = A_i$ and $B_i' = B_i$ applies for the rest, then overall this is a representation which is not equivalent to one based on $A_i' = A_i$ and $B_i' = B_i$ across all irreps. However, there is a further constraint to take into account, namely that the connectedness between blocks must be preserved. For example, if blocks 1 and 2 are connected then $A_1 = A_2 \pm \frac{1}{2}$ and $B_1 = B_2 \pm \frac{1}{2}$. But if $A_1' = B_1$ and $B_1' = A_1$ whilst $A_2' = A_2$ and $B_2' = B_2$ we would have only that $A_1' = B_2' \pm \frac{1}{2}$ and $B_1' = A_2' \pm \frac{1}{2}$ and connectedness in the alternative basis, which requires $A_1' = A_2' \pm \frac{1}{2}$ and $B_1' = B_2' \pm \frac{1}{2}$, is not automatically preserved. Careful consideration of the possible combinations of signs shows that connectedness is preserved only in two cases,

- $A_1 = B_1 = A_1' = B_1'$ and $A_2 = B_2 = A_2' = B_2'$, in which case the permutation $A \leftrightarrow B$ makes no difference, or,
- $(A_1', B_1') = (A_1 - 1, B_1 + 1)$ or $(A_1', B_1') = (A_1 + 1, B_1 - 1)$, but this case fails to preserve the minimum J eigenvalue, as we get $|A_1' - B_1'| = |A_1 - B_1 \pm 2|$, and so can be ruled out.

It is concluded that the two backbone structures, (A_i, B_i) based on the HLA, and (A_i', B_i') based on $SO(4)$, are identical. Thus any two realisations of the V_μ on the same backbone in the HLA basis have a unitary transformation to the same $SO(4)$ basis backbone, and hence

must have a unitary transformation which connects them, i.e., they are equivalent. In short, the representation on any given backbone, if it exists, is unique, up to equivalence.

12. The Structure of Reducible Representations and the Role of Duplicates

In terms of the connection diagram of A values against B values, we have seen (anticipating §13) that, in the absence of duplicates, the only irreps are either,

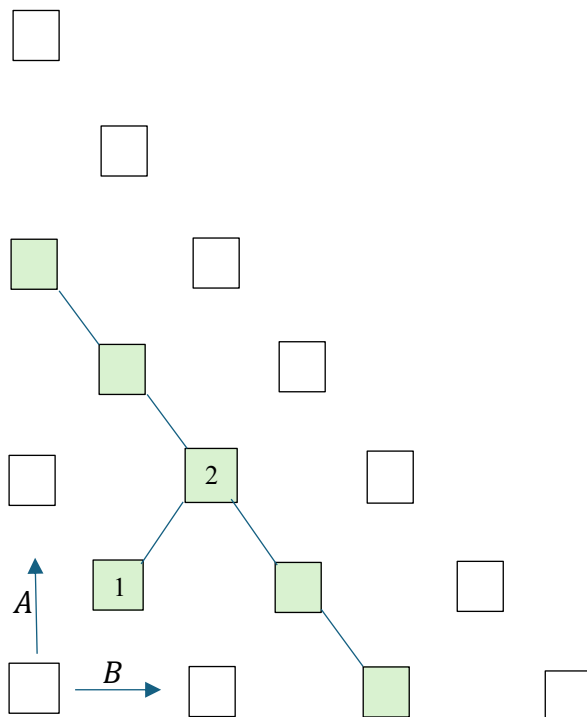
Based on Type 5.1a, whose backbone blocks lie on a trajectory of slope +1 and originate at the origin, $(0,0)$, or,

Based on Type 5.1b, whose backbone blocks lie on a trajectory of slope -1 and originate on $B = 0$ and terminate on $A = 0$.

A backbone which consists of the backbones of several different irreps but with non-intersecting trajectories, provides a representation which is the direct sum of the corresponding irreps, i.e., it is reducible, being already in reduced form. Figures 5a and 5b are examples.

But what happens if trajectories intersect? For example consider that shown in Figure 5.

Figure 7: An Example of Intersecting Backbone Trajectories. *Only shaded boxes are present, others shown below are merely unoccupied place holders. Blocks which are connected are shown linked. Numbers are block labels. There is no representation on the backbone displayed here (proved in text).*



But there is a problem with the proposed backbone of Figure 7. We have seen in §8.2 that the trajectory of slope -1, i.e., $(2,0) \oplus \left(\frac{3}{2}, \frac{1}{2}\right) \oplus (1,1) \oplus \left(\frac{1}{2}, \frac{3}{2}\right) \oplus (0,2)$, requires the t_{ij} along it to be $\frac{1}{2}$. But a conflict will arise at block 2 due to the link with block 1. If t_{12} is not zero it will compromise the self-consistency of the other t_{ij} . But if $t_{12} = 0$ block 1 is then a singleton and hence Figure 7 would also not provide a representation.

Figure 8: Another Example of Intersecting Backbone Trajectories. Only shaded boxes are present, others shown below are merely unoccupied place holders. Blocks which are connected are shown linked. Numbers are block labels. There is no representation on the backbone illustrated here.

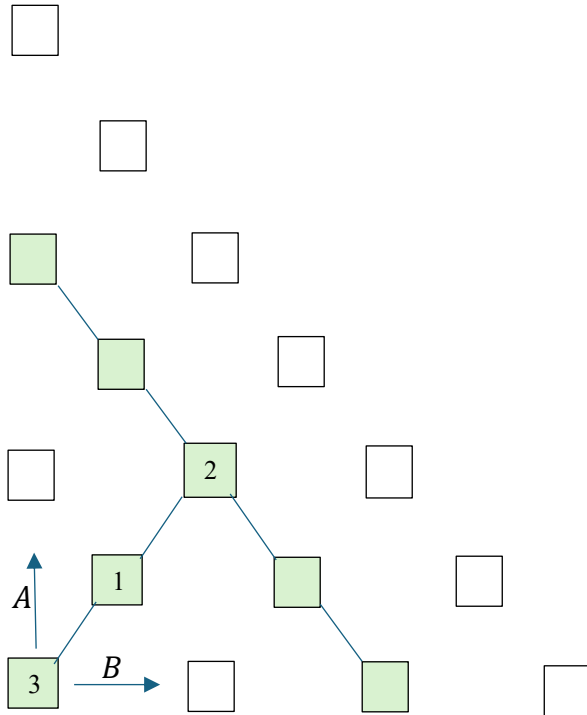
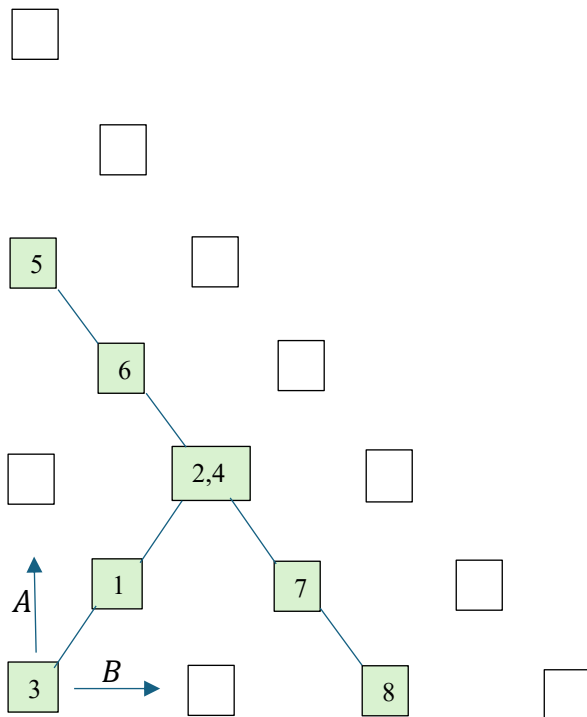


Figure 9: A Modified Version of the Example Case of Figure 8. There is now a duplicate block, number 4, at the same (A, B) as block 2. Numbers are block labels. A reducible representation results.

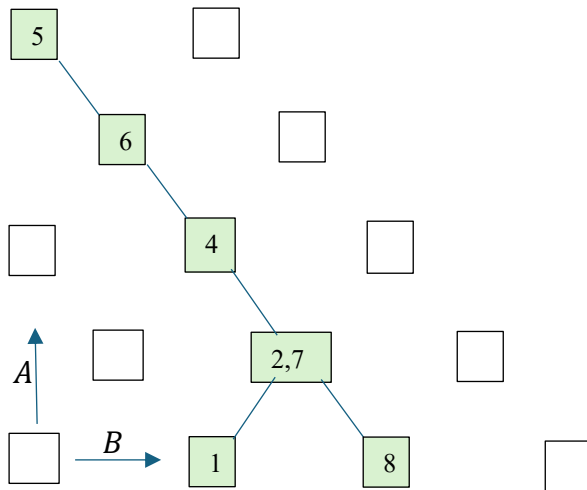


In fact, in Figure 7, the “dangling free end” at block 1 is not possible anyway, as follows from §9.2. We can, of course, overcome that problem by also including the block at the origin, (0,0), as shown in Figure 8 as block 3. However, the conflict at block 2 remains, with the 5-block Type 5.1b irrep with trajectory slope -1 and the 3-block Type 5.1a irrep with trajectory slope +1 requiring inconsistent t_{ij} values at block 2 where they intersect.

The problem resulting in Figure 7 failing to provide a representation can be overcome by assuming two blocks exist at the intersection, block 4 joining block 2 with the same (A, B) , which is (1,1) in this illustration. We now take block 1 to be connected to block 4 but not to block 2, whilst blocks 6 and 7 are connected to block 2 but not block 4. There is now no inconsistency, we merely have the backbone which provides both the irrep on backbone $(0,0) \oplus (\frac{1}{2}, \frac{1}{2}) \oplus (1,1)$ and that on $(2,0) \oplus (\frac{3}{2}, \frac{1}{2}) \oplus (1,1) \oplus (\frac{1}{2}, \frac{3}{2}) \oplus (0,2)$, i.e., Figure 9 provides a valid representation but it is reducible.

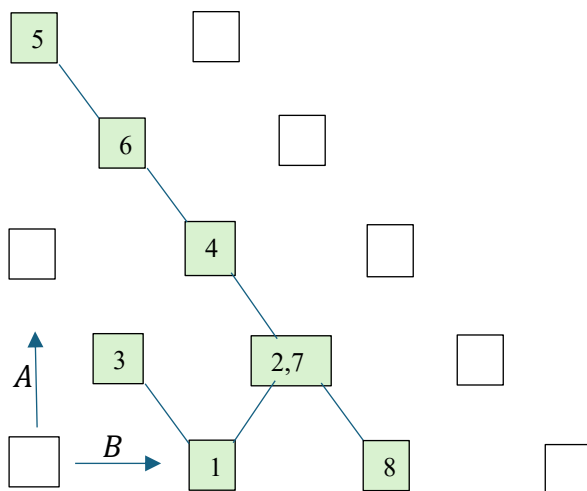
We may attempt a representation including the connectivity of Figure 10.

Figure 10: Another Failed Attempt to Find a Representation.



But this also fails because §9.2 has established that block 1, being connected only to blocks above and to the right must be at the origin.

Figure 11: And Another Failed Attempt to Find a Representation.

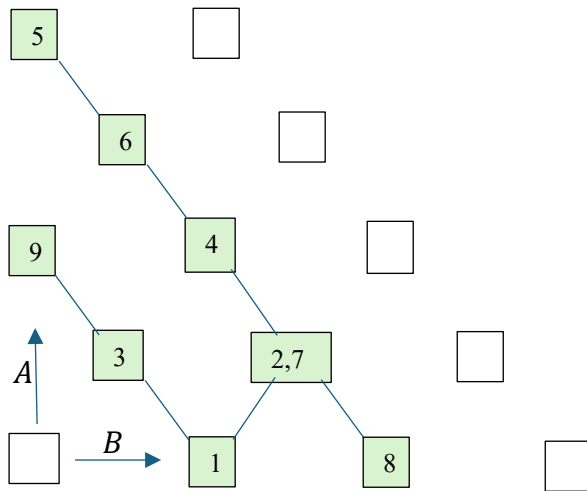


The problem with the attempt of Figure 10 may be avoided by adding block 3 shown in Figure 11. However, this merely pushes the problem to block 3 because §9.2 also showed that a block only connected to blocks to its right must have $B = 0$. This problem in turn can be avoided by also introducing a further block 9, as shown in Figure 12. However, to avoid a conflict at blocks 2,7 we must assume one of these (say block 2) is connected only to blocks 4 and 8, whilst only block 7 is connected to block 1. But now block 7 is a “dangling loose end” which introduces another inconsistency, namely that,

$$\frac{i}{4}[V_x, V_y]_{77} = t_{71}t_{17}[A_7b_7 + B_7a_7] \quad (12.1)$$

cannot be identical to $i(J_z)_{77} = -(a_7 + b_7)$.

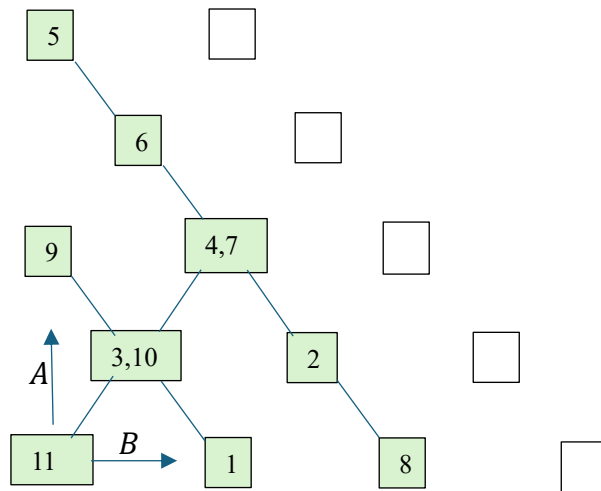
Figure 12: And Yet Another Failed Attempt to Find a Representation.



So we are driven to drop block 7 and the connection between block 1 and block 2, leaving Figure 12 to become the reducible representation consisting of the direct sum of irreps $(1,0) \oplus \left(\frac{1}{2}, \frac{1}{2}\right) \oplus (0,1)$ and $(2,0) \oplus \left(\frac{3}{2}, \frac{1}{2}\right) \oplus (1,1) \oplus \left(\frac{1}{2}, \frac{3}{2}\right) \oplus (0,2)$.

A connection between these two block sequences can be introduced between blocks 3 and 4, but only if the block at the origin is also included, and only if sufficient duplicates are included to avoid conflicts, as shown in Figure 13. This 3-irrep reducible representation consists of $(1,0) \oplus \left(\frac{1}{2}, \frac{1}{2}\right) \oplus (0,1)$ and $(2,0) \oplus \left(\frac{3}{2}, \frac{1}{2}\right) \oplus (1,1) \oplus \left(\frac{1}{2}, \frac{3}{2}\right) \oplus (0,2)$ and $(0,0) \oplus \left(\frac{1}{2}, \frac{1}{2}\right) \oplus (1,1)$. Note that all these are bosonic.

Figure 13: A Backbone which Produces a 3-Irrep Reducible Representation.



The general case should now be clear. The most general representation consists of the direct sum of all the following,

- Irreps with backbones of length r blocks and of Type 1b (connection slope -1), n_r copies of each ($r \geq 2$), and,
- Irreps with backbones of length s blocks and of Type 1a (connection slope +1), \tilde{n}_s copies of each ($s \geq 2$).

Where the Type 1a and Type 1b connection lines meet or cross, duplicate blocks are required, the total number of blocks at a given (A, B) being equal to the number of irreps which include (A, B) . Figures 14a and 14b present a random illustration of how this works, Figure 14a for the bosonic cases and Figure 14b for the fermionic cases. Note that the numbers in Figures 14a,b are the number of duplicate blocks at that (A, B) .

Figure 14a illustrates the backbone for the reducible representation consisting of the direct sum of representations based on the following irreps,

- 2 blocks $(1,0) \oplus \left(\frac{1}{2}, \frac{1}{2}\right) \oplus (0,1)$;
- 4 blocks $(2,0) \oplus \left(\frac{3}{2}, \frac{1}{2}\right) \oplus (1,1) \oplus \left(\frac{1}{2}, \frac{3}{2}\right) \oplus (0,2)$;
- 7 blocks $(3,0) \oplus \left(\frac{5}{2}, \frac{1}{2}\right) \oplus (2,1) \oplus \left(\frac{3}{2}, \frac{3}{2}\right) \oplus (1,2) \oplus \left(\frac{1}{2}, \frac{5}{2}\right) \oplus (0,3)$;
- 5 blocks $(0,0) \oplus \left(\frac{1}{2}, \frac{1}{2}\right)$;
- 1 block $(0,0) \oplus \left(\frac{1}{2}, \frac{1}{2}\right) \oplus (1,1)$;
- 1 block $(0,0) \oplus \left(\frac{1}{2}, \frac{1}{2}\right) \oplus (1,1) \oplus \left(\frac{3}{2}, \frac{3}{2}\right)$.

Figure 14b illustrates the backbone for the reducible representation consisting of the direct sum of representations based on the following irreps,

- 2 copies of $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$;
- 4 copies of $\left(\frac{3}{2}, 0\right) \oplus \left(1, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 1\right) \oplus \left(0, \frac{3}{2}\right)$;
- 7 copies of $\left(\frac{5}{2}, 0\right) \oplus \left(2, \frac{1}{2}\right) \oplus \left(\frac{3}{2}, 1\right) \oplus \left(1, \frac{3}{2}\right) \oplus \left(\frac{1}{2}, 2\right) \oplus \left(0, \frac{5}{2}\right)$.

Figure 14a: The General Case of Any Bosonic Representation Showing its Irrep Structure. *The numbers indicate the number of copies of each block (an arbitrary illustration). Bosonic cases may be of Type 5.1a or 5.1b.*

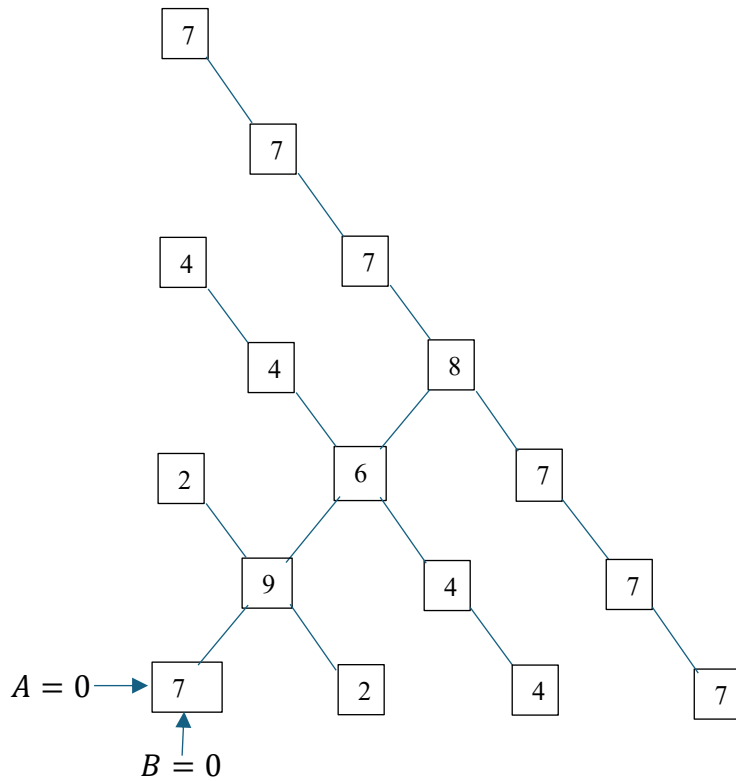
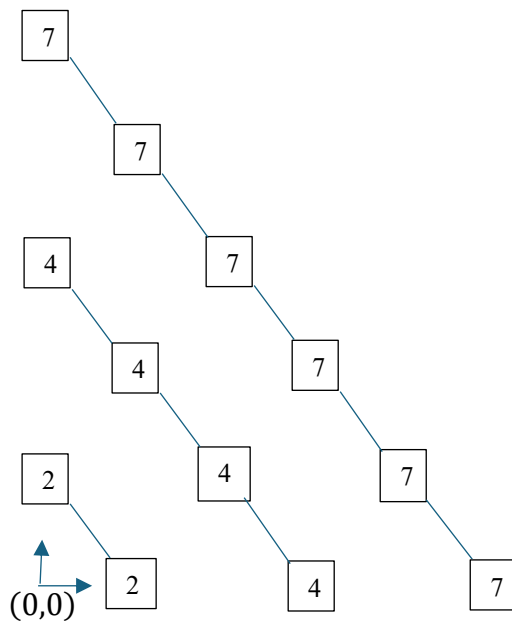


Figure 14b: The General Case of Any Fermionic Representation Showing its Irrep Structure. *The numbers indicate the number of copies of each block (an arbitrary illustration). Fermionic representations are of Type 5.1b only (because (0,0) is not fermionic so Type 5.1a is excluded).*



13. The Casimir Matrices

The Casimir operators are,

$$C1 = V_t^2 + K^2 - J^2 - V^2 \quad (13.1)$$

$$C2 = (\bar{K} \cdot \bar{J})^2 - (\bar{V} \cdot \bar{J})^2 + J^2 \quad (13.2)$$

where $V^2 = V_x^2 + V_y^2 + V_z^2$, etc., and,

$$\bar{J} = V_t \bar{J} + \bar{K} \times \bar{V} \quad (13.3)$$

The Casimir operators commute with all the generators of the Lie algebra and hence their representations within an irrep equals a unit matrix times a constant, the Casimir invariants. The possible values of the Casimir invariants for the irreps can be derived by explicit substitution of the complete representation matrices derived above. For C1 this is facilitated by the following algebraically equivalent expression, (13.4)

$$C1 = K_z^2 - J_z^2 + \frac{1}{2} \{ (K_+ K_- + K_- K_+) - (J_+ J_- + J_- J_+) \} - 2 \{ (V_+ V_- + V_- V_+) + (W_+ W_- + W_- W_+) \}$$

Each of the terms in (13.4) can now be evaluated using (4.2a-e) and Tables 2 and 4. Doing so confirms that the Casimir matrices are indeed multiples of the unit matrix and hence confirms that the representations based on the backbones of (5.1a,b) are irreps (and, we now know, the only irreps). Direct sums of multiple copies of the same irrep would also produce Casimir matrices which were multiples of the unit matrix, but their reducible nature is then betrayed by the presence of duplicates. The absence of duplicates in (5.1a,b) confirms them to be the basis of irreps.

To find the Casimir invariants it suffices to evaluate block 11 for the irreps of (5.1a,b). A diagonal matrix results, as it must, and we find the Casimir invariants to be

$$\text{For Type 5.1a:} \quad -C1 = 4A_1(A_1 + 1) + 8t_{12}^2 A_1^2 \quad (13.5a)$$

$$\text{For Type 5.1b:} \quad -C1 = 2A_1(A_1 + 1) + 8t_{12}^2 A_1 \quad (13.5b)$$

As noted in §1, work on the infinite dimensional unitary representations has resulted in C1 being expressed in terms of two parameters, which, in the case of the discrete representations, take positive integral or half-integral values. Various notations have been used. In that of Dixmier and Enayati et al, the expression is,

$$-C1 = p(p + 1) + (q + 1)(q - 2) \quad (13.6)$$

in which p takes values from the infinite range $p \in \left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots \right\}$ and q takes values from the finite range $q \in \left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots, p \right\}$. (Infinite dimensional unitary representations are also found with the same expression for C1 but in which q takes values from a continuum range).

Recall that for irreps Type 5.1b we have $t_{21} = t_{12} = \frac{1}{2}$, whereas for irreps Type 5.1a Table 4 gives us $t_{12} = -t_{21} = \frac{1}{2\sqrt{A_1}}$. We thus find,

$$\text{For Type 5.1a:} \quad -C1 = 2A_1(2A_1 + 3) = p(p + 1) - 2 \quad (13.7a)$$

$$\text{where,} \quad p = 2A_1 + 1 \text{ and } q = 0 \text{ and } p \in \{2, 3, 4 \dots\} \quad (13.7b)$$

Here the value of q assigned makes (13.7a) consistent with the infinite dimensional case, (13.6). The infinite dimensional unitary representations also provide a parametrised expression for $C2$,

$$-C2 = p(p + 1)q(q - 1) \quad (13.8)$$

This expression implies that $C2 = 0$ for Type 5.1a for which $q = 0$. This is confirmed by direct computation.

Expressions (13.6) and (13.8) for the two Casimir invariants are also found to apply for Type 5.1b but now with $q = p$, thus,

$$\text{For Type 5.1b:} \quad -C1 = 2A_1(A_1 + 2) = 2(p^2 - 1) \quad (13.9a)$$

$$\text{where,} \quad p = q = A_1 + 1 \quad \text{and} \quad p \in \left\{ \frac{3}{2}, 2, \frac{5}{2}, 3 \dots \right\} \quad (13.9b)$$

Hence, the finite representations of interest here have both Casimir invariants expressible in terms of a single integral or half-integral parameter, p , as given explicitly by (13.7a) and (13.9a). The range of possible p values is more restricted than for the unitary case, the values $\frac{1}{2}$ and 1 being excluded.

Table 6 summarises the values taken by the Casimir scalars for the first ten irreps.

Table 6: The Values of the Casimir Scalars for the First Ten Irreps. The reference number follows Table 1. Note that $q = 0$ could equally be $q = 1$.

Ref	p	q	-Casimir1	-Casimir2
Type 5.1b				
1	1.5	1.5	5 / 2	45 / 16
3	2	2	6	12
5	2.5	2.5	21 / 2	525 / 16
7	3	3	16	72
9	3.5	3.5	45 / 2	2205 / 16
Type 5.1a				
2	2	0	4	0
4	3	0	10	0
6	4	0	18	0
8	5	0	28	0
10	6	0	40	0

14. Conclusion

The complete set of all the finite irreps of the de Sitter Lie algebra with either Hermitian or anti-Hermitian generators have been found to have HLA backbones given by (5.1a) or (5.1b). Their matrix elements have been specified, being defined by (4.4a-f), Table 2 and Table 4 with equation (8.13). The two Casimir invariants may be specified in terms of a single integral or half-integral parameter, p , as given by (13.8a,b) and (13.9a,b). Any representations based on any other HLA backbones must be reducible.

15. References

- [1] Thomas, L.H. (1941). *On Unitary Representations of the Group of De Sitter Space*. Annals of Mathematics, 42(1) 113-126 (January 1941). Available online at <https://www.jstor.org/stable/1968990>.
- [2] Newton, T.D. (1950). *A Note on the Representations of the De Sitter Group*. Annals of Mathematics, 51(3) 730-733 (May 1950). Available online at <https://www.jstor.org/stable/1969376>.
- [3] Dixmier, J. (1961). *Représentations intégrables du groupe de De Sitter*. Bull. Soc. Math. France, 89, 9-41 (1961), p. 9-41. http://www.numdam.org/item?id=BSMF_1961__89__9_0
- [4] Enayati, M., Gazeau, J.P., Pejhan, H., and Wang, A. *The de Sitter (dS) Group and its Representations; An Introduction to Elementary Systems and Modeling the Dark Energy Universe* (2nd edition), Springer Nature (2024). <https://doi.org/10.1007/978-3-031-56552-6>.
- [5] Shurleff, R. (2007). *A Derivation of Vector and Momentum Matrices*. [arXiv:math-ph/0401002](https://arxiv.org/abs/math-ph/0401002) (version 3 dated 2 July 2007).