Prove: (a) $\nabla^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$ (2D polars)

(b) $\begin{cases} \sigma_r = \frac{1}{r^2}\phi_{,\theta\theta} + \frac{1}{r}\phi_{,r} & \sigma_\theta = \phi_{,rr} \\ \tau_{r\theta} = \frac{1}{r^2}\phi_{,\theta} - \frac{1}{r}\phi_{,r\theta} \end{cases}$

$x = r\cos\theta$
$y = r\sin\theta$
$\tan\theta = y/x$

$$\frac{\partial}{\partial x} = \left(\frac{\partial r}{\partial x}\right)_y \frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial x}\right)_y \cdot \frac{\partial}{\partial \theta} \quad ; \quad r^2 = x^2 + y^2$$

$$r\left.\frac{\partial r}{\partial x}\right)_y = x \Rightarrow \left.\frac{\partial r}{\partial x}\right)_y = \frac{x}{r} = \cos\theta$$

$$\sec^2\theta \cdot \left.\frac{\partial\theta}{\partial x}\right)_y = -\frac{y}{x^2} = -\frac{r\sin\theta}{r^2\cos^2\theta} \Rightarrow \left.\frac{\partial\theta}{\partial x}\right)_y = -\frac{\sin\theta}{r}$$

$$\Rightarrow \partial_x = \cos\theta\,\partial_r - \frac{\sin\theta}{r}\,\partial_\theta$$

$$\partial_y = \left.\frac{\partial r}{\partial y}\right)_x \partial_r + \left.\frac{\partial\theta}{\partial y}\right)_x \partial_\theta \quad ; \quad r\left.\frac{\partial r}{\partial y}\right)_x = y \Rightarrow \left.\frac{\partial r}{\partial y}\right)_x = \frac{y}{r} = \sin\theta$$

$$\sec^2\theta \cdot \left.\frac{\partial\theta}{\partial y}\right)_x = \frac{1}{x} = \frac{1}{r\cos\theta} \Rightarrow \left.\frac{\partial\theta}{\partial y}\right)_x = \frac{\cos\theta}{r}$$

$$\Rightarrow \partial_y = \sin\theta\cdot\partial_r + \frac{\cos\theta}{r}\cdot\partial_\theta$$

$$\therefore \nabla^2 = \cos^2\theta\,\partial_r^2 + \frac{\sin\theta}{r}\partial_\theta\left(\frac{\sin\theta}{r}\partial_\theta\right) - \cancel{\sin\theta\cos\theta\,\partial_r\left(\frac{1}{r}\partial_\theta\right)} - \frac{\sin\theta}{r}\partial_\theta\left(\cos\theta\cdot\partial_r\right)$$
$$+ \sin^2\theta\,\partial_r^2 + \frac{\cos\theta}{r}\partial_\theta\left(\frac{\cos\theta}{r}\partial_\theta\right) + \cancel{\sin\theta\cos\theta\,\partial_r\left(\frac{1}{r}\partial_\theta\right)} + \frac{\cos\theta}{r}\partial_\theta\left(\sin\theta\cdot\partial_r\right)$$

$$= \partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r}\partial_r \qquad \checkmark \text{OK}$$

$$\sigma_r = \cos^2\theta \cdot \sigma_x + \sin^2\theta \cdot \sigma_y + 2\sin\theta\cos\theta \cdot \tau_{xy}$$
$$\sigma_\theta = \sin^2\theta \cdot \sigma_x + \cos^2\theta \cdot \sigma_y - 2\sin\theta\cos\theta \cdot \tau_{xy}$$
$$\tau_{r\theta} = (\cos^2\theta - \sin^2\theta)\tau_{xy} + \cos\theta\sin\theta \cdot (\sigma_y - \sigma_x)$$

$$\Rightarrow \sigma_r = \Big[\underbrace{\cos^2\theta \cdot \partial_y^2 + \sin^2\theta \cdot \partial_x^2 - 2\sin\theta\cos\theta\, \partial_x\partial_y}_{[\sim]}\Big]\phi$$

$$(\cos\theta \cdot \partial_y - \sin\theta \cdot \partial_x)^2 = [\sim] + \cos\theta \cdot (\partial_y \cos\theta)\partial_y + \sin\theta \cdot (\partial_x \sin\theta) \cdot \partial_x$$
$$- \sin\theta(\partial_x \cos\theta) \cdot \partial_y - \cos\theta \cdot (\partial_y \sin\theta) \cdot \partial_x$$

$$\cos\theta \cdot (\partial_y \cos\theta) - \sin\theta(\partial_x \cos\theta) = \cos\theta\left(\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right)\cos\theta - \sin\theta\left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right)\cos\theta$$
$$= \cos\theta\left(-\frac{\sin\theta\cos\theta}{r}\right) + \sin\theta\left(-\frac{\sin^2\theta}{r}\right) = -\frac{\sin\theta}{r}(\sin^2\theta + \cos^2\theta) = -\frac{\sin\theta}{r}$$

$$\sin\theta \cdot (\partial_x \sin\theta) - \cos\theta(\partial_y \sin\theta) = \sin\theta\left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right)\sin\theta - \cos\theta\left(\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right)\sin\theta$$
$$= \sin\theta\left(-\frac{\sin\theta\cos\theta}{r}\right) - \cos\theta\left(\frac{\cos^2\theta}{r}\right) = -\frac{\cos\theta}{r}(\sin^2\theta + \cos^2\theta) = -\frac{\cos\theta}{r}$$

$$\therefore [\sim] = (\cos\theta\,\partial_y - \sin\theta\,\partial_x)^2 + \cancel{\frac{\sin\theta + \cos\theta}{r}}\ \frac{\sin\theta}{r}\partial_y + \frac{\cos\theta}{r}\partial_x$$

But, $\cos\theta\,\partial_y - \sin\theta\,\partial_x = \frac{1}{r}\partial_\theta$

and, $\sin\theta\,\partial_y + \cos\theta\,\partial_x = \partial_r$

Hence, $[\sim] = \left(\frac{1}{r}\partial_\theta\right)^2 + \frac{1}{r}\partial_r$ QED for σ_r.

$$\sigma_\theta = \Big[\sin^2\theta \cdot \partial_y^2 + \cos^2\theta \cdot \partial_x^2 + 2\sin\theta\cos\theta\,\partial_x\partial_y\Big]\phi$$

$$[\sim] = (\sin\theta \cdot \partial_y + \cos\theta \cdot \partial_x)^2 - \sin\theta \cdot (\partial_y \sin\theta) \cdot \partial_y - \cos\theta \cdot (\partial_x \cos\theta) \cdot \partial_x$$
$$- \cos\theta(\partial_x \sin\theta) \cdot \partial_y - \sin\theta(\partial_y \cos\theta) \cdot \partial_x$$

$$\sin\theta(\partial_y \sin\theta) + \cos\theta(\partial_x \sin\theta) = \sin\theta\left(\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right)\sin\theta + \cos\theta\left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right)\sin\theta$$

$$= \sin\theta\left(\frac{\cos^2\theta}{r}\right) + \cos\theta\left(-\frac{\sin\theta\cos\theta}{r}\right) = \underline{\underline{0}}$$

$$\cos\theta(\partial_x \cos\theta) + \sin\theta(\partial_y \cos\theta) = \cos\theta\left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right)\cos\theta + \sin\theta\left(\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right)\cos\theta$$

$$= \cos\theta\left(\frac{\sin^2\theta}{r}\right) + \sin\theta\left(-\frac{\sin\theta\cos\theta}{r}\right) = \underline{\underline{0}}$$

But, $\sin\theta . \partial_y + \cos\theta . \partial_x \equiv \partial_r$

$$\therefore \quad \underline{\underline{\sigma_\theta = \partial_r^2 \phi}} \qquad \text{QED.}$$

$$\tau_{r\theta} = \left[\sin\theta\cos\theta\left(\partial_x^2 - \partial_y^2\right) - \left(\cos^2\theta - \sin^2\theta\right)\partial_x\partial_y\right]\phi$$

$$[\sim] = \left(\cos\theta . \partial_x + \sin\theta . \partial_y\right)\left(\sin\theta . \partial_x - \cos\theta . \partial_y\right)$$

$$- \cos\theta(\partial_x \sin\theta)\partial_x + \cos\theta(\partial_x \cos\theta)\partial_y - \sin\theta(\partial_y \sin\theta)\partial_x + \sin\theta(\partial_y \cos\theta)\partial_y$$

$$\cos\theta(\partial_x \sin\theta) + \sin\theta(\partial_y \sin\theta) = \cos\theta\left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right)\sin\theta + \sin\theta\left(\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right)\sin\theta$$

$$= \cos\theta\left(-\frac{\sin\theta\cos\theta}{r}\right) + \sin\theta\left(\frac{\cos^2\theta}{r}\right) = \underline{\underline{0}}$$

$$\cos\theta(\partial_x \cos\theta) + \sin\theta(\partial_y \cos\theta) = \cos\theta\left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right)\cos\theta + \sin\theta\left(\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right)\cos\theta$$

$$= \cos\theta\left(\frac{\sin^2\theta}{r}\right) + \sin\theta\left(-\frac{\sin\theta\cos\theta}{r}\right) = \underline{\underline{0}}$$

$$\therefore \quad [\sim] \equiv \left(\partial_r\right)\left(-\frac{1}{r}\partial_\theta\right) \equiv \frac{1}{r^2}\partial_\theta - \frac{1}{r}\partial_r\partial_\theta$$

$$\text{i.e.,} \quad \underline{\underline{\tau_{r\theta} = \frac{1}{r^2}\phi_{,\theta} - \frac{1}{r}\phi_{,r\theta}}} \qquad \underline{\text{QED}}$$