

Prove: (a) $\nabla^2 = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2$ (2D polar)

$$(b) \begin{cases} \sigma_r = \frac{1}{r^2} \phi_{,\theta\theta} + \frac{1}{r} \phi_{,r} \\ \sigma_\theta = \phi_{,rr} \\ \chi_{r\theta} = \frac{1}{r^2} \phi_{,r\theta} - \frac{1}{r} \phi_{,r\theta} \end{cases} \quad \begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ \tan \theta &= \frac{y}{x} \end{aligned}$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial r}{\partial x} \right) \frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial x} \right) \frac{\partial}{\partial \theta} \quad ; \quad r^2 = x^2 + y^2$$

$$\left(\frac{\partial r}{\partial x} \right) = x \Rightarrow \left(\frac{\partial r}{\partial x} \right) = \frac{x}{r} = \cos \theta$$

$$\sec^2 \theta \cdot \left(\frac{\partial \theta}{\partial x} \right) = -\frac{y}{x^2} = -\frac{r \sin \theta}{r^2 \cos^2 \theta} \Rightarrow \left(\frac{\partial \theta}{\partial x} \right) = -\frac{\sin \theta}{r}$$

$$\Rightarrow \partial_x = \cos \theta \cdot \partial_r - \frac{\sin \theta}{r} \cdot \partial_\theta$$

$$\partial_y = \left(\frac{\partial r}{\partial y} \right) \partial_r + \left(\frac{\partial \theta}{\partial y} \right) \partial_\theta \quad ; \quad \left(\frac{\partial r}{\partial y} \right) = y \Rightarrow \left(\frac{\partial r}{\partial y} \right) = \frac{y}{r} = \sin \theta$$

$$\sec^2 \theta \cdot \left(\frac{\partial \theta}{\partial y} \right) = \frac{1}{x} = \frac{1}{r \cos \theta} \Rightarrow \left(\frac{\partial \theta}{\partial y} \right) = \frac{\cos \theta}{r}$$

$$\Rightarrow \partial_y = \sin \theta \cdot \partial_r + \frac{\cos \theta}{r} \cdot \partial_\theta$$

$$\therefore \nabla^2 = \cos^2 \theta \partial_r^2 + \frac{\sin \theta}{r} \partial_\theta \left(\frac{\sin \theta}{r} \partial_\theta \right) - \sin \theta \cos \theta \partial_r \left(\frac{1}{r} \partial_\theta \right) - \frac{\sin \theta}{r} \partial_\theta (\cos \theta \cdot \partial_r) + \sin^2 \theta \partial_r^2 + \frac{\cos \theta}{r} \partial_\theta \left(\frac{\cos \theta}{r} \partial_\theta \right) + \sin \theta \cos \theta \partial_r \left(\frac{1}{r} \partial_\theta \right) + \frac{\cos \theta}{r} \partial_\theta (\sin \theta \cdot \partial_r)$$

$$= \partial_r^2 + \frac{1}{r^2} \partial_\theta^2 + \frac{1}{r} \partial_{rr} \quad \underline{\underline{\text{VOK}}}$$

$$\begin{aligned} \sigma_r &= \cos^2 \theta \cdot \sigma_x + \sin^2 \theta \cdot \sigma_y + 2 \sin \theta \cos \theta \cdot \tau_{xy} \\ \sigma_\theta &= \sin^2 \theta \cdot \sigma_x + \cos^2 \theta \cdot \sigma_y - 2 \sin \theta \cos \theta \cdot \tau_{xy} \\ \tau_{r\theta} &= (\cos^2 \theta - \sin^2 \theta) \tau_{xy} + \cos \theta \sin \theta \cdot (\sigma_y - \sigma_x) \end{aligned}$$

$$\Rightarrow \sigma_r = \left[\cos^2 \theta \cdot \sigma_y^2 + \sin^2 \theta \cdot \sigma_x^2 - 2 \sin \theta \cos \theta \cdot \sigma_x \sigma_y \right] \phi$$

$$\begin{aligned} (\cos \theta \cdot \sigma_y - \sin \theta \cdot \sigma_x)^2 &= \left[\sigma_y^2 \cos^2 \theta + \sigma_x^2 \sin^2 \theta - 2 \sin \theta \cos \theta \sigma_x \sigma_y \right] \\ &+ \cos \theta \cdot (\partial_y \cos \theta) \sigma_y \\ &+ \sin \theta \cdot (\partial_x \sin \theta) \cdot \sigma_x \\ &- \sin \theta \cdot (\partial_x \cos \theta) \cdot \sigma_y - \cos \theta \cdot (\partial_y \sin \theta) \cdot \sigma_x \end{aligned}$$

$$\begin{aligned} \cos \theta \cdot (\partial_y \cos \theta) - \sin \theta \cdot (\partial_x \sin \theta) &= \cos \theta \left(\sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta \right) \cos \theta \\ &- \sin \theta \left(\cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta \right) \sin \theta \\ &= \cos \theta \left(-\frac{\sin \theta \cos \theta}{r} \right) + \sin \theta \left(-\frac{\sin^2 \theta}{r} \right) = -\frac{\sin \theta}{r} (\sin^2 \theta + \cos^2 \theta) = \underline{\underline{-\frac{\sin \theta}{r}}} \end{aligned}$$

$$\begin{aligned} \sin \theta \cdot (\partial_x \sin \theta) - \cos \theta \cdot (\partial_y \cos \theta) &= \sin \theta \left(\cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta \right) \sin \theta \\ &- \cos \theta \left(\sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta \right) \cos \theta \\ &= \sin \theta \left(-\frac{\sin \theta \cos \theta}{r} \right) - \cos \theta \left(\frac{\cos^2 \theta}{r} \right) = -\frac{\cos \theta}{r} (\sin^2 \theta + \cos^2 \theta) = \underline{\underline{-\frac{\cos \theta}{r}}} \end{aligned}$$

$$\therefore [m] = (\cos \theta \sigma_y - \sin \theta \sigma_x)^2 + \frac{\sin \theta}{r} \sigma_y + \frac{\cos \theta}{r} \sigma_x$$

But, $\cos \theta \sigma_y - \sin \theta \sigma_x = \frac{1}{r} \partial_\theta$
 and, $\sin \theta \sigma_x + \cos \theta \sigma_y = \partial_r$

Hence, $[m] = \left(\frac{1}{r} \partial_\theta \right)^2 + \frac{1}{r} \partial_r$ Q.E.D for σ_r .

$$\begin{aligned} \sigma_\theta &= \left[\sin^2 \theta \cdot \sigma_y^2 + \cos^2 \theta \cdot \sigma_x^2 + 2 \sin \theta \cos \theta \cdot \sigma_x \sigma_y \right] \phi \\ [m] &= (\sin \theta \cdot \sigma_y + \cos \theta \cdot \sigma_x)^2 - \sin \theta \cdot (\partial_y \sin \theta) \cdot \sigma_y \\ &- \cos \theta \cdot (\partial_x \cos \theta) \cdot \sigma_x \\ &+ \cos \theta \cdot (\partial_x \sin \theta) \cdot \sigma_y - \sin \theta \cdot (\partial_y \cos \theta) \cdot \sigma_x \end{aligned}$$

$$\begin{aligned} \sin\theta (\partial_y \sin\theta) + \cos\theta (\partial_x \sin\theta) &= \sin\theta \left(\sin\theta \partial_r + \frac{\cos\theta}{r} \partial_\theta \right) \sin\theta \\ &\quad + \cos\theta \left(\cos\theta \partial_r - \frac{\sin\theta}{r} \partial_\theta \right) \sin\theta \\ &= \sin\theta \left(\frac{\cos^2\theta}{r} \right) + \cos\theta \left(-\frac{\sin\theta \cos\theta}{r} \right) = \underline{\underline{0}} \end{aligned}$$

$$\begin{aligned} \cos\theta (\partial_x \cos\theta) + \sin\theta (\partial_y \cos\theta) &= \cos\theta \left(\cos\theta \partial_r - \frac{\sin\theta}{r} \partial_\theta \right) \cos\theta \\ &\quad + \sin\theta \left(\sin\theta \partial_r + \frac{\cos\theta}{r} \partial_\theta \right) \cos\theta \\ &= \cos\theta \left(\frac{\sin^2\theta}{r} \right) + \sin\theta \left(-\frac{\sin\theta \cos\theta}{r} \right) = \underline{\underline{0}} \end{aligned}$$

but, $\sin\theta \cdot \partial_y + \cos\theta \cdot \partial_x \equiv \partial_r$

$\therefore \underline{\underline{\sigma_\theta = \partial_r^2 \phi}}$ Q.E.D.

$$\tau_{r\theta} = \left[\sin\theta \cos\theta (\partial_x^2 - \partial_y^2) - (\cos^2\theta - \sin^2\theta) \partial_x \partial_y \right] \phi$$

$$\begin{aligned} [\dots] &= (\cos\theta \cdot \partial_x + \sin\theta \cdot \partial_y) (\sin\theta \cdot \partial_x - \cos\theta \cdot \partial_y) \\ &\quad - \cos\theta (\partial_x \sin\theta) \partial_x + \cos\theta (\partial_x \cos\theta) \partial_y - \sin\theta (\partial_y \sin\theta) \partial_x + \sin\theta (\partial_y \cos\theta) \partial_y \\ \cos\theta (\partial_x \sin\theta) + \sin\theta (\partial_y \sin\theta) &= \cos\theta \left(\cos\theta \partial_r - \frac{\sin\theta}{r} \partial_\theta \right) \sin\theta \\ &\quad + \sin\theta \left(\sin\theta \partial_r + \frac{\cos\theta}{r} \partial_\theta \right) \sin\theta \\ &= \cos\theta \left(-\frac{\sin\theta \cos\theta}{r} \right) + \sin\theta \left(\frac{\cos^2\theta}{r} \right) = \underline{\underline{0}} \end{aligned}$$

$$\begin{aligned} \cos\theta (\partial_x \cos\theta) + \sin\theta (\partial_y \cos\theta) &= \cos\theta \left(\cos\theta \partial_r - \frac{\sin\theta}{r} \partial_\theta \right) \cos\theta \\ &\quad + \sin\theta \left(\sin\theta \partial_r + \frac{\cos\theta}{r} \partial_\theta \right) \cos\theta \\ &= \cos\theta \left(\frac{\sin^2\theta}{r} \right) + \sin\theta \left(-\frac{\sin\theta \cos\theta}{r} \right) = \underline{\underline{0}} \end{aligned}$$

$$\therefore [\dots] \equiv (\partial_r) \left(-\frac{1}{r} \partial_\theta \right) \equiv \frac{1}{r^2} \partial_\theta - \frac{1}{r} \partial_r \partial_\theta$$

$\therefore \underline{\underline{\tau_{r\theta} = \frac{1}{r^2} \phi_{,\theta} - \frac{1}{r} \phi_{,r\theta}}}$ Q.E.D.