A Tale of Two Tails

RAWB, 24/7/14

A question which arises in respect of the correctness of R5V4/5 ccg calculations is,

> Is it correct to estimate ccg by inserting the calculated $C(t)$ into an empirical formula obtained by vetoing experimental data prior to t_{red} and hence based solely on C^*?

This question cannot be answered by carrying out R5V4/5 assessments of the lab tests if the comparison is made only for $t > t_{red}$, i.e., using only what is usually defined as 'valid' data, when the trend line has been achieved. However, if the tests are ever $C(t)$ controlled this will occur in the tails. Hence the question to be addressed here is,

> What characteristic of the tails indicates $C(t)$ control?

Here I derive the following result by assuming the usual ductility exhaustion based continuum damage mechanics,

> If the tail lies above the trend line then this indicates $C(t)$ control.
>
> If the tail lies below the trend line then this indicates that $C(t)$ is not controlling.

The deformation behaviour is assumed to be of the form $\dot{\varepsilon}^c = Bt^m \sigma^n$ where $-1 < m \le 0$

Secondary creep is obtained for $m = 0$. Secondary creep is more likely to apply near the crack tip due to the higher stresses. Hence we distinguish between the near-tip and bulk behaviour by writing,

$$\dot{\varepsilon}^c_{tip} = Bt^{m_{tip}} \sigma^n \text{ and } \dot{\varepsilon}^c_{bulk} = Bt^{m_{bulk}} \sigma^n \tag{1}$$

The creep HRR field gives the creep strain rate near the crack tip as,

$$\dot{\varepsilon}^c_{tip}(t,r) = \left(Bt^{m_{tip}}\right)^{1-q} \left(\frac{C(t)}{I_n r}\right)^q \tilde{\varepsilon}(\theta, n) \tag{2}$$

where $q = n/(n+1)$. In (2), $C(t)$ may be replaced by C^* if the latter is controlling. Consider a point initially at a distance r_c from the crack tip (on $\theta = 0$, assumed to be the direction of growth). The crack grows to this point in time t_c. Ductility exhaustion therefore requires,

$$\varepsilon^*_f = \int_0^{t_c} \dot{\varepsilon}^c_{tip}(t, r(t)) \cdot dt \tag{3}$$

where ε^*_f is the creep ductility under the triaxial stress state prevailing at this near-tip point. In (3), $r(t)$ is an unknown function required to have initial and final values $r(0) = r_c$ and $r(t_c) = 0$.

The analysis that follows applies over periods sufficiently short that the crack growth is small compared with the ligament, so that the reference stress and SIF are not significantly affected by the change in crack length. This is likely to be reasonable in the tails.

(A) Assume Elastic-Creep with $t << t_{red}$ and $C(t)$ Control

In this case, in (2) we can put,

$$C(t) = \frac{K^2}{(n+1)Et} \tag{4}$$

Substitution of the result into (3) gives,

$$\varepsilon_f^* = \int_0^{t_c} \left(Bt^{m_{tip}}\right)^{1-q} \left(\frac{K^2}{I_n(n+1)Ert}\right)^q \tilde{\varepsilon}(\theta,n)\cdot dt = D\int_0^{t_c} \frac{t^{m_{tip}(1-q)-q}}{r(t)^q} dt \tag{5}$$

where,
$$D = B^{1-q}\left(\frac{K^2}{I_n(n+1)E}\right)^q \tilde{\varepsilon}(\theta,n) \tag{6}$$

The integral equation, (5), is simply solved by guessing the solution to be of the form,

$$r = r_c\left(1-\tau^{\alpha}\right) \quad \text{where,} \quad \tau = \frac{t}{t_c} \tag{7}$$

Substitution of (7) into (5) gives,

$$\varepsilon_f^* = \Im D \frac{t_c^{(1-q)(1+m_{tip})}}{r_c^q} \qquad \text{where,} \quad \Im = \int_0^1 \frac{\tau^{m_{tip}(1-q)-q}}{\left(1-\tau^{\alpha}\right)^q} d\tau \tag{8}$$

But r_c is the crack growth, Δa, which occurs in time $t = t_c$, so we write,

$$\Delta a^q = \frac{\Im D}{\varepsilon_f^*} t^{(1-q)(1+m_{tip})} \tag{9}$$

i.e.,
$$\Delta a = r_c - r = r_c\left(\frac{t}{t_c}\right)^{\alpha} = \left(\frac{\Im D}{\varepsilon_f^*}\right)^{1/q} t^{(1-q)(1+m_{tip})/q} \tag{10}$$

Hence (7) is indeed a solution of (5) and we can identify the index as,

$$\alpha = \frac{(1-q)(1+m_{tip})}{q} \tag{11}$$

Because $-1 < m \le 0$ and $0.5 < q < 1$ it follows that $0 < \alpha < 1$. Hence the crack growth rate,

$$\dot{a} = \alpha\left(\frac{\Im D}{\varepsilon_f^*}\right)^{1/q} t^{\alpha-1} \tag{12}$$

will be ***decreasing***. To derive the trajectory on the $\dot{a} - C^*_{\exp}$ plot note that the experimental estimate of $C^*_{\exp}$ is such that,

$$C^*_{\exp} \propto \dot{\Delta}_c \propto t^{m_{bulk}} \tag{13}$$

Since $-1 < m \le 0$ this means that $C^*_{\exp}$ will be ***decreasing***. Using (13) in (12) gives,

$$\dot{a} \propto \left(C^*_{\exp}\right)^{\beta} \qquad \text{where,} \quad \beta = \frac{(\alpha-1)}{m_{bulk}} \tag{14}$$

For example, taking $q = 0.9$ $(n = 9)$ and $m_{bulk} = -0.5$ then assuming secondary creep at the crack tip $(m_{tip} = 0)$ gives $\beta = 1.78$. Alternatively, assuming primary creep at the crack tip, $m_{tip} = -0.5$, gives $\beta = 1.89$. The salient point is that $\beta > q$ so the slope of the $\dot{a} - C^*_{\exp}$ trajectory exceeds the slope of the trend line and has both $\dot{a}$ and $C^*_{\exp}$ decreasing. It thus lies above the trend line as shown on Figure 1 (line marked A(i)).

When the bulk material later enters secondary creep $C^*_{\exp}$ becomes constant whilst $\dot{a}$ continues to reduce, resulting in the vertical trajectory labelled A(ii) on Figure 1.

Figure 1

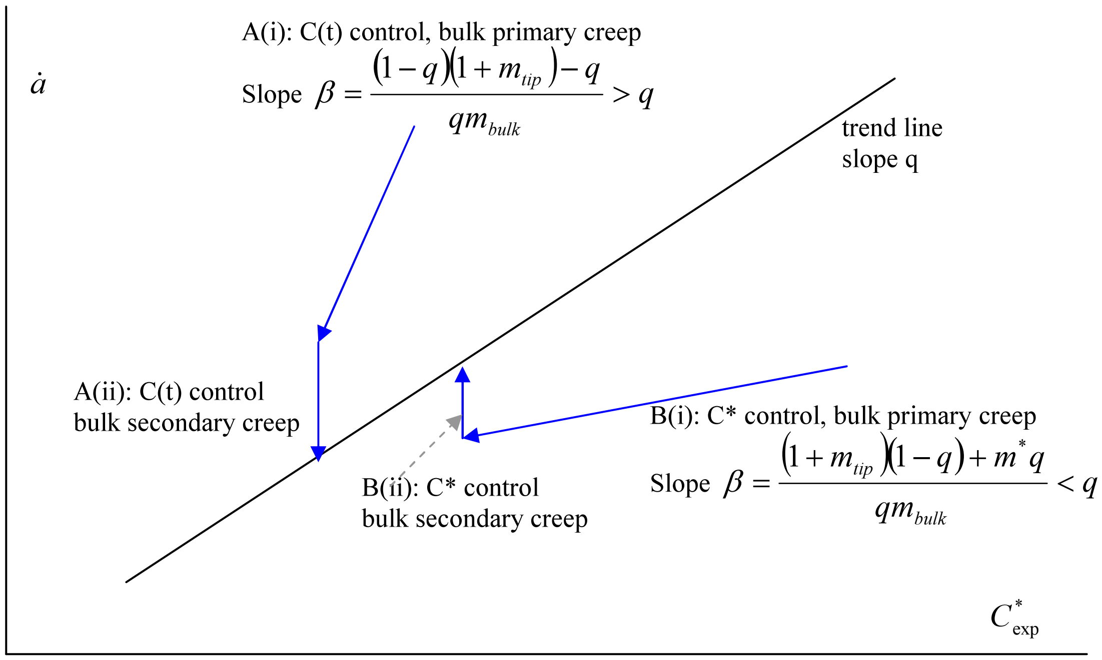

Table 1

Case	Controlled by	Bulk Creep	$\dot{a} - C^*_{\exp}$ Slope	$\dot{a}$
A(i)	C(t)	Primary	$\frac{(1-q)(1+m_{tip}) - q}{qm_{bulk}}$	decreasing
A(ii)	C(t)	Secondary	infinite	decreasing
B(i)	C*	Primary	$\frac{(1+m_{tip})(1-q) + m^* q}{qm_{bulk}}$	decreasing
B(ii)	C*	Secondary	infinite	increasing

NB: C here means the quantity obtained using the reference stress formula and the combined load, hence time varying if relaxation of secondary loads occurs. It is also time dependent for pure primary loading whilst primary creep prevails.*

(B) Assume C^* Control

By C^* here we mean the parameter estimated using the combined load, hence time dependent, reference stress, σ_{ref}, and corresponding reference strain rate, $\dot{\varepsilon}^c_{ref}$, and corresponding LEFM SIF based on the total, time-varying, loading, K_{TOT}, thus,

$$C^* = \frac{\dot{\varepsilon}^c_{ref}}{\sigma_{ref}} K^2_{TOT} \tag{15}$$

The distinction between this C^* and $C(t)$ is entirely due to the "$C(t)$ spike", i.e., the function $f(\tilde{\tau})$ of the dimensionless time $\tilde{\tau} = t / t_{red}$, where,

$$C(t) = C^* f(\tilde{\tau}) \qquad \text{and} \qquad f(\tilde{\tau}) = \frac{(1+\tilde{\tau})^{n+1}}{(1+\tilde{\tau})^{n+1} - \phi} \tag{16}$$

and where $\phi = 1$ for elastic-creep. Plant assessments to R5V4/5 use $C(t)$, obtained essentially as in (16) including the $f(\tilde{\tau})$ factor (though the exact formulation of f in R5V4/5 is slightly more complicated, but this can be ignored for our purposes). This factor can cause $C(t)$ to exceed C^* enormously when $t << t_{red}$, and hence to result in far faster growth rate estimates.

In this section we explore the implications of assuming that growth in test specimens is in fact C^* controlled even for $t < t_{red}$, i.e., in the tail. Ignoring relaxation of secondary stresses, e.g., residual stresses, the time dependence of C^* results only from primary creep and hence,

$$C^* \propto t^{m_{bulk}} \tag{17}$$

However, relaxation of residual stresses will cause C^* to reduce even faster. Rather than attempt a complicated allowance for relaxation, here we simply assume that,

$$C^* \propto t^{m^*} \tag{18}$$

where $m^* < 0$ is an effective index with $|m^*| > |m_{bulk}|$ which accounts for the combined effect of primary creep and relaxation.

The creep HRR field gives the creep strain rate near the crack tip as,

$$\dot{\varepsilon}^c_{tip}(t,r) = \left(Bt^{m_{tip}}\right)^{1-q} \left(\frac{C^*}{I_n r}\right)^q \tilde{\varepsilon}(\theta,n) \propto t^{m_{tip}(1-q)} \left(\frac{t^{m^8}}{r(t)}\right)^q \tag{19}$$

Substitution into (3), and again assuming a solution of the form given by (7), gives,

$$\varepsilon^*_f \propto \int_0^{t_c} \frac{t^{m_{tip}(1-q)+m^* q}}{r(t)^q} dt = \tilde{\Im} \frac{t_c^{m_{tip}(1-q)+m^* q+1}}{r_c^q} \tag{20}$$

where, $$\tilde{\Im} = \int_0^1 \frac{\tau^{m_{tip}(1-q)+m^* q}}{(1-\tau^\alpha)^q} d\tau \tag{21}$$

and hence, since $r_c = \Delta a$, we find the index in this case to be,

$$\alpha = \frac{m_{tip}(1-q) + m^* q + 1}{q} \tag{22}$$

and the crack growth rate is,

$$\dot{a} \propto t^{\alpha - 1} \tag{23}$$

<u>Bulk in Primary Creep</u>

In this case m^* is non-zero and (13) gives $C^*_{\exp} \propto t^{m_{bulk}}$ and so $C^*_{\exp}$ is ***decreasing***. Substitution in (23) gives,

$$\dot{a} \propto \left(C^*_{\exp}\right)^\beta \quad \text{where,} \quad \beta = \frac{(1 + m_{tip})(1-q) + m^* q}{q m_{bulk}} \tag{24}$$

As an example take $q = 0.9$ $(n = 9)$ and $m^* = m_{bulk} = -0.5$ then assuming secondary creep at the crack tip $(m_{tip} = 0)$ gives $\alpha = 0.61$ and $\beta = 0.78$. Alternatively assuming primary creep at the crack tip, say $m_{tip} = -0.5$, then $\alpha = 0.56$ and $\beta = 0.89$. Hence the crack growth rate is again ***decreasing***. However, unlike case A(i), $\beta < q$ so the slope of the $\dot{a} - C^*_{\exp}$ trajectory is less than the slope of the trend line, although both $\dot{a}$ and $C^*_{\exp}$ are decreasing. It thus lies below the trend line as shown on Figure 1 (line marked B(i)).

<u>Bulk in Secondary Creep</u>

When the bulk material later enters secondary creep $C^*_{\exp}$ becomes constant and we take $m_{bulk} = m^* = 0$.

If the bulk is in secondary creep then the crack tip is certainly also in secondary creep, and hence $m_{tip} = 0$. (22) then gives,

$$\alpha = \frac{1}{q} \quad \text{and} \quad \dot{a} \propto t^{\frac{1}{q} - 1} \tag{25}$$

Since $q < 1$ this means that the index for the crack growth is positive and hence the crack growth is ***increasing***. Since $C^*_{\exp}$ is constant this results in the vertical upward trajectory on the $\dot{a} - C^*_{\exp}$ diagram, as shown labelled B(ii) on Figure 1.

Hence,

If the tail lies above the trend line then this indicates $C(t)$ control.
If the tail lies below the trend line then this indicates C^* control (in the sense defined above)

<u>**QED**</u>